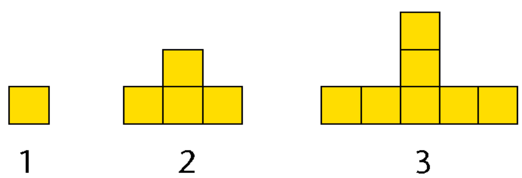
19. INVESTIGATIONS

PATTERNS IN MATHEMATICS

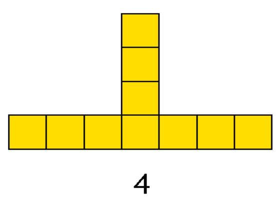
Mathematics has often been referred to as the study of patterns and relationships. In this chapter, we shall focus on this aspect of mathematics. In particular, we will be investigating number sequences and spatial patterns. In doing so, we will learn how to predict the next term in a sequence and to derive the rule for the sequence.

Continuing a pattern

A basic skill in studying patterns is the ability to predict the next term in the sequence. A term here refers to a number or a shape. Here is a basic sequence of shapes in which the first three shapes are drawn.



To draw the next shape in the pattern we observe what has changed from Figure 1 to Figure 2 and this change must be repeated from Figure 2 to Figure 3. In this sequence, three squares are added to the previous shape – two at opposite ends and one on the top. The fourth shape will look like this.



If we are required to determine the number of squares in a given pattern we need to generate a sequence in which the number of squares is counted and recorded.

Pattern Number	Number of squares (by counting)	Number of squares (by pattern)
1	1	1
2	4	$1+3$
3	7	1+3+3
4	10	1+3+3+3
12		1+ 11(3)
n		$1+(n-1)\times 3$ $=3n-2$

In the third column, we observe that:

- each term in the sequence starts with one.
- the number of three's to be added is always one less than the pattern number.

This pattern enables us to determine the number of squares in the 12th pattern as or any desired pattern number, say, n.

We could have deduced the n^{th} term in the sequence using another method. Since each term exceeds the preceding one by at least 3, we can deduce that the pattern is of the form:

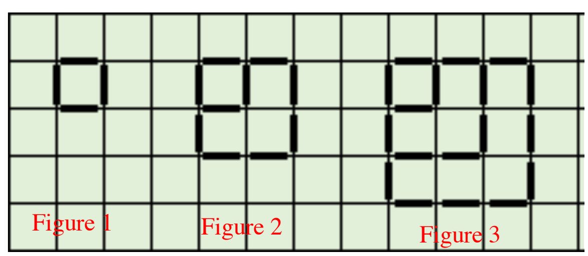
$$t=3n+m$$

where t is the total number of squares in the pattern number, n is the pattern number and m is a constant to be determined.

We can now substitute values of n and t given in the table find these constants.

Consider $t=3n+m$	
When $n=2, t=4$: So, $4=3(2)+m$, $m=4-6=-2$	Check when $n=3, t=7$ So, $7=3(3)+m$, $m=7-9=-2$
We confirm that $t=3m-2$	

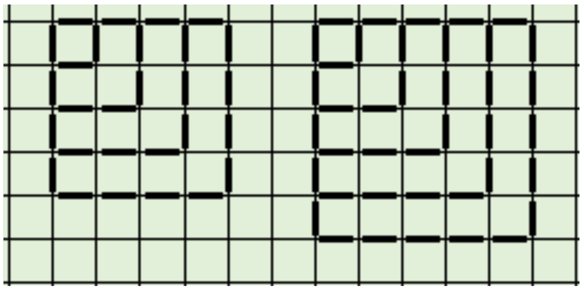
Now let us examine another pattern. Assume that matchsticks are used to form the shapes in the three figures below.



To draw the next two shapes in the pattern we observe that the change in length of the side of each square in the sequence. These lengths would form the number sequence $1, 2, 3, \ldots$

Hence, the next two shapes in the sequence would be squares with sides 4 and 5.

We can now draw the next two shapes to continue the pattern.



We next examine the number of matchsticks in each pattern. This forms the sequence:

4, 10, 18, …

In the table below, we enter the Figure number in the first column.

In column 2 we record the length of the side of each square – note this sequence is the same as the Figure numbers, 1, 2, 3, …, etc.

Figure Number	Length of side of square	Number of matchsticks in the figure (m)
1	1	4
2	2	4 + 6 =10
3	3	10 + 8 =18
4	4	18 + 10 =28
5	5	28 + 12 =40

In column 3 we record the number of matchsticks in each figure. Notice that unlike the previous sequence, the numbers do not increase by a fixed value. However, there is a pattern – the second term can be obtained by adding 6, the third by adding 8, the fourth by adding 10, the fifth by adding 12, and so on.

This pattern enables us to calculate a term provided that we know the previous term but it does not give us the rule for the sequence. To establish a rule for the *n*th term of this sequence will require some more analysis.

When the pattern is not a simple one, some information is given so that the rule can be predicted. Examine the patterns shown in columns 3. In column 3, the total number of matchsticks, *m* is expressed as:

$$m_1 = Figure\ Number \times 4$$
$$m_2 = Figure\ Number \times 5$$
$$m_3 = Figure\ Number \times 6$$

and so on.

In column 4 below. the number of matchsticks in any Figure is expressed as:

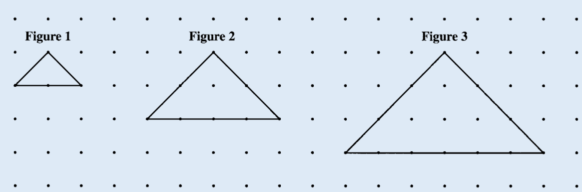
$$m = Figure\ No. \times (3 + Figure\ No.)$$
$$m = n(3 + n)$$

We can now complete the table for the number of matchsticks in the 10th and *n*th term of the sequence.

Figure Number	m	m Pattern 1	m Pattern 2
1	4	1×4	$1 \times (3 + 1)$
2	10	2×5	$2 \times (3 + 2)$
3	18	3×6	$3 \times (3 + 3)$
4	28	4×7	$4 \times (3 + 4)$
5	40	5×8	$5 \times (3 + 5)$
10	130	10×13	$10(3 + 10)$
n			$n(3 + n)$

Example 1

The diagram below shows the first three figures in a sequence. Each figure is an isosceles triangle made with a rubber band stretched around pins in a geoboard. The pins are one unit apart.

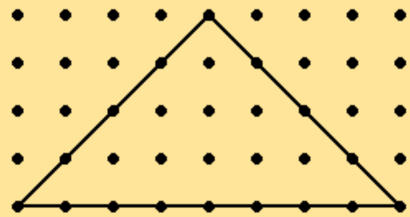


(a) Draw the fourth figure in the sequence.
(b) Complete the table by inserting the missing values at the rows marked (i), (ii), (iii) and (iv).

Figure	**Area of Triangle**	**No. of Pins on Base**
1	1	$(2 \times 1) + 1 = 3$
2	4	$(2 \times 2) + 1 = 5$
3	9	$(2 \times 3) + 1 = 7$
4		
	100	
20		
n		

Solution

(i) The fourth figure in the sequence of figures is:



(ii) The completed table is:

Figure	Area of Triangle	No. of Pins on Base
1	1	$(2 \times 1) + 1 = 3$
2	4	$(2 \times 2) + 1 = 5$
3	9	$(2 \times 3) + 1 = 7$
4	$4^2 = 16$	$(2 \times 4) + 1 = 9$
$\sqrt{100} = 10$	100	$(2 \times 10) + 1 = 21$
20	$20^2 = 400$	$(2 \times 20) + 1 = 41$
n	n^2	$2n + 1$

Example 2

The diagram below shows the first three figures in a sequence of figures. Each figure is made up of squares of side 1 unit.

Figure 1

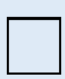


Figure 2

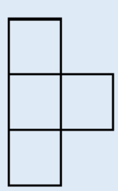
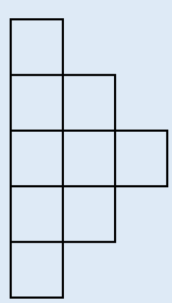


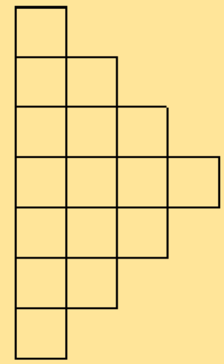
Figure 3



(i) Draw the fourth figure in the sequence.

Solution

(i) Fourth figure



Example 2 (ii)

(ii) Study the patterns in the table shown below, and complete the rows numbered a), b), c) and d).

	Figure	Area	Perimeter
	1	1	4
	2	4	10
	3	9	16
a)	4		
b)	8		
c)	25		
d)	n		

Solution 2(ii)

The numbers in the first column represent the number of squares in the pattern. By squaring these numbers, we obtain the values for the area.

The values in the column labelled perimeter are all increasing by 6.

Pattern Number	Perimeter
2	10 = 4+6
3	16 = 4+6+6
4	22 = 4+6+6+6

Notice the number of sixes to be added each time is one less than the pattern number.

Solution 2(ii)- continued

If n represents the figure number, then the perimeter is

$$4 + (n - 1) \times 6$$

This can be simplified to $4 + 6n - 6 = 6n - 2$

Therefore, when $n = 8, P = 6 \times 8 - 2 = 46$

Hence, when $n = 20, P = 6 \times 20 - 2 = 118$

The completed table is shown below.

	Figure	Area	Perimeter
	1	1	4
	2	4	10
	3	9	16
a)	4	16	22
b)	8	64	46
c)	20	400	118
d)	n	n^2	$6n - 2$

Example 3

A number sequence may be formed by counting the number of dots shown on the perimeter of hexagons.

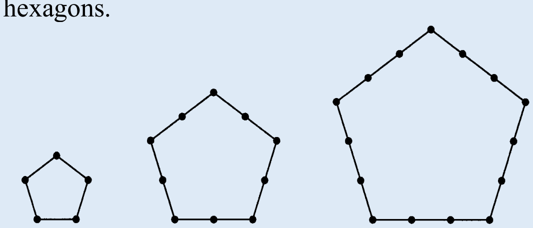


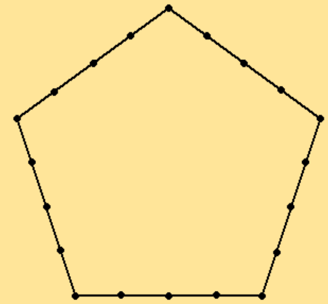
Figure 1 Figure 2 Figure 3

(a) Draw Figure 4, the next figure in the sequence.
(b) Complete the table below by inserting the information in the rows numbered (i) and (iii)

Figure (f)	Total Number of Dots	
	Formula	Number (n)
1	$5\times2-5$	5
2	$5\times3-5$	10
3	$5\times4-5$	15
4	Not Required	Not Required
5		
6		

Solution- Example 3(a) and (b)

(a) Figure 4



(b) We notice that in the column for formula, the expression is always
5 (No. of the figure + 1) – 5
Hence the completed table is:

Figure (f)	Total Number of Dots	
	Formula	Number (n)
1	$5\times2-5$	5
2	$5\times3-5$	10
3	$5\times4-5$	15
4	Not Required	Not Required
5	$5\times$ (5 + 1) -5	25
6	$5\times$ (6 + 1) -5	30

Example 3 (c) and (d)

(c) Write an expression in f for the number (n) of dots used in drawing the fth figure.
(d) Which figure in the sequence contains 145 dots

Solution 3 (c) and (d)

(c) The number of dots, n, in the fth figure is given by (as shown above).

$$n = 5f$$

(d) The figure in the sequence containing 145 dots.

The number of dots $n = 145$

$145 = 5f$

$f = 145 \div 5$

$= 29$

There are 145 dots in the 29th figure.