13. SIMULTANEOUS EQUATIONS

SOLUTION OF A LINEAR EQUATION

In our study of algebra so far, we have solved equations of the form $ax + b = c$, where a, b and c are constants. Such equations are linear equations in one variable (or unknown quantity).
In general, the solution of a linear equation in one variable can be obtained by applying basic principles of algebra. We would have seen this before in the chapter on linear equations.

For example, to solve the equation $2x - 4 = 8$, we would have used the following procedures to obtain a value for the unknown, x.

$$2x - 4 = 8$$
$$2x = 8 + 4$$
$$2x = 12$$
$$x = \frac{12}{2}$$
$$x = 6$$

Our result indicates that $x = 6$ satisfies the equation, $2x - 4 = 8$.
We should also note that $x = 6$ is the **only** value of x that satisfies this equation.
The solution to this equation is certainly unique. We can, therefore, state the following rule.

> A linear equation, in one unknown, has only one solution.

Solution of linear equations in two variables

An equation can have two variables or unknown quantities, such as $ax + by = c$ (where a, b and c are constants). Such equations are linear equations in two variables, x and y. Their solutions must consist of values for x and for y that satisfies the equation.

For example, the solution of the equation, $x + y = 5$, will consist of any pair of numbers whose sum is 5.

Consider, $x = 1$ and $y = 4$. By substituting these values in the given equation, we obtain a solution, since 1+4 = 5. But, is this the only possible solution?
Let us record some more solutions by selecting any arbitrary value for x and calculating the corresponding value of y. In each case, we ensure that the sum of x and y is 5. For simplicity in calculation, let us choose a few positive whole number values (though any real number may be chosen). Some of these solutions are presented in the table below.

x	y	$x + y$
0	5	5
5	0	5
1	4	5
2	3	5
3	2	5
4	1	5

We could have presented our solutions as a set of ordered pairs, such as:

(0, 5), (5, 0), (1, 4), (2, 3), (3, 2), (4, 1), …

We can state, with certainty, that there are many values of x and y that satisfy the equation $x + y = 5$.
There would be solutions with negative values as well, such as
(–6, 11), (7, –2), (–9, 14), (12, –7), …

There would be solutions with fractional values, such as
(1.2, 3.8), (2.9, 2.1), (–8.3, 13.3), (12.9, –7.9), …
The solution to this equation is certainly not unique. We can, therefore, state the following rule.

> A linear equation in two unknowns has many solutions. These solutions can be represented as a set of ordered pairs. In fact, the number of solutions is infinite.

Solution of a pair of linear equations in two variables

Now, let us introduce another linear equation in two variables. For example, $x - y = 3$.
By a similar argument, this equation will also have an infinite number of solutions. Solutions will include any two numbers, x and y, and whose x value is 3 more than its y value.

For example, $x = 5$ and $y = 2$ is one such pair of values.
Once more, we select a set of arbitrary values of x and calculate the corresponding value of y so that $x - y$ is always 3. We will use only positive number values for simplicity in the arithmetic.

We can record our solutions using a table like the one above. Some solutions are shown below.

x	y	$x-y$
0	−3	3
3	0	3
5	2	3
4	1	3
3	0	3
7	4	3

We could have presented our solutions as a set of ordered pairs as shown below.

(0, −3), (3, 0), (5, 2), (4, 1), (3, 0), (7, 4), …

Assuming that we wish to find a solution that satisfies both of the above equations, that is, we wish to find the unique solution that satisfies the pair of equations, $x+y=5$ and $x-y=3$. Let us refer to these equations as equation (1) and (2) respectively.

$x+y=5$ (1)

$x-y=3$ (2)

We are now interested in obtaining the pair of numbers whose sum is 5 [equation (1)] and whose difference is 3 [equation (2)].
We already have an infinite set of solutions to equation (1).
We also have an infinite set of solutions for equation (2).

We wish to find the common solution to both equations.
By observation, we can deduce the solution that satisfies both equations. We can compare both sets of ordered pairs and look for this common pair. The ordered pairs of equation 1 and equation 2 respectively are:

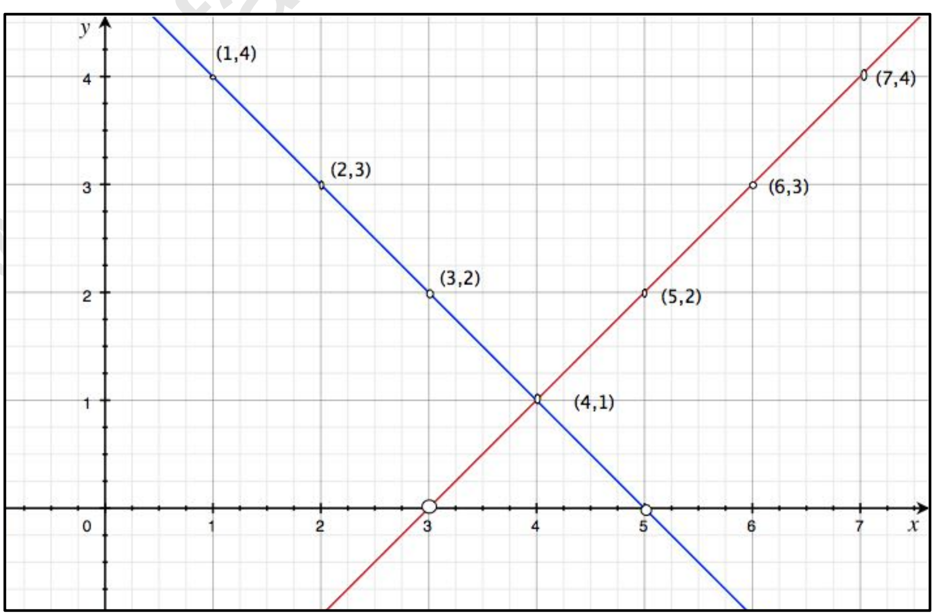
(0, 5), (5, 0), (1, 4), (2, 3), (3, 2), (4, 1), …

(0, −3), (3, 0), (5, 2), (4, 1), (3, 0), (7, 4), …

Notice that (4, 1) is the only pair that is common to both sets. The solution of the pair of equations is therefore unique. We can, therefore, state the following rule.

A pair of linear equation in two unknowns say x and y, has a unique solution. The solution can be represented by an ordered pair, (x, y).

We may solve a pair of linear equations together to obtain a unique set of values of x and y that satisfy both equations. Since we must solve these equations together, that is at the same time or simultaneously, they are called simultaneous equations.
The above method of finding a common solution to a pair of linear method is more efficient when we plot the ordered pairs (draw the graphs) for both functions and locate the point of intersection of the straight lines



As shown above, the point (4, 1) is the only point common to both lines and represents the solution to the pair of equations when solved simultaneously. This method is called the graphical method of solving a pair of linear equations in two variables.

The solution to the above pair of linear equations could also have been obtained algebraically. In so doing, we will examine two methods:
These are
 i. Elimination method
 ii. Substitution method

Solving simultaneous equations -elimination

Using the same equations from above, we will illustrate how to solve for x and y using the method of elimination.

$x + y = 5$ (1)
$x - y = 3$ (2)

In the method of elimination, we examine both equations and see if it is possible to eliminate a variable by either adding or subtracting like terms.

In our example, this is possible. We notice that y can be eliminated by adding both equations. This is shown below.

Method of elimination	
$x + y = 5$ $x - y = 3$ $2x = 8$ $x = 4$	By adding both equations, we have eliminated one unknown, y. The resulting equation, $2x = 8$ has only one unknown, x. This can be easily solved to obtain $x = 4$.
$4 + y = 5$ $y = 5 - 4$ $y = 1$	We can substitute $x = 4$ in either (1) or (2) to solve for y. We chose equation (1) in this case and obtained $y = 1$.

This solution may be written as the ordered pair (4, 1). Note that these values satisfy both equations.

Example 1

Solve for x and y in
$3x - 2y = 11$ (1)
$4x + 2y = 10$ (2)

Solution

Notice that if we add equations (1) and (2), y will be **eliminated.**

$3x - 2y = 11$ (1)
$4x + 2y = 10$ (2)
$7x = 21$ Equation (1) + equation (2)
$x = 3$

We are now left with an equation in x only.
To obtain the value of y, we now substitute the known value of x in any of the above equations. If we choose equation (2), then we would obtain,

$$4(3) + 2y = 10$$
$$2y = 10 - 12$$
$$2y = -2$$
$$y = -1$$

Hence $x = 3$ and $y = -1$

The addition of two given equations does not always eliminate one of the unknowns. In some cases, we may find it convenient to subtract the equations to eliminate one unknown.

Example 2

Solve for x and y in
$5y + 2x = 9$ (1)
$2y + 2x = 6$ (2)

Solution

If we subtract one equation from the other, we can eliminate the unknown, x.

$5y + 2x = 9$ (1)
$2y + 2x = 6$ (2)
$3y = 3$ Equation (1) – Equation (2)
$y = 1$

To obtain the value of x, we substitute the known value of y in any of the above equations. If we choose equation (2), then we would obtain,

$$2(1) + 2x = 6$$
$$2x = 6 - 2$$
$$2x = 4$$
$$x = 2$$

Hence $x = 2$ and $y = 1$

In some cases, neither addition nor subtraction would result in the elimination of a variable. And so, we must apply another strategy prior to elimination.

Example 3

Solve for x and y in

$5x - 4y = 7$ (1)

$7x - 2y = 17$ (2)

Solution

Notice, if we add or subtract both equations, we would still obtain an equation in x and y. It would be convenient if our second equation has $4y$ instead of $2y$. This situation is rectified if we multiply the entire equation (2) by 2.

So, our new equation (2), which we may now call equation (3) becomes:

$14x - 4y = 34$ (3)

We will now replace equation (2) by equation (3), which is really an equivalent form of equation (2). We can now solve the following pair of equations:

$5x - 4y = 7$ (1)

$14x - 4y = 34$ (3)

Subtracting equation (3) from equation (1), we obtain:

$$\begin{aligned} 5x - 4y &= 7 \quad (1) \\ 14x - 4y &= 34 \quad (3) \\ \hline -9x &= -27 \\ x &= 3 \end{aligned}$$

To obtain the value of y, we substitute the known value of x in any of the above equations. If we choose equation (1), then we would obtain,

$$\begin{aligned} 5(3) - 4y &= 7 \\ -4y &= 7 - 15 \\ -4y &= -8 \\ y &= 2 \end{aligned}$$

Hence, $x = 3$ and $y = 2$.

Example 4

Solve for x and y given

$3x + 4y = 10$

$5x - 7y = 3$

Solution

In this example, it is not possible to eliminate an unknown by adding the equations, subtracting one from the other or even changing one equation to an equivalent form. We must, therefore, change both equations.

If we choose to eliminate y, then consider the y coefficients of the two equations which are 4 and 7. We need to have a common coefficient. The best choice is the LCM of 4 and 7, which is 28. So, we can multiply each equation by a number so that both y coefficients become 28.

This means, equation (1), in which the y coefficient was 4, must be multiplied by 7 and equation (2) in which the y coefficient was -7, must be multiplied by 4. Our two new equations are called equation (3) and (4) respectively.

$3x + 4y = 10$ …(1)

$5x - 7y = 3$ …(2)

Equation (1) $\times$ 7

$21x + 28y = 70$ …(3)

Equation (2) $\times$ 4

$20x - 28y = 12$ …(4)

Adding (3) and (4) we obtain

$$\begin{aligned} 21x + 28y &= 70 \\ 20x - 28y &= 12 \quad + \\ \hline 41x &= 82 \\ x &= \frac{82}{41} \\ &= 2 \end{aligned}$$

Solving simultaneous equations -substitution

The method of substitution is an important one and it is especially useful when solving a pair of equations when one of them is not linear. However, in this chapter, we will continue to solve a pair of linear equations simultaneously.

Example 5

Solve for x and y in

$$x+3y=7...(1)$$
$$2x+5y=8...(2)$$

Solution

The first step involves expressing one variable in terms of the other. We may choose **any** of the given equations.

By examination, the first equation is a better choice because we can isolate x easily.

From Equation (1) $x=7-3y$

Substituting the expression $x=7-3y$ in equation (2), we have

$$2(7-3y)+5y=8$$

This produces **ONE** equation with **ONE** unknown quantity, which can be easily solved.
Solving for y:

$$2(7-3y)+5y=8$$
$$14-6y+5y=8$$
$$-y=-6$$
$$y=6$$

This value of y is now substituted in any of the given equations or the expression obtained from the first step, to obtain the value of the other unknown.

$$x=7-3y$$
$$x=7-3(6)$$
$$x=7-18$$
$$x=-11$$

Hence $x=-11$ and $y=6$.

Example 6

Solve for x and y in

$$3x+4y=10 \quad \ldots(1)$$
$$5x-7y=\ 3 \quad \ldots(2)$$

Solution

We can choose any of the given equations and express one of the unknown quantities in terms of the other. We choose equation 2.

$$5x-7y=3$$
$$5x=3+7y$$
$$x=\frac{3+7y}{5}$$

This expression is now substituted into equation 1.

Substitute $x=\frac{3+7y}{5}$ into the equation (1)

$$3\left(\frac{3+7y}{5}\right)+4y=10$$

This produces **ONE** equation with **ONE** unknown quantity, which can be solved. The algebra here requires a knowledge of algebraic fractions. To simplify the equation we multiply both sides by 5.

$$\frac{3(3+7y)}{5}+\frac{4y}{1}=\frac{10}{1}$$

$\times 5$

$$3(3+7y)+5(4y)=(5\times 10)$$
$$9+21y+20y=50$$
$$\therefore 41y=50-9$$
$$41y=41$$
$$y=\frac{41}{41}$$
$$=1$$

This y value is now substituted in any of the given equations or the expression obtained from the first step, to obtain the value of the other unknown.

$$x=\frac{3+7y}{5}$$
$$=\frac{3+7(1)}{5}$$
$$=\frac{10}{5}$$
$$=2$$

Hence, $x=2$ and $y=1$.

Word problems involving two variables

We can now apply our knowledge of solving equations simultaneously to word problems involving two variables. In these problems, we are given information in words from which we need to create our pair of equations. After we create the pair of equations, we use any of the methods for solving simultaneous to obtain the value of each variable.

Example 7

The cost of 4 apples and 5 pears is \$23 and the cost of 3 apples and 7 pears is \$27. Find the cost of 1 apple and the cost of 1 pear.

Solution

Let the cost of 1 apple be $\$x$.
Let the cost of 1 pear be $\$y$.
If 4 apples and 5 pears cost \$23, then

$(4\times x)+(5\times y)=23$

and so $4x+5y=23 \ldots(1)$

Similarly,

$(3\times x)+(7\times y)=27$

$3x+7y=27 \quad \ldots(2)$

Solving by the method of elimination:

$4x+5y=23\ldots(1)$
$3x+7y=27\ldots(2)$

$12x+15y=69\ldots(3)$ Eq (1) × 3
$12x+28y=108\ldots(4)$ Eq (2) × 4
$13y = 39$ Eq (4) – Eq (3)
$y=3$

Substituting for y in equation (1)

$4x+5(3)=23$
$4x+15=23$
$4x=23-15$
$4x=8$
$x=2$

The cost of 1 apple is \$2
The cost of 1 pear is \$3.

Example 8

The sum of two numbers is 72 and their difference is 50. Find the two numbers.

Solution

Let the larger number be p.
And let the small number be q.

Hence, from the data

$p+q=72 \ldots(1)$
$p-q=50 \ldots(2)$
$2p=122$ Adding Eq (1) and (2)
$p=61$

Substituting $p=61$ in equation (1)

$61+q=72$
$q=72-61$
$q=11$

The smaller number is 61
The larger number is 11

Example 9

Lollies cost $\$x$ per pack and ice-cream costs $\$y$ per cup.
One pack of lollies and 2 cups of ice cream cost \$8.
3 packs of biscuits and 1 cup of ice cream cost \$9.
Find the cost of 1 pack of lollies and 1 cup of ice-cream.

Solution

1 pack of lollies and 2 cups of ice cream cost \$8

$1\times x+2\times y=8$
$x+2y=8$

3 packs of lollies and 1 cup of ice cream cost \$9.

$3\times x+1\times y=8$
$3x+y=9$

$x+2y=8 \quad \ldots(1)$
$3x+y=9 \quad \ldots(2)$

Using the method of substitution, from equation (1) $x=8-2y \quad \ldots(3)$

Substitute $x=8-2y$ in (2)

$3(8-2y)+y=9$
$24-6y+y=9$
$24-5y=9$
$24-9=5y$
$5y=15$
$y=3$

Substituting $y=3$ in (3), $x=8-2(3)=2$

1 pack of lollies cost \$2
1 cup of ice cream cost \$3

Example 10

Solve for x and y in

$$x + 2y = 3 \qquad \ldots(1)$$
$$5x - 4y = 1 \qquad \ldots(2)$$

by a graphical method.

Solution

To obtain a set of values for x and y that satisfies both equations, we need to draw the graph of each equation separately and on the same axes.

Consider equation (1).

$$x + 2y = 3 \qquad \ldots(1)$$
$$2y = 3 - x$$

When $x = -1$ $2y = 3 - (-1)$ $2y = 3 + 1 = 4$ $y = 2$	When $x = 5$ $2y = 3 - 5$ $2y = -2$ $y = -1$	The points are x: -1, 5 y: 2, -1

Likewise, we can do the same for equation (2), and obtain a table of values that satisfies the equation, $5x - 4y = 1$.

Consider equation (2)

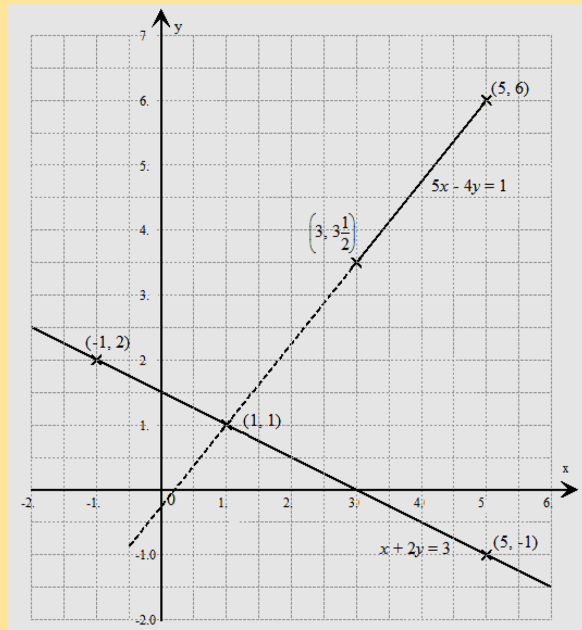
$$5x - 4y = 1$$
$$5x - 1 = 4y$$
$$y = \frac{5x - 1}{4}$$

When $x = 3$ $y = \frac{5(3) - 1}{4}$ $y = \frac{14}{4}$ $y = 3\frac{1}{2}$	When $x = 5$ $y = \frac{5(5) - 1}{4}$ $y = \frac{24}{4}$ $y = 6$	The points are x: 3, 5 y: $3\frac{1}{2}$, 6

These values of x and y from the tables are plotted on the coordinate plane to obtain two straight lines. The graphs are shown below shows the straight lines joining points:

$$(-1, 2) \text{ and } (5, -1)$$
$$\left(3, 3\frac{1}{2}\right) \text{ and } (5, 6)$$

The point of intersection gives the solution of the equations. In this example, the point of intersection is (1, 1).



Hence, $x = 1$ and $y = 1$.

Some points to note when solving a pair of linear equations simultaneously

1. It is sometimes possible to obtain two equations such that when drawn, they represent parallel lines. Since parallel lines do not intersect, there will be no solutions. For example, there are no solutions to the pair of equations:

$$5x + 6y = 15$$
$$5x + 6y = 25$$

2. If the two equations are equivalent, for example, the two lines are actually the same and we have an infinite number of solutions. This is different to the equations of parallel lines. A pair of equivalent equations is shown below.

$$3x + 2y = 5$$
$$6x + 4y = 10$$