9. PLANE GEOMETRY

PLANE FIGURES

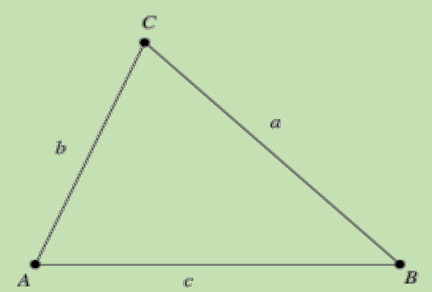
In mathematics, a **plane** is a flat or two-dimensional surface that has no thickness that and so the term 'plane figures' is used to describe figures that are drawn on a plane. Circles, ellipses, triangles, quadrilaterals and other polygons are some examples of plane figures. All plane figures are two-dimensional in nature and the study of these shapes is known as plane geometry or Euclidean geometry.

Triangles

A triangle is a plane figure bounded by three straight lines. It has three sides and three angles. Hence, there are six elements in a triangle that can be measured. The sides meet at three points called the vertices (singular vertex). There are many different types of triangles and although all triangles have some common properties, different types have their own special additional properties as well.

Naming triangles

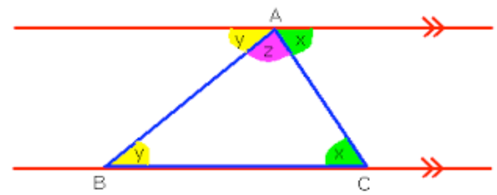
A triangle is named using the three letters that refer to their vertices. Vertices are named using 'upper case' letters and sides opposite the vertex are named using the corresponding lower case letters.



In the triangle, the vertices are A, B and C, and so, the triangle is named ABC.
The side opposite vertex A, BC is denoted a, the side opposite vertex B is denoted b and the side opposite angle C is named c.

Sum of the angles in a triangle

The three angles at the vertices of a triangle are called interior angles. In the diagram below, x, y and z are the interior angles of the triangle.
Angle y at vertex B is the same as angle y at vertex A (alternate angles).
Angle x at vertex C is the same as angle x at vertex A (alternate angles).
At the vertex A, $x + y + z = 180^0$ (angles on a straight line).
But, x, y and z are the interior angles of the triangle ABC. Hence, the sum of the interior angles of a triangle must be 180^0.

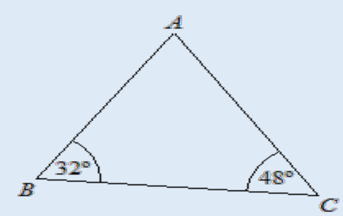


We can now state a general rule connecting the three interior angles of a triangle.

The sum of the interior angles of a triangle is 180^0.

Example 1

In the triangle, ABC, $\hat{B} = 32°$ and $\hat{C} = 48°$; calculate the size of $\hat{A}$.



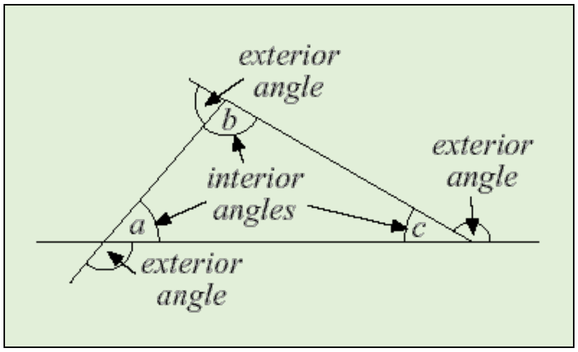
Solution

$$\hat{A} + 32° + 48° = 180°$$
$$\therefore \hat{A} = 180° - (32° + 48°)$$
$$= 100°$$

(The sum of the interior angles in a triangle $= 180°$)

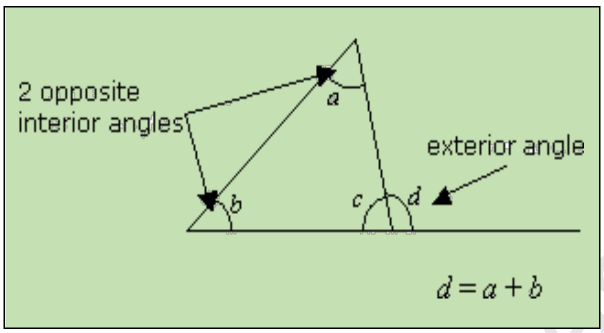
Exterior angle of a triangle

If we extend the side of a triangle, the angle created between the extended side and the 'next' side of the triangle is called an exterior angle.
A triangle has 3 interior angles and three exterior angles. The exterior angles are formed by extending each side, moving either in a clockwise direction or in an anticlockwise direction. Exterior angles are shown below.



Exterior Angle Theorem

In the triangle below, d is an exterior angle and a, b and c, are interior angles. We refer to the angles a and b as the interior opposite angles since they are opposite to the exterior angle, d.



We already know that the sum of the interior angles in a triangle is 180°. That is,

$a + b + c = 180°$

Also, $d + c = 180^0$ (d and c lie on a straight line)

Therefore, $a + b + c = d + c = 180^0$

Or $(a + b) + c = d + c$

Subtracting c from each side of the equation, we have

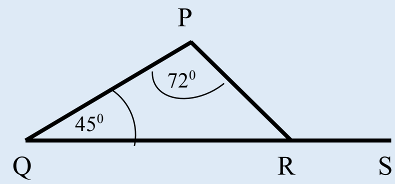
$(a + b) = d$

We can now state a general rule connecting the exterior of a triangle and the interior opposite angles of the triangle.

Exterior Angle Theorem:
The measure of the exterior angle of a triangle is equal to the sum of the measures of the two interior opposite angles.

Example 2

In ΔPQR, angle $PQR = 45°$, and angle $QPR = 72°$. Calculate the measure of the measure of the exterior angle at R.



Solution

The exterior angle at R, angle PRS, is equal to the sum of the two interior angles at R and at Q,

$PRS = 45^0 + 72^0$
$PRS = 117^0$

General properties of triangles

The following relationships are true for all triangles, regardless of the shape or size.

1. The shortest side is always opposite the smallest interior angle.
2. The longest side is always opposite the largest interior angle.
3. If two sides are equal, then the equal sides are opposite equal angles.
4. If all sides are equal, then all three angles have the same measure of 60^0.
5. The sum of the lengths of any two sides of a triangle is greater than the length of the third side (triangle inequality theorem).

Types of triangles

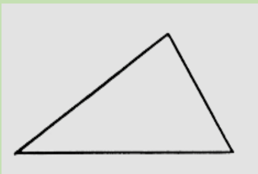
In describing types of triangles, we should note that there are two classifications. We can classify triangles by:

(a) the relationship between the length of their sides
(b) the relationship between the measure of their angles

As we shall soon see, these classifications are not mutually exclusive as it is possible to have a triangle that belongs to both sets.

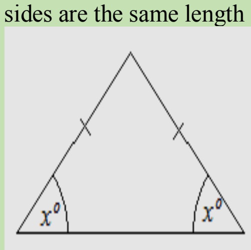
Classification of triangles by their sides

Scalene Triangle
No sides of the same length



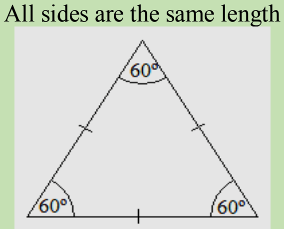
No equal angles

Isosceles triangle Two sides are the same length



Angles opposite equal sides are equal

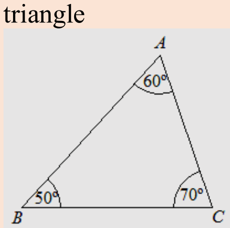
Equilateral triangle
All sides are the same length



All angles are equal to 60^0

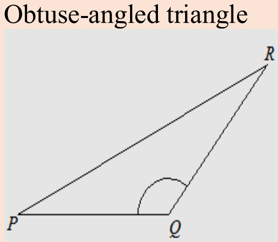
Classification of triangles by their angles

Acute-angled triangle



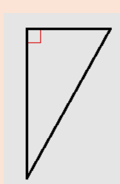
All angles are acute

Obtuse-angled triangle



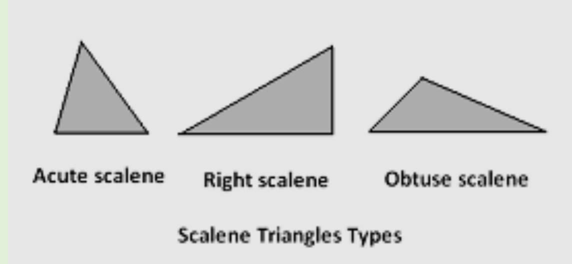
One angle is obtuse

Right-angled triangle



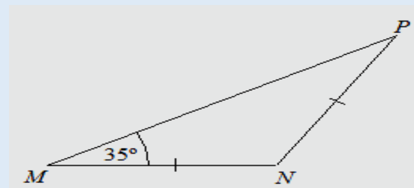
One angle is right

As mentioned above, a given triangle can have properties that come from both sets. For example, consider the following scalene triangles in the diagram below. each one is different in terms of the type of angles.



Example 3

In the figure, triangle MNP is isosceles with $NP = NM$. If $\hat{M} = 35°$, calculate the measure of $\hat{P}$ and $\hat{N}$.



Solution

$\hat{P} = 35°$ (The base angles of an isosceles triangle are equal)

$\hat{N} = 180° - (35° + 35°)$

$\quad = 110°$

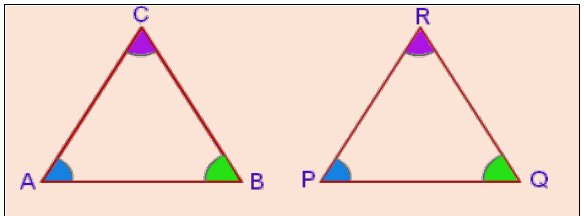
(The sum of the angles in a triangle is 180º)

Congruence

Two plane shapes are said to be congruent if they have the exact same shape and size. Hence, if either one is placed atop the other, there is no overlap. In other words, congruent shapes are the same in all respects and either one can replace the other without detection.

Congruent triangles

Before we introduce the idea of congruence in triangles, it is important to understand certain key terms. We used the term corresponding angles with reference to parallel lines earlier in this chapter. In the same way, we can refer to corresponding angles or even corresponding sides with reference to triangles. If the congruent triangles, shown below, are superimposed, **corresponding sides** will coincide with each other and **corresponding angles** will coincide. For example, in the diagram below, the side AB will coincide with PQ and so we say that AB **corresponds** to PQ. So too, the angle at B will coincide with the angle at Q and we say that angle B **corresponds** to angle Q and so on.



Corresponding angles	Corresponding sides
$\angle A$ and $\angle P$	AB and PQ
$\angle B$ and $\angle Q$	AC and PR
$\angle C$ and $\angle R$	BC and QR

Conditions for congruence

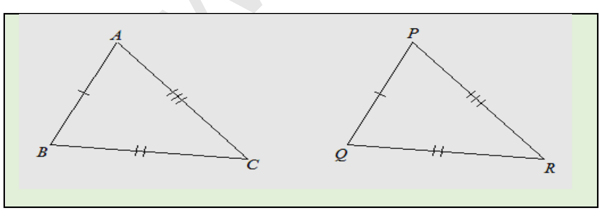
When two triangles are congruent, the three corresponding sides and the three corresponding angles are equal. However, if we wish to prove that two triangles are congruent, it is not necessary to prove that all six elements are identical.

We can apply certain tests for congruency and this will allow us to determine very easily if two triangles are congruent. These tests outline a set of minimum conditions that must be satisfied for congruency to be concluded.

There are four different tests named below. Each test comprises a set of three conditions, which determine whether or not a pair of triangles are congruent. If any one of these tests is satisfied then the triangles are congruent. These tests are described below.

1. Three sides (SSS)

If three pairs of sides of two triangles are equal in length, then the triangles are congruent.



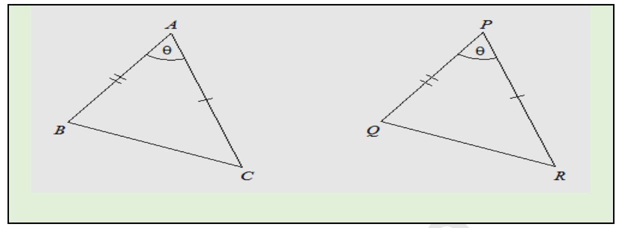
In triangle ABC and PQR

1. $AB = PQ$
2. $BC = QR$
3. $AC = PR$.

Therefore, $\Delta ABC \cong \Delta PQR$.
The symbol $\cong$ means 'is congruent to'.

2. Two sides and the included angle (SAS)

If two pairs of sides of two triangles are equal in length, and the measures of the included angles are equal, then the triangles are congruent.



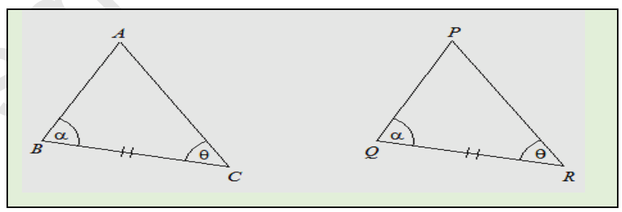
In triangle ABC and PQR

1. $AB = PQ$
2. $AC = PR$
3. $\angle BAC = \angle QPR$.

Therefore, $\Delta ABC \cong \Delta PQR$.

3. Two angles and the corresponding side (ASA).

If two pairs of angles of two triangles are equal in measurement, and the included sides are equal in length, then the triangles are congruent.



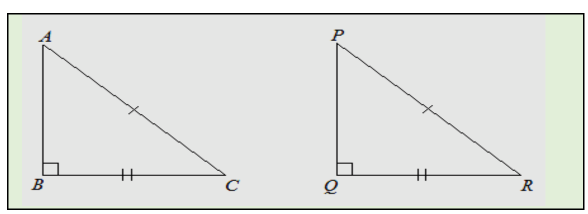
In triangle ABC and PQR

1. $BC = QR$
2. $\angle ABC = \angle PQR$
3. $\angle ACB = \angle PRQ$.

Therefore, $\Delta ABC \cong \Delta PQR$.

4. Right angle, hypotenuse and side (RHS).

If two right-angled triangles have their hypotenuses equal in length, and a pair of shorter sides are also equal in length, then the triangles are congruent.



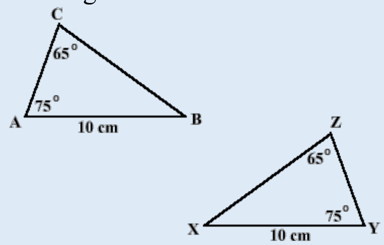
In triangle ABC and PQR

1. $\angle ABC = \angle PQR = 90^0$
2. $BC = QR$
3. $AC = PR$.

Therefore, $\Delta ABC \cong \Delta PQR$.

Example 4

State, with reason, whether the triangles shown below are congruent.



Solution

In triangles ACB and XYZ:

1. $\angle ACB = \angle XZY$
2. $\angle CAB = \angle XYZ$
3. $AB = XY$

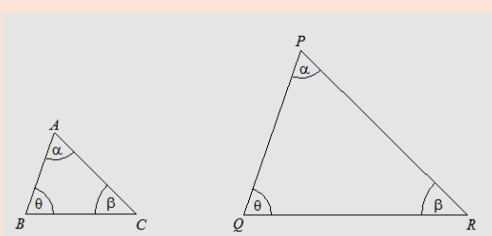
Therefore, triangles ACB and XYZ are congruent (two angles and the corresponding side).

Similar Figures

Similar figures have the same shape but differ in size. Since their shapes are the same, similar figures are also said to be equiangular, that is, their corresponding angles are equal.

The triangles below are similar because $\hat{A} = \hat{P}$, $\hat{B} = \hat{Q}$ and $\hat{C} = \hat{R}$. That is, they are equiangular.

We may say that ΔPQR is an enlargement of ΔABC or ΔABC is a reduction of ΔPQR.



Since similar figures have the same shape, they have the property of the geometrical transformation, **enlargement**, and their sides are in a fixed proportion.

Hence, we can conclude that

$$\frac{PQ}{AB} = \frac{QR}{BC} = \frac{PR}{AC} = \frac{k}{1}$$

where k is the scale factor of the enlargement. Hence, the length of a side of triangle PQR is k times the length of the corresponding side of triangle ABC.

If we reversed the order of comparison, then we would have

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = \frac{1}{k}$$

In stating the ratios, one must be consistent in maintaining the order. Notice that the above ratio compares the sides of the smaller triangle with the sides of the larger triangle.

Similar Figures

If any two plane figures are similar, then

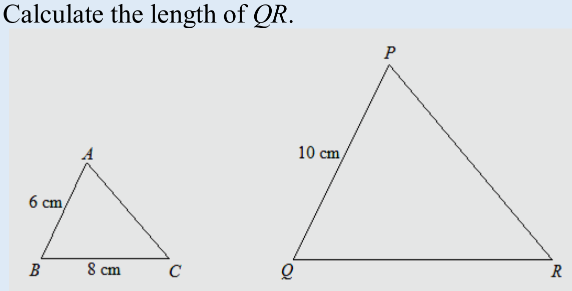
i. the figures are equiangular
ii. the ratios of their corresponding sides are the same.

The converse is also true, so if we know that a pair of figures have corresponding sides in a fixed ratio, then the figures are similar and their corresponding angles are equal.

Example 5

ΔABC is similar to ΔPQR such that $\hat{A} = \hat{P}, \hat{B} = \hat{Q},$ and $\hat{C} = \hat{R}$.

Calculate the length of QR.



Solution

The ratio of their corresponding sides is the same, so if we compare the smaller triangle with the larger triangle, we obtain

$$\frac{AB}{PQ} = \frac{BC}{QR}$$

$$\frac{6}{10} = \frac{8}{QR}$$

$$QR \times 6 = 8 \times 10$$

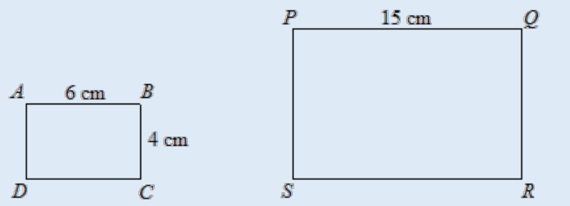
$$QR = \frac{8 \times 10}{6}$$

$$\text{QR} = \frac{40}{3} = 13\tfrac{1}{3}\,cm$$

We can apply the concept of similarity to solve problems on other plane figures as well.

Example 6

Rectangle *ABCD* is similar to rectangle *PQRS*. Calculate the length of *QR*.



Solution

The ratio of their corresponding sides is the same, so comparing the smaller square with the larger square, we have:

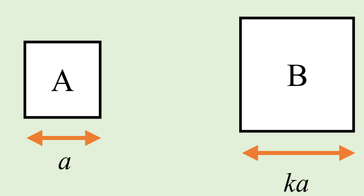
$$\frac{AB}{PQ}=\frac{BC}{QR}$$

$$\frac{6}{15}=\frac{4}{QR}$$

$$QR=\frac{4\times 15}{6}=10 \text{ cm}$$

Ratio of the area of similar figures

Consider the two squares, A and B shown below. They are similar figures since they have the same shape.
Let the side of the smaller square, A be a units.
Let the side of the larger square, B be ka units.



Area $=a^2$	Area $=(ka)^2$

Comparing the sides and the areas of both squares, we have.

The ratio of side B to side A $=\frac{ka}{a}=k$

The ratio of the area of B to the area of A is

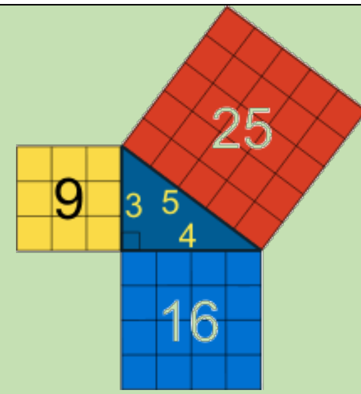
$$\frac{(ka)^2}{a^2}=\frac{k^2a^2}{a^2}=k^2$$

The ratio of the areas of similar figures is equal to the square of the ratio of their corresponding sides.

Pythagoras' Theorem

The relationship between the length of the sides of right-angled triangles was derived by the Greek mathematician, Pythagoras. In a right-angled triangle, the longest side is called the hypotenuse.

We will not show a formal proof of this theorem but the following diagram illustrates how this theorem works.



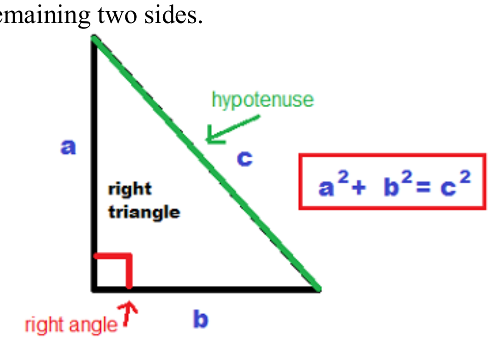
In the right-angled triangle shown, the sides are 3, 4 and 5 units. When these numbers are squared, we obtain the square numbers 9, 16 and 25. The square numbers are illustrated by drawing squares on the three sides of the triangle.

The theorem states that the square on the hypotenuse (area of the large square) is equal to the sum of the squares on the other two sides (areas of the two smaller squares). Note that 9 + 16 = 25.

The set of numbers {3, 4, 5} is called a Pythagorean triple or triplet. Other examples of Pythagorean triples are {6, 8, 10}, {5, 7, 12} and {8, 15, 17}.

Pythagoras' Theorem

The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the remaining two sides.

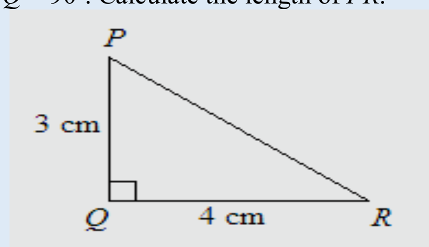


This **theorem** enables us to calculate the length of any side of a right-angled triangle if we know the length of any two sides. We can re-arrange the formula to calculate any unknown side of a right-angled triangle.

$$c^2 = a^2 + b^2$$
$$c = \sqrt{a^2 + b^2}$$
$$a = \sqrt{c^2 - b^2}$$
$$b = \sqrt{c^2 - a^2}$$

Example 7

Triangle PQR is such that $PQ = 3$ cm, $QR = 4$ cm and $Q = 90^0$. Calculate the length of PR.

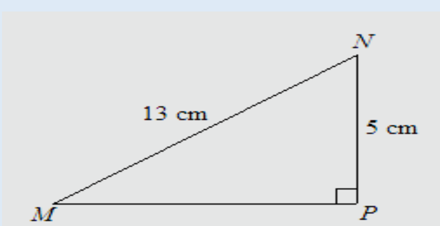


Solution

$$\begin{aligned} PR^2 &= PQ^2 + QR^2 \quad \text{(Pythagoras' Theorem)} \\ &= (3)^2 + (4)^2 \\ &= 25 \\ PR &= \sqrt{25} \\ &= 5 \text{ cm} \end{aligned}$$

Example 8

In the figure MN is 13 cm, NP is 5cm and the angle at P is a right angle. Calculate the length of MP.

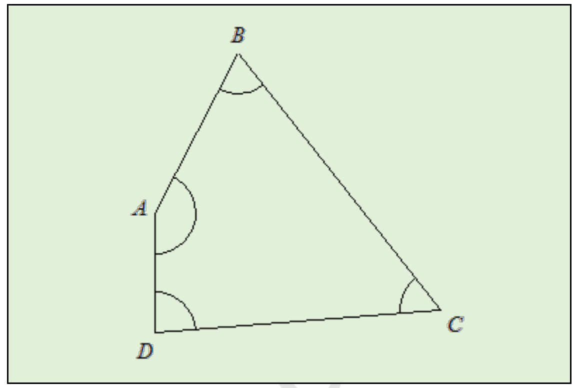


Solution

$$\begin{aligned} MP^2 + (5)^2 &= (13)^2 \quad \text{(Pythagoras' Theorem)} \\ MP^2 &= (13)^2 - (5)^2 \\ &= 144 \\ MP &= \sqrt{144} \\ &= 12 \text{ cm} \end{aligned}$$

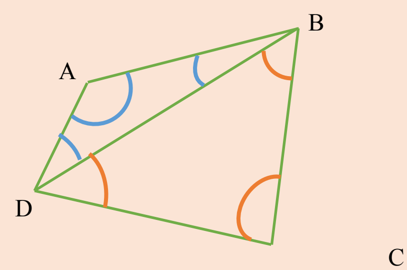
Quadrilaterals

A quadrilateral is a plane figure bounded by four (4) straight lines. Hence, it has four interior angles.



Sum of the interior angles of a quadrilateral

By dividing a quadrilateral into two triangles, we can deduce that the sum of the interior angles is 360^0 as shown below:



The sum of the angles in triangle ABD is 180^0.
The sum of the angles in triangle BDC is 180^0.
Therefore, the sum of the angles in quadrilateral ABCD is
$180^0 + 180^0 = 360^0$
OR $\hat{A} + \hat{B} + \hat{C} + \hat{D} = 360°$

We can now state a general rule connecting the four interior angles of a quadrilateral.

The sum of the four interior angles of a quadrilateral is 360^0.

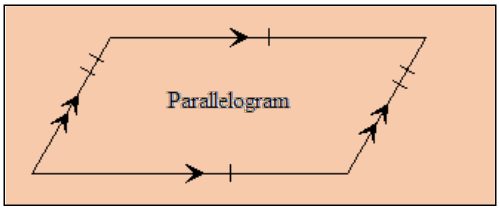
Like triangles, quadrilaterals can have different properties depending on the relationship between the length of their sides and angles. Since quadrilaterals have four sides, an additional property comes into play. Sometimes, one or even both pairs of opposite sides can be parallel. This property introduces more variability when studying the types of quadrilaterals.

Types of quadrilaterals

We will begin our study of quadrilaterals by examining the parallelogram since many types of quadrilaterals are special parallelograms. We will identify the minimum criteria that must be present to define the quadrilateral.

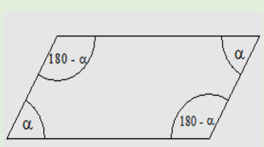
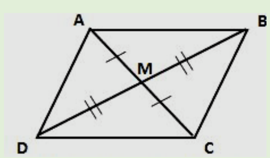
Parallelogram

A parallelogram is defined as a quadrilateral with opposite sides parallel and equal.



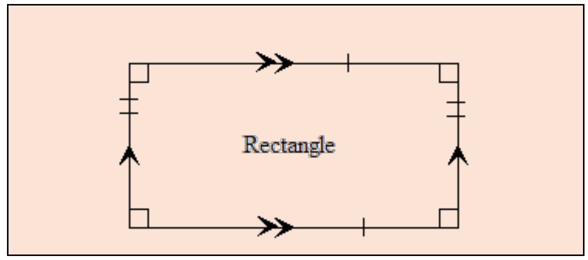
While the above definition is commonly used, it should be noted that it is sufficient to state that a quadrilateral is a parallelogram if any one of these is true:

1. opposite sides parallel
2. opposite sides equal
3. one pair of opposite sides are both parallel and equal
4. opposite angles are equal
5. adjacent angles are supplementary
6. diagonals bisect each other
7. any diagonal divides the parallelogram into a pair of congruent triangles.

Properties 4 and 5	Properties 6 and 7
	
$\propto +(180^0 - \propto) = 180^0$	$\Delta ABM \cong \Delta DCM$ $\Delta ADM \cong \Delta BCM$

Rectangle

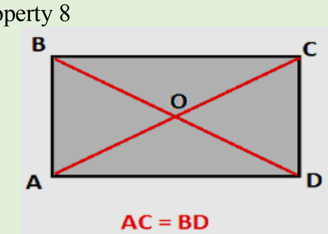
A rectangle is a parallelogram with **four right angles**. A rectangle is, therefore, a special parallelogram and has all the seven properties of a parallelogram mentioned above.



In addition, a rectangle has the following additional property:

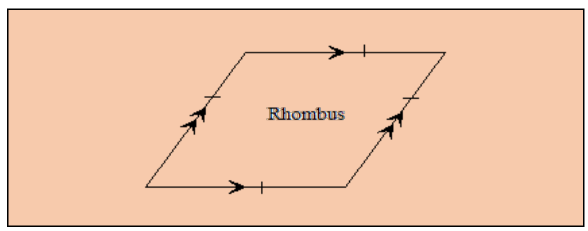
8. the diagonals are equal.

Property 8



Rhombus

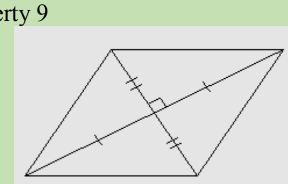
A rhombus is a parallelogram with **four equal sides**. A rhombus is, therefore, a special parallelogram and has all the seven properties of a parallelogram.



In addition, a rhombus has the following additional property:

9. The diagonals of a rhombus bisect each other at right angles.

Property 9

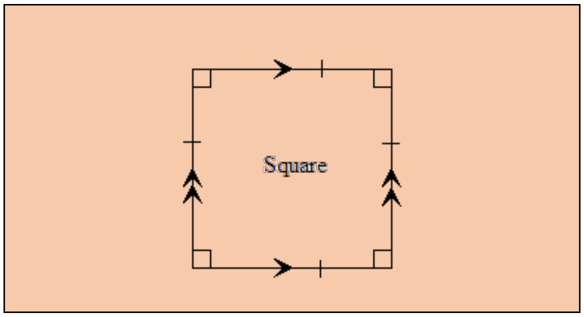


Square

A square is a parallelogram with four right angles and four equal sides. Note that a square can also be defined as

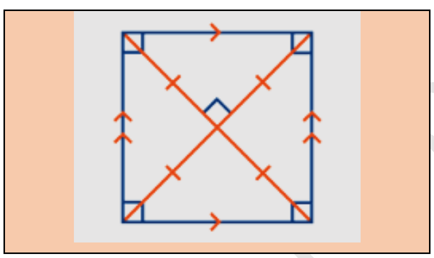
- a rectangle with four equal sides
- a rhombus with four right angles

Therefore, a square has all the properties of a rectangle and a rhombus. The properties are listed as above as properties 1-9.



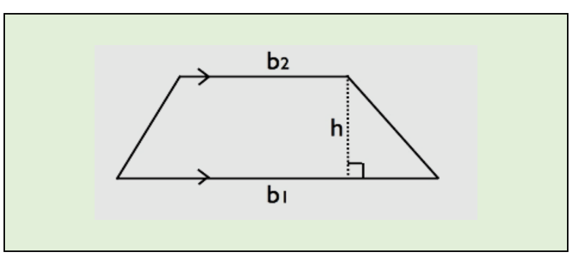
In addition, a square has the following additional property:

10. The diagonals of a square divide the square into four congruent isosceles triangles.

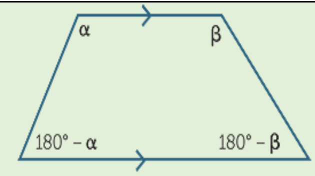


Trapezium

The trapezium is a quadrilateral with only one pair of parallel sides. The height of a trapezium is the perpendicular distance between the parallel sides. The parallel sides are called the bases of the trapezium. The plural of the word trapezium is trapezia.



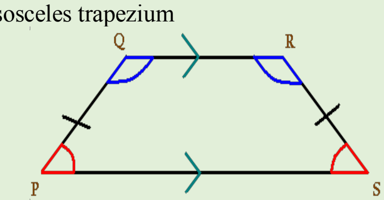
The adjacent angles are supplementary in a trapezium.



$$\propto +(180^0 - \propto) = 180^0$$

If the non-parallel sides of a trapezium are equal, it is an isosceles trapezium.

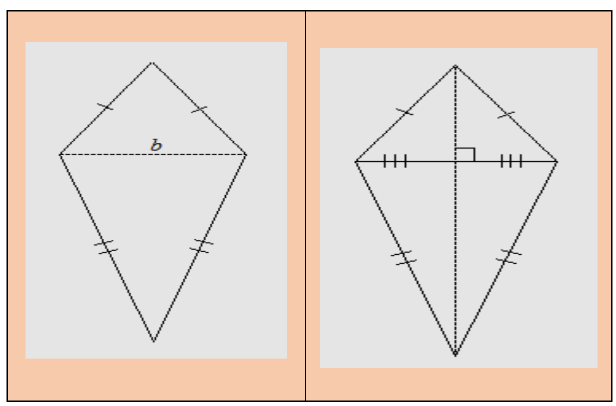
An isosceles trapezium



$$\angle P = \angle S$$
$$\angle Q = \angle R$$

Kite

A kite is a quadrilateral with two pairs of equal adjacent sides. It may be described as a combined figure consisting of two isosceles triangles with the same base placed with their bases together and the triangles lying opposite to each other.



The diagonals of a kite cut at right angles but only one diagonal is bisected. The diagonals divide the kite into two pairs of congruent, right-angled triangles.

Polygons

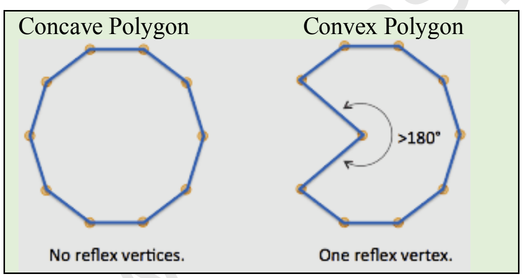
Triangles and quadrilaterals are members of the set of polygons. A polygon is a two-dimensional figure bounded by straight lines or line segments. These line segments are called the sides and the point where any two of these sides meet is called a vertex. The word polygon is a general name and does not indicate the number of sides of the figure. In fact, a polygon composed of n sides is called an n-gon.

Hence, from the definition, the least number of sides that a polygon can have is three (3). Polygons are named according to the number of sides – the names of some polygons are displayed in the table below.

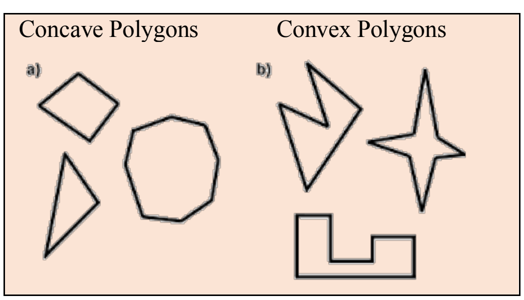
Number of sides	Name
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon **or** Septagon
8	Octagon
9	Nonagon
10	Decagon
n	n-gon

Convex and concave polygons

If all the interior angles of a polygon are less than 180º, the polygon is called convex. If **one or more** of the interior angles of a polygon is reflex, the polygon is called concave or re-entrant.



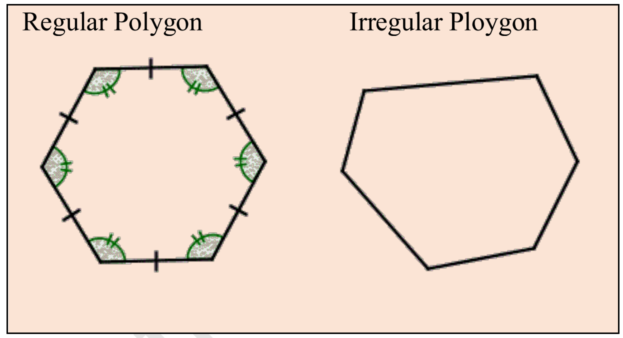
Concave and convex polygons are shown below.



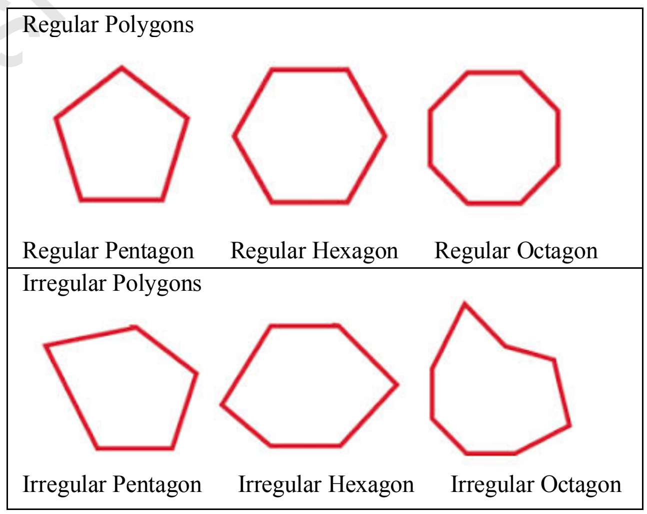
Regular and irregular polygons

If all the sides of a polygon are equal in length, then the polygon is called regular or equiangular. In this case, it follows all the angles, both interior and exterior, are also equal to each other.

The converse of this rule is also true, that is, if all the angles of a polygon are equal, then all the sides are also equal and the polygon is a regular one.



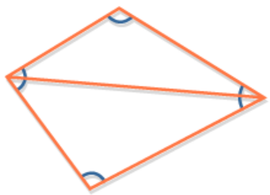
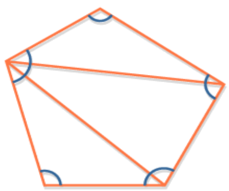
Some regular and irregular polygons are shown below.



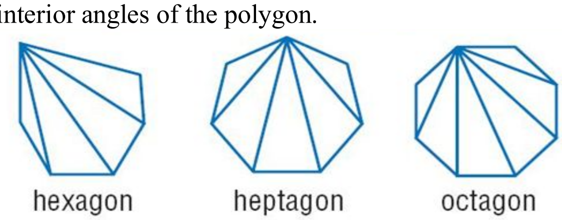
Sum of the interior angles of a polygon

The sum of the interior angles of a polygon varies in accordance with the number of sides of the polygon. In a polygon of n sides, there are n interior angles. To determine the sum of the interior angles in a polygon, we divide each polygon into a set of non-overlapping triangles (shown below).
We already know the sum of the interior angles in a triangle is 180^0. We can use this fact to derive the sum of the angles in any polygon.

We begin by dividing each polygon into a set of non-overlapping triangles as shown below. We start with a 4-sided polygon (quadrilateral). This quadrilateral has been divided into two triangles, as shown in the table below.

Quadrilateral, $n = 4$ 	Sum of the interior angles of the quadrilateral is equal to the sum of the interior angles in two triangles. $=2 \times 180 = 360°$.
Pentagon, $n = 5$ 	Sum of the interior angles of the pentagon is equal to the sum of the interior angles in three triangles. $=3 \times 180 = 540°$.

We continue the process for 6-sided, 7-sided, and 8-sided polygons making sure that the diagonals do not overlap. This is because we need to have distinct triangles whose angles add up to the sum of the interior angles of the polygon.



The results are summarised in a table and a pattern emerges.

Polygon	No. of sides (n)	No. of triangles	Sum of the interior angles
Triangle	3	1	1×180^0
Quadrilateral	4	2	2×180^0
Pentagon	5	3	3×180^0
Hexagon	6	4	4×180^0
Heptagon	7	5	5×180^0
Octagon	8	6	6×180^0
n-gon	n	$n-2$	$(n-2) \times 180^0$

If a polygon has n sides, the sum of its n interior angles is $(n-2)\times 180° = (2n-4)\times 90°$

Example 9

Calculate the measure of the interior angle in a regular pentagon.

Solution

In a pentagon, n=5.

$\therefore$ The sum of the 5 interior angles is

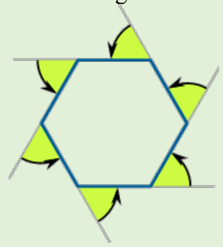
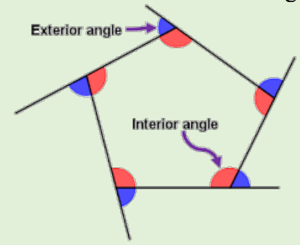
$\{2(5)-4\}\times 90° = 540°$

A regular pentagon has 5 equal sides and 5 equal interior angles.

Each angle is $540^0 \div 4 = 108^0$.

Sum of the exterior angles of a polygon

If the sides of a polygon are extended as shown, exterior angles are formed. At each vertex of the polygon lies an interior angle and an exterior angle. This is shown diagrammatically below.

Exterior angles	Interior and exterior angles
	
The sum of the interior and exterior angles is 180^0.	

We will now use the fact that the sum of the interior and exterior angles is 180^0 to determine the sum of the exterior angles for any polygon.

For a polygon with n sides, there are n vertices.

The sum of the exterior and interior angles in a polygon with n sides

$= n \times 180^0$

But, the sum of the interior angles in any polygon

$= (n-2) \times 180^0$

Hence, the sum of the exterior angles

$= n \times 180^0 - (n-2) \times 180^0$

$= 180n - (180n - 360^0)$

$= 180n - 180n + 360^0$

$= 360^0$

If a polygon has n sides, the sum of its n exterior angles is 360^0.

Example 10

Calculate the measure of each exterior angle in a regular hexagon.

Solution

In a regular hexagon, each of the six (6) exterior angles will be $\frac{360^\circ}{6} = 60^\circ$.

Example 11

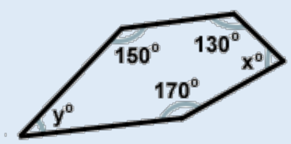
Calculate the number of sides of the regular polygon whose exterior angles are 45º.

Solution

Sum of the exterior angles = 360º
Each angle is 45º.
$\therefore$ The number of angles of the polygon
$= 360^0 \div 45^0 = 8$
$\therefore$ The polygon has 8 sides

Example 12

In the diagram shown, determine if x and y are supplementary angles.

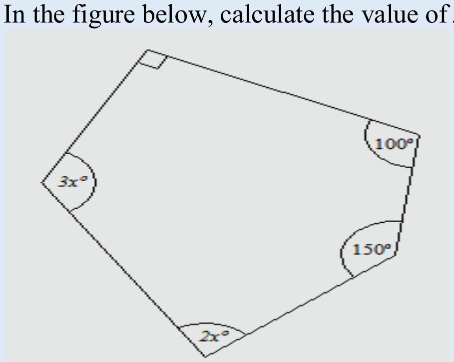


Solution

The polygon has 5 sides, so the sum of the interior angles is
$180^0(5 - 2) = 540^0$.
$x + y = 540^0 - (150^0 + 130^0 + 170^0) = 540^0 - 450^0 = 90^0 \neq 180^0$
Hence x and y are not supplementary.

Example 13

In the figure below, calculate the value of x.



Solution

The figure shown describes a pentagon.
The sum of the five interior angles is therefore
$\{2(5) - 4\} \times 90^\circ = 540^\circ$
Hence $90^0 + 3x^0 + 2x^0 + 150^0 + 100^0 = 540^0$
$5x + 340^0 = 540^0$
$5x = 200$
$x = 40$