6. ALGEBRAIC FACTORISATION AND FORMULAE

ALGEBRAIC FACTORISATION

In arithmetic, when we speak of the factors of a number, we refer to its divisors. When we express a number as a product of its prime factors, it is easy to identify the Highest Common Factor (HCF) of the numbers. For example,

$$12 = 2 \times 2 \times 3$$
$$18 = 2 \times 3 \times 3$$

The factors common to both numbers are 2 and 3, hence the HCF is $2 \times 3 = 6$.

Similarly, algebraic expressions may also have divisors and these are factors of the expression. For example, the expression, $5x^2y$, written as a product of all its factors as shown below.

$$5x^2y = 5 \times x \times x \times y$$

$5x^2y$ has 5 as a divisor and so 5 is a factor of this expression. In algebra, we can have factors that are symbols as well. Hence, x and y are also factors of $5x^2y$. This is because the expression can be divided by x as well as y. In addition, x^2 is also a factor because the expression can be divided by x^2.

Highest common factor in algebra

When we have two algebraic expressions, the highest factor common to *both* expressions is called the *highest common factor* (HCF), or *greatest common factor*. The principles applied in algebra are the same as those that are applied in arithmetic.

Example 1

Find the HCF of
(i) b^3 and b^5
(ii) abc and bcd

Solution

(i) HCF of b^3 and b^5 is b^3 since b^3 is the highest power of b common to both.

(ii) HCF of abc and bcd is bc since both terms contain the common factor bc.

Example 2

Find the HCF of
$15x^2y$ and $10x^3y^2z$,

Solution

To find the HCF of $15x^2y$ and $10x^3y^2z$, we consider each variable separately. Since there are constants in each expression we must pay attention to these as well. The following steps illustrate the process.

1. The highest common factor of 15 and 10 is 5, so the first term in our HCF is 5.
2. *The highest common factor of x^2 and x^3 is x^2 so the second term in our HCF is x^2.
3. The highest common factor of y and y^2 is y so the third term in our HCF is y.
4. There is no variable in z in the first expression, so z is not a common factor.
5. Hence, the HCF is $5x^2y$.

We can now check to see whether this answer is correct. This can be done by dividing both expressions by the HCF, and then examining the answer for confirmation.

$$\frac{15x^2y}{5x^2y} = 3 \quad \text{and} \quad \frac{10x^3y^2z}{5x^2y} = 2xyz$$

Since there are no common factors between the two quotients, 3 and $2xyz$, the HCF obtained is correct.

Example 3

Find the HCF of $21p^4q^5r^4$ and $14p^3q^6r^4$

Solution

The HCF of 21 and 7 is 7
The HCF of p^4 and p^3 is p^3
The HCF of q^5 and q^6 is q^5
The HCF of r^4 and r^4 is r^4
Hence, the HCF of $21p^4q^5r^4$ and $14p^3q^6r^4$ is $7p^3q^5r^4$

Common factor method

We can also use the highest common factor to rewrite an algebraic expression as a product of prime factors. This process of is called factorisation. Consider the following expression:

$3k^2h^3 + 6k^3h^2$

Since there are common factors in both terms, it is now possible to *factorise* this expression by finding the highest common factor and rewriting the expression as a product of two factors.

The HCF of $3k^2 h^3$ and $6k^3 h^2$ is $3k^2h^2$.

Dividing each term by the HCF, the expression can be written as:

$$3k^2h^3 + 6^3h^2 = 3k^2h^2\left(k+h\right)$$

Note that in the factorised form this expression now has one term with two factors and which are $3k^2h^2$ and $(k + h)$.

Example 4

Factorise completely the following expressions:

(i) $7x + 7y$

(ii) $9b^2 - 3b$

(iii) $6m^5n^3p + 8m^4n^2p^2$

Solution

(i) $7x + 7y = 7\left(x+y\right)$

(ii) $9b^2 - 3b = 3b\left(3b-1\right)$

(iii) $6m^5n^3p + 8m^4n^2p^2 = 2m^4n^2p\left(3mn+4p\right)$

Common factors with grouping

We use the above method when there are common factors in all the terms in the expression. In some expressions, there are no common factors in each term and one may conclude that it is not possible to factorise such expressions. However, if there are factors common to any pair of terms, it may be possible to factorise the expression. Consider the expression:

$7a + 14b + ca + 2cb$

Note that there are four terms, but there is no factor common to all four terms. However, the first two terms have 7 as a common factor and the last two terms have c as a common factor.
So, the expression can be grouped as follows:

$(7a + 14b) + (ca + 2cb)$
Now, each group has a common factor.

$=7(a + 2b) + c\,(a + 2b)$

We have reduced the expression from four terms to two terms. Clearly, $(a + 2b)$ is now a common factor of these two terms. The expression can be further factorised by dividing each term by $(a + 2b)$. Taking this common factor out from each group, we have

$=(a + 2b)\,(7 + c)$

The expression is now completely factorised, as it is expressed as a product of two algebraic expressions.

This factorisation technique is called grouping and it is used to factorise an expression consisting of four terms when there are common factors in the pairs of terms.

Example 5

Factorise completely:

(i) $6x + 9 + 2kx + 3k$

(ii) $5p + 10q - np - 2nq$

(iii) $m^2 - 2m - mn + 2n$

Solution

1. $6x + 9 + 2kx + 3k = 3\left(2x+3\right) + k\left(2x+3\right)$
$= \left(2x+3\right)\left(3+k\right)$

2. $5p + 10q - np - 2nq = 5\left(p+2q\right) - n\left(p+2q\right)$
$= \left(p+2q\right)\left(5-n\right)$

3. $m^2 - 2m - mn + 2n = m\left(m-2\right) - n\left(m-2\right)$
$= \left(m-2\right)\left(m-n\right)$

Quadratic expressions

Some algebraic expressions cannot be factorised using the methods we learned so far. One such expression is the quadratic expression which takes the form,
$\boldsymbol{ax^2 + bx + c}$.

However, a quadratic expression is really a product of two linear expressions. We can illustrate this by using the distributive law as follows:

$$
\begin{aligned}
(x + 4)(x + 2) &= x(x + 4) + 2(x + 4)\\
&= (x^2 + 4x) + (2x + 8)\\
&= x^2 + 4x + 2x + 8\\
&= x^2 + 6x + 8
\end{aligned}
$$

which is of the form $\boldsymbol{ax^2 + bx + c}$.

Quadratic factorisation involves reversing the above process, whereby, we start with the quadratic and obtain the two linear factors.
That is, we start with the left-hand side (the quadratic) and end with the right-hand side (the linear factors).

$$x^2 + 6x + 8 = (x + 4)(x + 2)$$

From the above example, we should note that when our quadratic is factorised;

1. The product of the constants is 8
2. The sum of the coefficients is $6x$.

Hence, to factorise $x^2 + 6x + 8$, we need to first separate $6x$ into two terms such that the product of the coefficients is 8 and their sum is 6. So, we choose $4x$ and $2x$.
Our expression now has four terms that can be grouped for common factors.

$$
\begin{aligned}
x^2 + 6x + 8 &= x^2 + 4x + 2x + 8\\
&= (x^2 + 4x) + (2x + 8)\\
&= x(x + 4) + 2(x + 4)\\
&= (x + 4)(x + 2)
\end{aligned}
$$

If our quadratic expression was $x^2 + 9x + 8$, we would now have to separate $9x$ into two terms such that the product of the coefficients is 8 and their sum is 9. In such as case, we choose $8x$ and $1x$.

$$
\begin{aligned}
x^2 + 9x + 8 &= x^2 + 8x + x + 8\\
&= (x^2 + 8x) + (x + 8)\\
&= x(x + 8) + 1(x + 8)\\
&= (x + 8)(x + 1)
\end{aligned}
$$

Example 6

Factorise $x^2 - 10x + 21$.

Solution

We need to separate $-10x$ into two terms such that the product of the coefficients is 21 and the sum is -10. This would be $-3x$ and $-7x$.

$$
\begin{aligned}
\therefore x^2 - 10x + 21 &\equiv x^2 - 7x - 3x + 21\\
&= x(x - 7) - 3(x - 7)\\
&= (x - 7)(x - 3)
\end{aligned}
$$

Example 7

Factorise $x^2 + x - 20$

Solution

We need to separate x into two terms such that product of their coefficients is -20 and their sum is 1. This would be $5x$ and $-4x$.

$$
\begin{aligned}
x^2 + x - 20 &= x^2 + 5x - 4x - 20\\
&= x(x + 5) - 4(x + 5)\\
&= (x + 5)(x - 4)
\end{aligned}
$$

When the coefficient of the first term is not 1, the procedure takes into consideration this coefficient as we shall see in the next example.

Example 8

Factorise $12x^2 - 20x + 3$.

Solution

In our previous examples, the coefficient of x^2 was one. When the coefficient is not one, the product of the coefficients will be the constant term multiplied by the coefficient of x^2.
Now we must separate the $-20x$ into two terms such that the product of their coefficients is $12 \times 3 = 36$ and their sum is -20.
This would be $-18x$ and $-2x$. And so,

$$
\begin{aligned}
12x^2 - 20x + 3 &= 12x^2 - 18x - 2x + 3\\
&= 6x(2x - 3) - 1(2x - 3)\\
&= (2x - 3)(6x - 1)
\end{aligned}
$$

Difference of two squares

When we multiply two linear expressions of the form $(x-y)(x+y)$, we obtain a quadratic expression that has only two terms,

$$(x+y)(x-y) = x^2 - xy + xy - y^2 = x^2 - y^2$$

$$(p+4)(p-4) = p^2 + 4p - 4p - 4^2 = p^2 - 16$$

$$(q+1)(q-1) = q^2 + q - q - 1^2 = q^2 - 1$$

The expressions on the right have only two terms separated by a subtraction sign. Such expressions are called the difference of two squares. Notice that both terms are perfect squares, that is, they have exact square roots. Such expressions are called the difference of two squares. In factorising such expressions, we can apply the following rule:

When an expression is written in the form, $a^2 - b^2$, then it can be easily factorised into $(a-b)(a+b)$.

Example 9

Factorise: $25 - x^2 \equiv (5)^2 - (x)^2$

Solution

This is now expressed as a difference of two squares

$\therefore 25 - x^2 = (5-x)(5+x)$

Example 10

Factorise: $1 - 9y^2 \equiv (1)^2 - (3y)^2$

Solution

This is now expressed as a difference of two squares

$\therefore 1 - 9y^2 = (1-3y)(1+3y)$

Example 11

Factorise $(m+3)^2 - n^2$

Solution

$$(m+3)^2 - n^2 = (m+3+n)(m+3-n)$$

FORMULAE

We have used many mathematical formulae in our study of mathematics thus far. In consumer arithmetic, we used formulae to calculate Simple Interest and in measurement, we used formulae to calculate area and perimeter of various shapes.

A formula is a mathematical rule or principle usually expressed as an equation involving more than one variable. In using formulae, it is sometimes necessary to rearrange it so as to obtain any one of the variables in terms of the other variables.

In solving equations, we would have used the additive and multiplicative inverses to isolate variables and so obtain their values. These principles involved in using these inverses also apply to rearranging simple formulae.

We will first look at simple formulae involving procedures that we already know from our study of equations.

Example 12

In the formula, $v = u + at$, make

a) u the subject
b) a the subject
c) t the subject

Solution

a) To make u the subject, all term(s) involving u are left or kept on one side of the equation.

$$v = u + at$$
$$v - at = u$$
$$u = v - at$$

b) To make a the subject, first isolate the term at, then solve for a.

$$v = u + at$$
$$v - u = at$$
$$at = v - u$$
$$\therefore a = \frac{v-u}{t}$$

c) We use the same principles in (b) to make t the subject

$$v = u + at$$
$$v - u = at$$
$$at = v - u$$
$$\therefore t = \frac{v-u}{a}$$

In some cases, the variable we want as the subject may appear on both sides of the equation.

Example 13

Make x the subject of $px-b=a-qx$

Solution

First, collect all terms in x on one side of the equation.

$$px-b=a-qx$$
$$\therefore px+qx=a+b$$

To isolate x, we use the technique of factorisation since x is a common factor to both terms.

$$x(p+q)=a+b$$
$$\therefore x=\frac{a+b}{(p+q)}$$

Formulae involving fractions

Sometimes the variable we wish to make the subject is either the numerator or denominator of a fraction. We can use different techniques to isolate the unknown as shown below.

Example 14

Make f the subject of the formula: $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$

Solution

First, isolate the term in f
Then cross multiply to remove fractions.
Next, make f the subject

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$$
$$\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$$
$$\frac{1}{f}=\frac{v+u}{uv}$$
$$f(v+u)=uv$$
$$f=\frac{uv}{v+u}$$

Alternatively, we may find the L.C.M. of u, v and f and multiply each term by the LCM. This technique enables us to remove the fractions at the start.

Alternative solution to Example 12

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$$

The LCM of u, v and f is uvf. So we multiply each term by uvf.

$\times uvf$

$$vf+uf=uv$$
$$f(v+u)=uv$$
$$f=\frac{uv}{v+u}$$

Formulae involving roots and powers

Sometimes the variables may involve term(s) involving roots or powers. We adopt the following inverse principles.

$$If\ a^2=b, then\ a=\sqrt{b}$$
$$If\ \sqrt{a}=b, then\ a=b^2$$

Example 15

For the formula, $v^2=u^2+2\,as$, make u the subject

Solution

To make u the subject, we must first isolate the term in u. But, since this happens to be u^2, we first isolate u^2 and afterwards take the square root to isolate u.

$$v^2=u^2+2as$$
$$\therefore v^2-2as=u^2$$
$$u^2=v^2-2as$$
$$\therefore u=\sqrt{v^2-2as}$$

Example 16

Make h the subject of the formula, $q\sqrt{r}=p$.

Solution

We first make $\sqrt{r}$ the subject, then square both sides

$$q\sqrt{r}=p$$
$$\sqrt{\boldsymbol{r}}=\frac{p}{q}$$

Squaring, we get $r=\frac{p^2}{q^2}$

Example 15

Make c the subject of the formula

$$p = \frac{1}{3}nmc^2$$

Solution:

Rewrite the equation in the form of two equivalent fractions to make it ready for cross multiplication.

$$p = \frac{1}{3}nmc^2$$

$$\frac{p}{1} = \frac{nmc^2}{3}$$

Then cross multiply to remove fractions. Afterwards, isolate c^2 then take the square root.

$$\therefore\ p \times 3 = nmc^2 \times 1$$

$$3p = nmc^2$$

$$\frac{3p}{nm} = c^2$$

$$c^2 = \frac{3p}{nm}$$

$$c = \sqrt{\frac{3p}{nm}}$$

Example 16

Make g the subject of the formula $T = 2\pi\sqrt{\frac{l}{g}}$.

Solution

Observe the following steps

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Divide by 2π to isolate square root

$$\frac{T}{2\pi} = \sqrt{\frac{l}{g}}$$

Square both sides

$$\frac{T^2}{4\pi^2} = \frac{l}{g}$$

Cross multiply

$$T^2 g = 4\pi^2 l$$

Isolate g

$$g = \frac{4\pi^2 l}{T^2}$$