

## 4. INTRODUCING ALGEBRA

### DIRECTED NUMBERS

A directed number is one which has a positive (+) or negative (−) sign attached to it, in order to show its direction. When we speak of direction, it is necessary to establish a reference point or origin, from which direction is fixed. On a horizontal number line, for example, numbers to the right of zero are assigned positive directions, while numbers to the left of zero are assigned negative directions.

If a number has a positive direction, then the positive sign is not usually displayed before the numeral. However, when a number has a negative direction, a negative sign must be displayed just before the numeral. Usually, the position of the negative sign is slightly raised. For example, negative 5 is written as  $-5$  instead of  $-5$ ; the latter meaning to take away or subtract 5.

### Addition and Subtraction

- When adding two numbers whose directions are the same, the result is the sum of the two numbers and the direction is maintained.

$$\begin{array}{l} +3 + +5 = +8 \\ -6 + -1 = -7 \end{array}$$

- When adding two numbers whose directions are different, the result is the difference between the numbers and the direction is the same as that of the larger number.

$$\begin{array}{l} -7 + +5 = -2 \\ +6 + -4 = +2 \end{array}$$

### Relationship between Addition and Subtraction

Recall that subtraction is the inverse of addition. From the above example, we know that  $+6 + -4 = +2$

Using the inverse property, this could have been written as a subtraction of positive 4.  $+6 - +4 = +2$

Any addition sentence can be written as a subtraction sentence provided that we change the sign of the number to be subtracted (subtrahend).

Conversely, we can rewrite any subtraction sentence as an addition sentence, provided that the sign of the subtrahend changes. In the examples below the subtrahend is positive and so its inverse is negative.

$$\begin{array}{l} +5 - +2 = +5 + -2 \\ +8 - +3 = +8 + -3 \\ -2 - +4 = -2 + -4 \end{array}$$

When the subtrahend is negative, its inverse is positive.

$$\begin{array}{l} +5 - -2 = +5 + +2 \\ +8 - -3 = +8 + +3 \\ -2 - -4 = -2 + +4 \end{array}$$

Subtracting a number is equivalent to adding its **additive inverse**. In general,

$$\begin{array}{l} a - (b) = a + (-b) \\ a - (-b) = a + (b) \end{array}$$

### Multiplication and Division

When multiplying or dividing two numbers whose directions are the same, the result is a positive number.

$$\begin{array}{ll} 4 \times 3 = 12 & -4 \times -3 = 12 \\ 8 \div 4 = 2 & -8 \div -4 = +2 \end{array}$$

When multiplying or dividing two numbers whose directions are different, the result is a negative number.

$$\begin{array}{ll} 4 \times -3 = -12 & -4 \times 3 = -12 \\ -8 \div 4 = -2 & 8 \div -4 = -2 \end{array}$$

When we multiply (or divide) two terms with same signs, the result is positive.

When we multiply (or divide) two terms with different signs the result is negative.

## ALGEBRAIC SYMBOLS

The rules that apply to arithmetic are the same as those that apply to algebra. However, in arithmetic, we perform operations on numbers to obtain a numerical value, whereas, in algebra we perform operations on symbols, as well as numbers, to obtain our final result.

An algebraic symbol represents an unknown number. We usually use letters of the alphabet to represent symbols. Symbols can take any value, unlike numbers whose values are known.

A quantity that is fixed is called a **constant**.

A quantity that varies is called a **variable** or an **unknown** quantity.

### Formulating algebraic expressions

An algebraic expression may comprise both variables and numbers. When formulating algebraic expressions, we actually translate from words or verbal phrases to algebraic symbols.

#### Addition and subtraction

The sum of  $a$  and  $b$  is written  $a + b$

$a$  added to  $b$  is written  $b + a$

$a$  subtracted from  $b$  is written  $b - a$

The difference between  $a$  and  $b$  may be  $a - b$  or  $b - a$

Three more than  $p$  is written  $p + 3$

Three less than  $p$  is written  $p - 3$

#### Multiplication and division

Three times  $p$  is written  $3 \times p = 3p$

The product of  $a$  and  $b$  is written  $a \times b = ab$

$a$  divided  $b$  is written  $\frac{a}{b}$

One-third of  $a$  is written as  $\frac{1}{3} \times a = \frac{1}{3}a$

It is important to point out that the multiplication sign is omitted and division is written using the fractional notation.

$5 \times (x + 3)$  is written as  $5(x + 3)$

$4 \div (x - 5)$  is written as  $\frac{4}{x - 5}$

$(x - 1) \times (x + 3)$  is written as  $(x - 1)(x + 3)$

$(x + 6) \div (x - 5)$  is written as  $\frac{(x + 6)}{x - 5}$

## Example 1

Assuming Adam is  $a$  years old.

Write down the ages of his friends and relatives given that:

- His cousin, Raveed is twice his age.
- His friend, Avinash is 5 years older than he is.
- His sister Anah is 4 years more than half his age.
- His father, Naim is 10 years less than three times his age.
- If he adds 6 years to his age and multiplies the answer by five, the result is his grandmother's age.

### Solution

(i) Raveed's age:  $2 \times a = 2a$

(ii) Avinash's age:  $a + 5$

(iii) Anah's age:  $\frac{1}{2}(a) + 4 = \frac{1}{2}a + 4$

(iv) Naim's age:  $(3 \times a) - 10 = 3a - 10$

(v) Grandmother's age:  $5 \times (a+6) = 5(a+6)$

### Simplifying algebraic expressions

An algebraic **expression** consists of a set of terms. A **term** can be a number, a variable or the product of a number and a variable (or variables). Addition or subtraction signs separate terms in an expression.

Sometimes we may wish to rewrite a complex algebraic expression in a simpler form. This process is called simplification. In simplifying an algebraic expression, we express or write it in the most compact or efficient manner, without changing the value of the expression. To do so, we need to be able to differentiate between **like and unlike terms**.

The following is an algebraic expression:

$$3a + 5ab + 2c$$

This expression has three terms, all three terms have variables but all variables have different symbols or combination of symbols, hence, there are no like terms.

Like terms are terms with the same variable or product of variables. In the expression above, the symbol,  $a$ , can represent a value that is quite distinct from the values represented by the symbols  $b$  and  $c$ . All three terms are **unlike terms**.

## Addition and subtraction of algebraic terms

Consider the following algebraic expression:

$$5a + 6ac + 8a - 4ab$$

This expression has four terms in which two of them are like terms. The terms  $5a$  and  $8a$  are multiples of the same variable,  $a$ , and therefore it is possible to combine these two terms to obtain one term as follows:

$$5a + 8a = (5 + 8)a = 13a$$

The expression can be simplified to obtain:

$$5a + 6ac + 8a - 4ab = 13a + 6ac - 4ab$$

This expression cannot be simplified any further since there are no more like terms. Note that even though the term  $6ac$  has  $a$  as part of it; it is different to the terms with only  $a$ , that is  $5a$  and  $8a$ .

The following expression has four terms separated by addition signs.

$$3a + bc + 4a + 5bc$$

To simplify this expression, we must first identify like terms, then combine them using addition or subtraction. We may wish to rewrite the expression, placing like terms next to each other, before collecting the like terms.

$$3a + bc + 4a + 5bc = 3a + 4a + bc + 5bc = 7a + 6bc$$

In the above example, we add  $3a$  to  $4a$  to obtain  $7a$ . Then we add  $bc$  to  $5bc$  to obtain  $6bc$ . The simplified expression has only two terms,  $7a + 6bc$ .

We may combine any number of like terms by adding or subtracting their coefficients. However, only **like terms** can be added or subtracted. When combining like terms, the sign of the coefficients must be considered, and it is always positioned on the immediate left of the coefficient. This is illustrated in the examples below.

$$\begin{aligned} 5a + 3a - 7a + 6a &= (5 + 3 - 7 + 6)a = 7a \\ 7x + 3x + 4x &= (7 + 3 + 4)x = 14x \\ 6b + 3b - 2b &= (6 + 3 - 2)b = 7b \\ 4x + 5y + 3x + 7y &= 7x + 12y \end{aligned}$$

## Variables with exponents

Sometimes variables are expressed in exponents, such as  $3x^2 + 5x^2 + 7xy^2 + 9x^2y$ .

Note that

$xy^2$  ( $x \times y \times y$ ) is **NOT** the same as  $x^2y$  ( $x \times x \times y$ ). Unlike terms such as these are left alone.

$$3x^2 + 5x^2 + 7xy^2 + 9x^2y = 8x^2 + 7xy^2 + 9x^2y$$

$$2a + a + a^2 + a^3 + 3a^2 = 3a + 4a^2 + a^3$$

Sometimes, we need to recall the commutative property in determining if terms are like terms. For example,  $ab$  and  $ba$  are like terms. So that,

$$2ab + 5ba = 2ab + 5ab = 7ab$$

Consider this property in simplifying the following:

$$ab^3 + 7b^3a - 6ab^2 + 10 = 8ab^3 - 6ab^2 + 10$$

## Simplifying expressions with brackets

When an expression has brackets (also called parentheses), we use the distributive law to remove the brackets before combining like terms.

$$3x + 2(x + 4) = 3x + 2x + 8 = 5x + 8$$

When a negative sign is in front of a bracket, it is implied that the factor to be multiplied is  $-1$ .

$$\begin{aligned} 3x - (5 - x) &= 3x + (-1)[5 + (-x)] \\ &= 3x + (-1)(5) + (-1)(-x) \\ &= 3x - 5 + x \\ &= 4x - 5 \end{aligned}$$

## Multiplication and division of algebraic terms

Recall that in algebra, we can omit the multiplication sign between two symbols or between a symbol and a number. The division sign is also omitted and replaced by fractional form.

### Multiplication

Unlike addition and subtraction, to perform multiplication, we do **not** need to have like terms. Instead, we simply omit the sign of the operation. Numerals are always placed first, for example,  $p \times 3$  is written as  $3p$ .

$$\begin{aligned}
 x \times y &= xy \\
 a \times b \times c &= abc \\
 2x \times 3y &= 6xy \\
 -4a \times 3b &= -12ab \\
 3x \times 2y \times -4z &= -24xyz
 \end{aligned}$$

### Division

In performing a division, we convert to fractional form, simplifying if possible.

$$\begin{aligned}
 5a \div b &= \frac{5a}{b} \\
 12m \div 6n &= \frac{12m}{6n} = \frac{2m}{n}
 \end{aligned}$$

### Multiplication and division involving indices

We know from arithmetic, that certain numbers can be expressed as the product of a set of repeated factors, for example,

$$\begin{aligned}
 10000 &= 10 \times 10 \times 10 \times 10 = 10^4 \\
 32 &= 2 \times 2 \times 2 \times 2 \times 2 = 2^5
 \end{aligned}$$

The same principle applies to algebra, where a symbol is used to replace a number, for example,

$$\begin{aligned}
 a \times a \times a \times a &= a^4 \\
 b \times b \times b \times b \times b &= b^5
 \end{aligned}$$

In general, if  $m$  is a positive integer, then  $a^m = a \times a \times a \times \dots \times a$  ( $m$  times)

By definition,  $a^m$  means  $a$  multiplied by itself  $m$  times where  $a$  is called the base and  $m$  is the index.

### Laws of Indices

We will now examine the laws that relate to performing operations on algebraic terms in which there are indices.

#### 1. Addition law of indices

Let us examine what happens when we **multiply** algebraic terms with the same base.

$$b^2 \times b^3 = (b \times b) \times (b \times b \times b)$$

$$b^2 \times b^3 = b^5$$

$$b^2 \times b^3 = b^{2+3}$$

$$p^5 \times p = (p \times p \times p \times p \times p) \times p$$

$$p^5 \times p = p^6$$

$$p^5 \times p = p^{5+1}$$

**Law 1:**  $a^m \times a^n = a^{m+n}$ , for all values of  $m$  and  $n$ .

#### 2. Subtraction law of indices

Let us examine what happens when we **divide** algebraic terms with the same base.

$$y^6 \div y^2 = \frac{y \times y \times y \times y \times y \times y}{y \times y}$$

$$y^6 \div y^2 = y \times y \times y \times y$$

$$y^6 \div y^2 = y^4$$

$$y^6 \div y^2 = y^{6-2}$$

$$r^6 \div r^3 = \frac{r \times r \times r \times r \times r \times r}{r \times r \times r}$$

$$r^6 \div r^3 = r \times r \times r$$

$$r^6 \div r^3 = r^3$$

$$r^6 \div r^3 = r^{6-3}$$

**Law 2:**  $a^m \div a^n = a^{m-n}$ , for all values of  $m$  and  $n$ .

#### 3. Multiplication law of indices

This law applies to the simplifying expressions of the form such as,  $(k^5)^3$ .

To simplify these expressions, we expand to remove the brackets and then apply the addition law as shown below.

$$(p^5)^2 = p^5 \times p^5 = p^{10}$$

$$(p^5)^2 = p^{5 \times 2}$$

$$(q^4)^3 = q^4 \times q^4 \times q^4 = q^{12}$$

$$(q^4)^3 = q^{4 \times 3}$$

**Law 3:**  $(a^m)^n = a^{m \times n}$ , for all values of  $m$  and  $n$ .

#### 4. The zero index

This law illustrates the meaning of an index of zero. We will illustrate this law by simplifying  $a^3 \div a^3$  using two methods.

We may choose to write this as a fraction and expand the numerator and denominator as shown below.

$$a^3 \div a^3 = \frac{a \times a \times a}{a \times a \times a} = 1$$

Alternatively, we can apply the second law of indices.

$$a^3 \div a^3 = a^{3-3} = a^0$$

Hence, we can conclude that  $a^3 \div a^3 = a^0 = 1$

**Law 4:**  $a^0 = 1$  for any value of  $a$ .

#### 5. Negative indices

This law illustrates the meaning of negative indices. Assume we wish to simplify  $a^3 \div a^5$ .

We may choose to write this as a fraction and expand the numerator and denominator as shown below.

$$a^3 \div a^5 = \frac{a^3}{a^5} = \frac{a \times a \times a}{a \times a \times a \times a \times a}$$

$$a^3 \div a^5 = \frac{1}{a^2}$$

Alternatively, we can apply the second law of indices.

$$a^3 \div a^5 = \frac{a^3}{a^5} = a^{3-5}$$

$$a^3 \div a^5 = a^{-2}$$

Since both results are the same we can conclude that  $\frac{1}{a^2} = a^{-2}$ .

If the negative is in the denominator of the fraction, we can still apply the rule.

$$\frac{1}{a^{-3}} = \frac{1}{\frac{1}{a^3}} = 1 \times \frac{a^3}{1} = a^3, \text{ So } \frac{1}{a^{-3}} = a^3$$

**Law 5:**  $\frac{1}{a^m} = a^{-m}$  or  $\frac{1}{a^{-m}} = a^m$

#### Summary of the laws of indices (integral indices)

1.  $a^m \times a^n = a^{m+n} \quad \forall m, n$
2.  $a^m \div a^n = a^{m-n} \quad \forall m, n$
3.  $(a^m)^n = a^{m \times n} \quad \forall m, n$
4.  $a^0 = 1 \quad \forall a$
5.  $a^m = \frac{1}{a^{-m}} \quad \forall m$

There are two more laws of indices that you may encounter later on and they relate to fractional indices.

6. The  $n^{\text{th}}$  root of a number can be expressed as the number raised to the power of  $\frac{1}{n}$ .

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

$$\begin{aligned} \sqrt{y} &= y^{\frac{1}{2}} & \sqrt[3]{a} &= a^{\frac{1}{3}} \\ \sqrt[4]{16y} &= 2y^{\frac{1}{4}} & \sqrt[3]{y^6} &= y^2 \end{aligned}$$

A term raised to the power  $\frac{m}{n}$  is the  $n^{\text{th}}$  root of the term raised to the power of  $m$ , and this can be done in any order.

$$7. \quad a^{\frac{m}{n}} = \sqrt[n]{a^m} \text{ or } (\sqrt[n]{a})^m$$

$$\begin{aligned} (8)^{\frac{2}{3}} &= \sqrt[3]{(8)^2} \text{ or } (\sqrt[3]{8})^2 = 4 \\ (81)^{\frac{3}{4}} &= \sqrt[4]{(81)^3} \text{ or } (\sqrt[4]{81})^3 = 27 \end{aligned}$$

#### Simplifying algebraic terms with indices

To simplify an expression like:

$$2ab^2 \times 3a^2b^3$$

We first rewrite this expression, inserting the multiplication signs where they were supposed to be.

$$2 \times a \times b^2 \times 3 \times a^2 \times b^3$$

Then we place all the same variables together, with the constants at the front:

$$\begin{aligned} &= 2 \times 3 \times a \times a^2 \times b^2 \times b^3 \\ &= 6 \times a^3 \times b^5 \\ &= 6a^3b^5 \end{aligned}$$

Alternatively, we may choose to apply the laws of indices directly, as shown below.

$2x^2 \times 3y = 6x^2y$	$(a^2)^3 \div a^4 = a^{2 \times 3} \div a^4$
$-a \times 2b^2 \times c = -2ab^2c$	$= a^6 \div a^4$
$\frac{m^5n^6}{m^3n^2} = m^2n^4$	$= a^{6-4}$
	$= a^2$

### Example 2

Simplify  $8c^5d^3 \div 2c^2d$

#### Solution

$$\begin{aligned} \frac{8c^5d^3}{2c^2d} \\ &= \frac{8}{2} \times \frac{c^5}{c^2} \times \frac{d^3}{d} \\ &= 4 \times c^3 \times d^2 \\ &= 4c^3d^2 \end{aligned}$$

### Example 3

Simplify  $4x^2y \div 2xy^2$

#### Solution

$$\frac{4x^2y}{2xy^2} = \frac{4 \times x \times x \times y}{2 \times x \times y \times y} = \frac{2x}{y}$$

## SUBSTITUTION

This process of replacing variables by known values is called **substitution**. This replacement allows us to obtain a numerical value for our algebraic expression. In the expression,  $2l + 2b$ , both  $l$  and  $b$  are the variables. This expression may represent the perimeter of a rectangle, whose length is  $l$  and breadth is  $b$ .

Using this expression, we can calculate the perimeter of a rectangle whose length,  $l$  is 10 cm and breadth,  $b$  is 6 cm. We substitute these values and so obtain a value of 32 cm for the perimeter.

$$2(10) + 2(6) = 32$$

In substitution, when we replace a symbol by a number, it is necessary to introduce brackets strictly for the avoidance of costly numerical errors.

When substituting values in expressions involving a set of terms, we must **simplify each term separately**, then combine the terms using addition and/or subtraction. In the example below, there are three terms, the terms are separated by addition and subtraction signs. In order to simplify the expression, we should treat each term separately, as shown below.

When  $p = 2$ ,  $q = 4$  and  $r = 1$

$$5p + 3q - 6r = 5(2) + 3(4) - 6(1)$$

$$5(2) + 3(4) - 6(1) = 10 + 12 - 6 = 16$$

If the expression has brackets, then we compute that which is inside the brackets first. For example, using the same values of  $p$ ,  $q$ , and  $r$ , we can evaluate the following expression:

$$\begin{aligned} (pq)^2 + (p - r) \\ &= [2 \times (4)]^2 + (2 - 1) \\ &= (8)^2 + 1 \\ &= 64 + 1 \\ &= 65 \end{aligned}$$

### Example 4

Find the value of  $5m - 3n + 2$ , when  $m = 4$  and  $n = 6$ .

#### Solution

$$\begin{aligned} 5m - 3n + 2 &= 5(4) - 3(6) + 2 \\ &= 20 - 18 + 2 \\ &= 4 \end{aligned}$$

### Example 5

Evaluate  $\frac{x+3y}{x-2y}$  when,  $x = 4$  and  $y = 1$ .

#### Solution

$$\begin{aligned} \frac{x+3y}{x-2y} &= \frac{4+3(1)}{4-2(1)} \\ &= \frac{4+3}{4-2} \\ &= \frac{7}{2} = 3\frac{1}{2} \end{aligned}$$

**Example 6**

If  $a = 1$  and  $b = 2$ , evaluate

- (i)  $ab^2$
- (ii)  $a^2b$

**Solution**

(i)  $ab^2$  is actually  $a \times b^2$ .

The brackets remind us to square only  $b$ .

$$ab^2 = 1 \times (2)^2 = 1 \times 4 = 4$$

(ii)  $a^2b$  is actually  $a^2 \times b$ .

Square  $a$  first and then multiply the result by 2.

$$a^2b = (1)^2 \times (2) = 1 \times 2 = 2$$

**BINARY OPERATIONS**

A binary operation is simply a rule for combining two numbers to obtain another number. Since an operation is performed between **two** variables or numbers, we refer to the operation as a binary operation. For example, each of the four arithmetical operations of  $+$ ,  $=$ ,  $\times$  and  $\div$ , are examples of binary operations.

We can use symbols to replace operations in algebra. However, when we do so we must define the operation precisely so that it is possible to evaluate the expression.

Assuming we use the symbol  $*$  to represent an algebraic operation, we may choose to define  $*$  as say,

$$a * b = 2a + b$$

Using this definition of  $*$ , we can evaluate expressions involving this binary operation by simply substituting for  $a$  and  $b$  as follows:

$a * b = 2a + b$
$5 * 4 = 2(5) + 4 = 14$
$7 * 3 = 2(7) + 3 = 17$
$-8 * 20 = 2(-8) + 20 = 4$

**Example 7**

If  $p * q = p^2 - q$  evaluate

- (i)  $5 * 4$
- (ii)  $4 * 5$
- (iii)  $(3 * 7) * 1$
- (iv)  $4 * (3 * 6)$

**Solution**

(i)  $5 * 4 = 5^2 - 4 = 21$

(ii)  $4 * 5 = 4^2 - 5 = 11$

(iii)  $(3 * 7) * 1 = (3^2 - 7) * 1$   
 $= 2 * 1 = 2^2 - 1 = 3$

(iv)  $4 * (3 * 6) = 4 * (3^2 - 6)$   
 $= 4 * 3 = 4^2 - 3 = 13$

**Example 8**

If  $m \bullet n = 2mn - 5$

Evaluate:

- (i)  $7 \bullet 3$
- (ii)  $(5 \bullet 3) \bullet (1 \bullet 4)$

**Solution**

(i)  $7 \bullet 3 = 2(7)(3) - 5 = 42 - 5 = 37$

(ii)  $(5 \bullet 3) \bullet (1 \bullet 4) = [2(5)(3) - 5] \bullet [2(1)(4) - 5]$   
 $= (30 - 5) \bullet (8 - 5)$   
 $= 25 \bullet 3$   
 $= 2(25)(3) - 5$   
 $= 150 - 5$   
 $= 145$