FAS-PASS Maths

3. CONSUMER ARITHMETIC

PERCENTAGES

Concept of percent

A percent is a fraction with a denominator of 100. The word percent actually means 'per hundred' or 'hundredths'. It is denoted by the symbol, %.

Fraction, decimals and percents are basically rational numbers expressed in different forms. It is often convenient to make conversions among these forms.

Percent to fraction

A percent is really a fraction written in another form. So we can think of 48% as 48 hundredths. We may, however, wish to reduce the fraction to its lowest terms where possible.

$48\% = \frac{48}{100} = \frac{12}{25}$ $\qquad$ $164\% = \frac{164}{100} = \frac{41}{25} = 1\frac{16}{25}$

Percent to decimal

To convert a percent to a decimal, we express it as hundredths, then convert to decimal form by dividing the numerator by 100.

$23\% = \frac{23}{100} = 0.23$ $\qquad$ $159\% = \frac{159}{100} = 1.59$

Fraction to percent

Since one whole is equal to 100%, we can convert any fraction to a percent by multiplying the fraction by 100%.

1 whole = 100%

1 half $= \frac{1}{2} \times 100\% = 50\%$

1 quarter $= \frac{1}{4} \times 100 = 25\%$, and so on.

Example: Convert $\frac{3}{8}$ to a percent

$\frac{3}{8}$ of a whole $= \frac{3}{8} \times 100\% = 37\frac{1}{2}\%$

Calculating the percent of a quantity

Since percentages are special fractions, working with percentages is very similar to working with fractions. In calculating the percent of a quantity, we express the percent as a fraction and calculate the fraction of the quantity.

To calculate 25% of 56, we express 25% as 25 hundredths and then multiply by 56.

$\frac{25}{100} \times 56 = 14$

Example 1

Calculate: a) $12\frac{1}{2}$ % of 128 $\qquad$ b) 115% of 56?

Solution

a)
$= \frac{12.5}{100} \times 128$
$= \frac{1600}{100}$
$= 16$

b)
$= \frac{115}{100} \times 56$
$= \frac{6440}{100}$
$= 64.4$

Expressing one quantity as a percent of another

In comparing two quantities, it is always necessary to identify the base of the comparison. It is helpful to replace the word percent with fraction to identify the base of the comparison (whole).

Example 2

Express 15 as a **percent** of 25.

Solution

In this case, the base or whole is 25. Expressing 15 as a fraction of 25, our result is $\frac{15}{25}$. Converting our result to a percent gives $\frac{15}{25} \times 100\% = 60\%$

Example 3

What percent of 80 is120?

Solution

In this case, the base or whole is 80. Expressing 120 as a fraction of 80, we obtain $\frac{120}{80}$.
Converting our result to a percent gives
$\frac{120}{80} \times 100\% = 150\%$

Calculating the whole, given a part, expressed as a percent

If we know some percent of a quantity, we can calculate the whole, from which it came, by using basic proportions.

Example 4

If 25% of a quantity is 28, what is the quantity?

Solution

25% of the quantity $=28$

1% of the quantity $= \dfrac{28}{25}$

100% of the quantity $= \dfrac{28}{25}\times 100$

$=112$

Example 5

If 115% of a quantity is 90, what is the quantity?

Solution

$115\% \equiv 90$

Therefore, $1\% \equiv \dfrac{90}{115}$

And $100\% \equiv \dfrac{90}{115}\times 100 = 78.3$

The quantity is therefore 78.3, correct to 1 decimal place.

Profit and loss

If an article is sold for more than its cost, then it is sold at a profit.

If an article is sold for less than its cost, then it is sold at a loss.

Profit = Selling price – Cost price
Loss = Cost price – Selling price

Profit or loss may be expressed as a percentage of the cost price. This is called, percentage profit or percentage loss.

$$\text{Percentage Profit} = \frac{\text{Profit}}{\text{Cost Price}}\times 100\%$$

$$\text{Percentage Loss} = \frac{\text{Loss}}{\text{Cost Price}}\times 100\%$$

Example 6

A house was bought for \$150 000 and later sold for \$180 000. Calculate the percentage profit.

Solution

Profit = \$180 000 - \$150 000 = \$30 000

$$\text{Percentage Profit} = \frac{\text{Profit}}{\text{Cost Price}}\times 100\%$$

$$= \frac{\$30\,000}{\$150\,000}\times 100\%$$

$$= 20\%$$

Example 7

A van, which cost \$20 000, was later sold for \$16 000. Find the percentage loss.

Solution

Loss = Cost price – Selling price
= \$20 000 - \$16 000 = \$4 000

$$\text{Percentage Loss} = \frac{\text{Loss}}{\text{Cost Price}}\times 100\%$$

$$= \frac{4\,000}{20\,000}\times 100\%$$

$$= 20\%$$

Discount

A merchant may sometimes allow a purchaser to purchase an item for less than the marked price. The deduction is called the discount.

Selling price = Marked price – Discount on the marked price

Discounts can often be expressed as a percentage of the marked price.

Example 8

An article is marked at \$40 and sold for \$36. What percent discount was given?

Solution

Discount = Marked price – Selling price
= \$40 \$36 = \$4

Percent discount is $\dfrac{4}{40}\times 100\% = 10\%$

Example 9

An item of jewellery marked at \$900 is discounted by 20% for cash payment. Calculate
(i) The discount.
(ii) The price that a customer will pay when paying with cash.

Solution

(i) The discount is 20% off the marked price of \$900

$$\text{Discount} = \frac{20}{100} \times \$900 = \$180$$

(ii) Price the customer pays
= Marked price – Discount
= \$900 - \$180
= \$720

Sales tax

In most countries, customers are charged a sales tax or Value Added Tax (VAT) on goods purchased. This is expressed as a percentage of the selling price of an item.

Example 10

A company sells a printer for \$340. If the sales tax in the country is 12%, how much will a customer pay for a printer?

Solution

$$\text{Sales Tax} = 12\% \text{ of } \$340 = \frac{12}{100} \times \$340 = \$40.80$$

The customer pays:
Marked Price + Tax = \$340 + \$40.80 = \$380.80

Example 11

The price of an item, inclusive of the VAT is \$125. If VAT is charged at a rate of 15%, what was the price of the item before VAT?

Solution

Since the original quantity or cost price was increased by 15%, we can let the price before VAT be represented as 100%.

$$115\% \equiv \$125$$

$$1\% = \frac{\$125}{115}$$

$$100\% = \frac{\$125}{115} \times 100 = \$108.70$$

Hence, the price of the item before VAT = \$108.70

Commission

Some companies offer incentives to their employees in the form of commission. The commission is usually calculated as a percentage of the value of the commodities sold.

Example 12

A car salesperson is paid 4% commission on the selling price of a vehicle. How much will she receive if she sells a car for \$20 000?

Solution

Commission

$$4\% \text{ of } \$20\,000 = \frac{4}{100} \times \$20\,000 = \$800$$

Example 13

Amanda receives 5% commission on her first \$5 000 in sales and 10 % on sales beyond \$5 000. If her sales amounted to \$13 000, what was her commission?

Solution

Commission on first \$5 000 = 5% of \$5 000 = \$250
Balance of sales = \$13 000 − \$5 000 = \$8 000
Commission on remaining balance = 10% of \$8 000 = \$800
Total commission = \$250 + \$800 = \$1 050

Simple and Compound Interest

Financial institutions pay interest to customers who deposit money and they also charge interest to customers who borrow money. There are two different types of interest rates – simple interest and compound interest. The principal is the sum of money borrowed or invested.

Calculating Simple Interest

In calculating Simple Interest, the principal remains constant over the period of the loan and the amount of interest earned (or charged) remains the same each year. If the principal, P is invested at a rate of R%, then the interest, I, for a period of one year will be:

$$I = \frac{R}{100} \times P$$

After, a period of T years, the simple interest, I will be:

$$I = \frac{R}{100} \times P \times T$$

It is convenient to state the formula in the form,

$$I = \frac{PRT}{100},$$

where, I = interest, P = principal, R = rate and T = time in years

Example 14

Calculate the interest to be paid on a loan of $5 000 for 4 years, at a simple interest rate of 10% per annum. Hence, calculate the total amount to be repaid.

Solution

$$I = \frac{PRT}{100} = \frac{\$5000 \times 10 \times 4}{100} = \$2000$$

The total amount to be repaid = $5 000 + $2 000 = $7 000

Example 15

The simple interest on an investment of $40 000 for 3 years amounted to $6 000. What rate of interest was paid?

Solution

We can use the formula and substitute for all the given values.

$$I = \frac{PRT}{100}$$

$$\$6000 = \frac{\$40000 \times R \times 3}{100}$$

$$\$40000 \times R \times 3 = \$6000 \times 100$$

$$R = \frac{\$6000 \times 100}{\$40000 \times 3}$$

$$R = 5\%$$

Calculating Compound Interest

If the interest earned or paid on investments is added to the principal at given intervals, then the interest is said to be compounded over the time period. This type of interest is called compound interest.

Example 16

Calculate the compound interest earned on the sum of $1 000 at the rate of 10% per annum over a period of three (3) years.

Solution

For the 1st year:
Principal = $1 000 Interest =10%

$\therefore$ Interest $= \frac{10}{100} \times \$1000 = \$100.$

Balance at the end of the 1st year = $\$1000 + \$100 =$ $1 100.

For the 2nd year:
Principal = $1 000 + Interest at the end of the 1st year
= $1 000 + $100 = $1 100

Interest in 2nd year $= \frac{10}{100} \times \$1100 = \$110$

$\therefore$ Balance at the end of the 2nd year = $ 1 100 + $ 110 = $1 210

For the 3rd year:
Principal = $1 210

Interest $= \frac{10}{100} \times \$1210 = \$121$

Balance at the end of the 3rd year
=$1210 +$121 =$1321

Hence the total Compound Interest earned = $1331 -$1000 = $331

Formula for calculating compound interest

The above procedure for calculating Compound Interest becomes tedious, especially if the values of principal, time and rate are fractional values. When the period of the loan or the investment is over a long period, it is wise to use a formula to calculate the compound interest.

The sum of money earned or paid at the end of the transaction is called the amount (A). The difference between the amount and the principal (P) is called the compound interest.

For a rate of R% per annum over a period of n years, the formula is

$$A = P\left(1 + \frac{R}{100}\right)^n$$

Appreciation and Depreciation

In transactions involving Compound Interest, the amount invested grows each year and in this sense money invested in compound interest appreciates. Some investments decrease in value over time because of age or use and these are examples of depreciation.

Example 17

Alvin bought a new car for \$120 000.00. Its value depreciates at a rate of 10% per annum. Calculate the value of the car after (i) one year (ii) two years.

Solution

(i) The depreciation of the car after one year is
10% of \$120 000 = \$12 000
Hence the value of the car after one year is
\$120 000 – \$12 000 = \$108 000

(iii) The depreciation of the car after two years is
10% of \$108 000 = \$10 800
Hence the value of the car after 2 years
= \$108 000.00 − \$10 800.00
= \$97 200

Example 18

Calculate the compound interest on \$6 000 if invested for 3 years at $7\frac{1}{2}\%$ per annum.

Solution

$$A = P\left(1+\frac{R}{100}\right)^n$$
$$= \$6000\left(1+\frac{7.5}{100}\right)^3$$
$$= \$7453.78$$

Hence, the compound interest
$= \$7453.78 - \$6000 = \$1453.78$

Example 19

An item of furniture, originally valued at \$1 000, depreciates by 10% each year. What is its expected value after two (2) years?

Solution

$P = \$1000 \quad R = 10\% \quad n = 2$

A is the expected value or the amount after the period.

$$A = 1000\left(1-\frac{10}{100}\right)^2$$

By using a calculator, $A = \$810$

Hire purchase

Instead of taking a loan to buy an item, some customers may buy directly from the company and pay on terms, that is, over a period of time. This type of transaction is called hire purchase (HP). It allows the customer to acquire the item while still paying for it. Full ownership is only achieved when the customer has completed all payments. The customer is usually expected to provide a down payment and the balance is paid off in monthly or weekly installments.

Example 20

A company advertises the cash price of a computer as \$4800. If it is bought on hire purchase, the customer must pay a down payment of 10% of the cash price and then 12 monthly installments of \$364 each. Calculate the hire-purchase price of the item.

Solution

10% of \$4800 = \$480
Total paid in installments = Down-payment + Total Installments
Total in instalments = 12 × \$364 = \$4368.
Hence, the hire purchase price of the item is \$480 + \$4368 = \$4848

RATIOS

Ratios are used to make quantitative comparisons between or among measures of the same kind.

Definition and notation

A ratio is a quantitative relation between two quantities that are expressed in the same units. It shows the number of times one quantity contains (or is contained in) the other.

The ratio *a* to *b* is written as *a*:*b*, read as *a* to *b*.
The ratio *a* to *b* to *c* is written as *a*:*b*:*c*, where *a*, *b* and *c* represent the quantities.

Ratios have the following properties:
1. A ratio is a multiplicative comparison.
2. The quantities to be compared are measured in the same units.
3. The order is important in a ratio.
4. Ratios are expressed in its lowest terms.
5. We use only integer values in expressing ratios.

Sharing a quantity in a given ratio

To share a quantity in a ratio, say *a*:*b*, we first add *a* to *b* to determine the number of shares. Then, divide the quantity by the number of shares to obtain the size of one share. We can now calculate the value of each share using multiplication as shown below.

Share \$54 in the ratio 4:5.
Total number of shares: 4 + 5 = 9
9 shares = \$54
1 share = \$6
4 shares = \$6 × 4 = \$24
5 shares = \$6 × 5 = \$30
Hence, the ratio is 24:30, and which is clearly 4:5.

Finding the missing part of a ratio

Two lengths are in the ratio 3:7. If the longer piece is 35 cm, what is the length of the shorter piece?
Since the longer piece represents 7 parts, we can equate as follows:
7 parts = 35 cm
1 part = (35 ÷ 7) cm = 5 cm
The shorter length = 3 parts = 3 × 5 cm = 15 cm

Example 21

Randy has 12 more CDs than Jackie. The ratio of the number of Randy's CD to the number of Jackie's CD's is 5:1. How many CDs does Randy have?

Solution

Randy has 4 more shares than Jackie. But, Randy has 12 more CDs than Jackie. We can, therefore, equate as follows:
4 shares = 12 CD's
1 share = 3 CD's
Hence, Randy has 5 (3) = 15 CD's

RATES

Like ratios, rates involve comparison of quantities but these are expressed different units. The following are examples of rates.

1. Mangoes are sold at a rate of 5 for \$8.00
2. The rate at which a car consumes fuel is 400 km per 30 litres.
3. The speed of a vehicle is 80 km/hour or 80 kmh^{-1}.

A **unit rate** is the rate per one unit. In the first two examples, the unit rate is not given. The third statement is an example of a unit rate, the speed of the vehicle gives the distance travelled (80 km) in one unit of time (one hour).

In solving problems involving rates, we may choose any of the methods illustrated below.

Unitary method

This method involves calculating a unit rate using division and then calculating the for any number of units using multiplication.

Mangoes are sold at a rate of 5 for \$8.00. To calculate the selling price of 6 mangoes, we use division to obtain the unit rate.
5 mangoes are sold for \$8.00
1 mango will be sold for \$8.00 ÷ 5 = \$1.60
Then use multiplication to obtain the selling price of 6 mangoes.
6 mangoes will be sold for × 6 = \$9.60

Equivalent rates (or ratios) method

Proportions

Sometimes it is convenient to solve problems involving rates and ratios by setting up a proportion. A proportion is a statement of equality between two ratios or two rates.

Since a rate (or ratio) involves two quantities, a pair of equivalent rates will involve four quantities. In solving for an unknown quantity, three out of these four quantities are given, and we use the basic principle of equivalent fractions to find the unknown quantity.

It should be noted, however, that this method is applicable to situations involving direct proportion. In direct proportion, two quantities vary such that as one quantity increases (or decreases), the other increases (or decreases).

The following is an example of direct proportion since as distance increases the volume of fuel increases.

A car consumes 30 litres in travelling a distance of 400 km. Calculate the distance covered in consuming 1 litre of fuel.
Let x represent the unknown distance.

Using equivalent ratios,	Using equivalent rates,
$\frac{400\text{ km}}{x\text{ km}} = \frac{30\text{ litres}}{1\text{ litres}}$	$\frac{400\text{ km}}{30\text{ litres}} \equiv \frac{x\text{ km}}{1\text{ litre}}$
$30 \times x = 400 \times 1$	$x \times 30 = 400 \times 1$
$x = \frac{400}{30}$	$x = \frac{400 \times 1}{30}$
$x = 13.3\text{ km}$	$= 13.3\text{ km}$

We will now illustrate how the method of setting up a proportion (using equivalent rates or ratios) can be used to find the rate for any number of units. This method involves finding cross- products and is sometimes referred to as the cross- product method. A comparison of the unitary method and the method of setting up a proportion is shown in the following example.

Example 22

If 2 litres of petrol cost \$5.40, calculate the cost of 5 litres of petrol.

Solution

Unitary method	**Using proportions**
If 2 litres cost \$5.40 Then 1 litre costs $\frac{\$5.40}{2} = \$2.70$ And 5 litres will cost $\$2.70 \times 5$ $= \$13.50$	$\frac{2L}{\$5.40} = \frac{5L}{\$x}$ $2 \times x = 5 \times 5.40$ $x = \frac{5 \times 5.40}{2}$ $= \$13.50$

Example 23

3 litres of paint cover 20 m^2. How many litres of paint will cover 160 m^2?

Solution

Unitary method:	**Using proportions**
20 m^2 used 3 litres of paint 1 m^2 will use $\frac{3}{20} litres$ 160 m^2 will use $\frac{3}{20} \times 160\ litres$ = 24 litres	Let x litres cover 160 m^2 $\frac{3 \text{ litres}}{20 \text{ m}^2} = \frac{x \text{ litres}}{160 \text{ m}^2}$ $20 \times x = 3 \times 160$ $x = \frac{3 \times 160}{20}$ $x = 24$ litres

Example 24

Two quantities are in the ratio 8: 7. If the smaller quantity is 35, what is the larger quantity?

Solution

The smaller quantity represents 7 parts, So, 7 parts = 35 1 part = 5 8 parts = 5 × 8 = 40	Let x represent the larger quantity Using equivalent ratios $\frac{8}{7} = \frac{x}{35}$ $7 \times x = 8 \times 35$ $x = \frac{8 \times 35}{7} = 40$

Indirect Proportion

In indirect proportion as one quantity increases the other decreases or vice versa. In solving problems on indirect proportion, we cannot use the cross-product method. This method is based on the assumption that as one quantity increases the other increases. In some situations, this is not the case. For example, an increase in the number of employees in a factory will result in a decrease in time required to complete the job. This is provided that the employees work at the same rate.

Problems of this nature are solved using the unitary method.

Example 25

12 men complete a job in 5 days. How long will it take 15 men to complete the job given that they work at the same rate?

Solution

12 men complete the job in 5 days
1 man will complete the job in $12 \times 5 = 60\ days$
15 men will complete the job in $60 \div 15 = 4\ days.$

Indirect proportion is treated in more depth in the chapter on *Variation* later on. In this chapter, an algebraic approach to solving problems on both direct and indirect proportion will be introduced.

Currency conversion

To convert from one currency to another, we can use any of the methods mentioned in the previous section. The unit rate method and cross product method are more popular when converting currencies.

Example 26

Water is leaking from a tank at the constant rate of 4 litres per minute. How long will it take to lose 32 litres of water?

Solution

The rate at which water leaks $= 4\ l$ per minute

Time taken to lose $4\ l = 1$ minute

Time taken to lose $1\ l = \frac{1}{4}$ minute

Time taken to lose $32\ l = 32 \times \frac{1}{4}$ minutes

$= 8$ minutes

Example 27

Jerry and Kerry are running a 60 km marathon. Jerry runs at 24 km per hour, Kerry at 20 km per hour.

(i) How long will it take Jerry to finish the race?

(ii) When Jerry finishes the race, how much further does Kerry have to run to reach the finish line?

Solution

(i) The time taken for Jerry to finish the race

$$= \frac{\text{Distance}}{\text{Speed}} = \frac{60\text{km}}{24\text{km/h}} = 2.5 \text{ hours}$$

(ii) When Jerry finishes the race, 2.5 hours would have elapsed. We must, therefore, find out the distance covered by Kerry after 2.5 hours. Kerry's distance can be calculated from:

Distance = Speed $\times$ Time

$= 20 \times 2.5$ km $= 50$ km

Since the marathon is 60 km, Kerry still has to run: (60 – 50) =10 km.

Overtime

When employees work beyond their normal schedule of hours than is expected, they are paid extra money for doing so. Overtime pay is usually a higher rate than the basic rate.

The wages earned for working extra time is called **overtime wages** and the rate of pay is called **overtime rate**. The normal wages earned without overtime is called the **basic rate** of pay.

Overtime rates vary, it can be one-half times the **basic rate** of pay for overtime work or two times or more, depending on the company.

Example 28

A company pays a basic wage of $900.00 for a 40-hour week. Overtime is paid at a time-and-a-half.

a) Calculate

i. The basic hourly rate.

ii. The overtime rate.

b) An employee worked 20 hours overtime in a certain week. Calculate

i. The employee's overtime earnings for the week.

ii. The employee's total earnings for the week.

Another employee earned $1 305.00 in a particular week. How many hours of overtime did she work?

Solution

a)(i) Basic hourly rate
= $900.00 ÷ 40 = $22.50

(ii) Overtime hourly rate
= $22.50 × 1.5 = $ 33.75

b)(i) Overtime earnings:
20 hours @ $33.75 per hour = $ 675.00

(ii) Basic weekly wage = $ 900.00

Total Earnings = $ 675 + $900
= $1 575.00

c) The total earnings of the worker
= $1 305. 00

The overtime earnings = $ 1 305 – $900 = $ 405.
Number of hours overtime = $405.00 ÷ $33.75
= 12 hours

Utility bills

Consumers are billed at timely intervals for goods, services and for utilities such as electricity, telephone, water, cable and internet access. Some companies charge a rental or fixed charge for the use of the service and add on a variable charge depending on how much of the service was used. The fixed charge can be for rental of the equipment such as a telephone, cable box or electricity meter. The variable charge is calculated based on rates set by the company. Some systems rely on a minimum charge for the first number of units used and further charges for use of additional units.

Example 29

A company charges the following monthly rates for all local telephone calls.

First 15 calls or less	45 cents per call
Remaining calls	30 cents per call

a) Calculate the total bill for each of the following customers.

i. Customer George, who made 12 local calls for the month.

ii. Customer Miriam, who made 52 local calls for the month.

Mr Paul was charged a total of \$28.05 for local calls in a certain month. How many calls did he make for the month?

Solution

a)(i) George's bill

$12 \times 0.45 = \$5.40$

(ii) Miriam's bill

$15 \times 0.45 = \$6.75$

52 calls − 15 calls = 37 calls

$37 \times 0.30 = \$11.10$

$\$11.10 + \$6.75 = \$17.85$

b) Mr Paul's calls cost

$\$28.05 - \$6.75 = \$21.30$

$\$21.30 \div 0.30 = 71$ calls

Total calls: $71 + 15 = 86$

Example 30

The information displayed in the table below is an extract from Mr Sandy's electricity bill for a period of one month.

Meter Reading (kWh)		Fuel Charge	Energy Charge	VAT
Previous	Present			
5833	6925	25 cents per kWh	20 cents per kWh	15% of the total bill

Calculate:

i. The number of units in kWh used by Mr Sandy for the month.

ii. Mr Sandy's bill, inclusive of VAT, for the month

Solution

(i) Number of units used = $6925 - 5833 = 1092$

(ii) Fuel Charge = $1092 \times \$0.25 = \$273$

Energy Charge = $1092 \times \$0.20 = \$218.40$

Total cost of electricity = $\$273 + \$218.40$
$= \$491.40$

VAT on charges $= \frac{15}{100} \times \$491.40 = \$73.71$

Mr. Sandy's bill will total:

$\$491.40 + \$73.71 = \$565.11$