2. NUMBER THEORY

TYPES OF NUMBERS

We can describe numbers as belonging to specific sets. Some of these sets can be listed while others have to be described as it is impossible to list the members.

Number sets

The following sets are described by listing the elements. We include three dots at the end to indicate that the set is infinite. This means that we can count forever and still will not be able to reach the end.

W denotes the set of whole numbers.
$W = \{0, 1, 2, 3, 4, ...\}$

N denotes the set of natural numbers or counting numbers.
$N = \{1, 2, 3, 4, ...\}$

Z denotes the set of integers - these are positive and negative whole numbers.
$Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational Numbers

Q denotes the set of rational numbers. A rational number is any number that can be expressed in the form $\frac{a}{b}$, where a and b are integers, $b \neq 0$. Rational numbers include integers, fractions, decimals that are either recurring or terminating. All rational numbers can be expressed in an exact form.

Some examples of rational numbers are $\frac{3}{4}$, $5\frac{3}{4}$, 0.2, 0.3, 23, $\frac{8}{9}$, $\frac{13}{7}$, 3.26, 0.8 and -5

Irrational numbers

Numbers that are not rational are irrational. Hence, any number that ***cannot*** be expressed in the form $\frac{p}{q}$ (where p and q are integers, $q \neq 0$), is an **irrational** number.
Some examples of irrational numbers are $\sqrt{2}$, $\sqrt{5}$, $\sqrt[3]{12}$, and π. When irrational numbers are written in root form, we refer to them as surds and in this form, they are considered as exact.
When irrational numbers are written as decimals, they are non-terminating and non-recurring and hence, they are not exact. The table below shows how irrational and rational numbers differ.

Irrational Numbers	Rational Numbers
$\sqrt{2} = 1.414213562$ $\sqrt{5} = 2.236067977$ $\pi = 3.141592654$ In each case, there is no predictable pattern of recurring digits.	$\frac{3}{7} = 0.428571428571...$ $\frac{4}{11} = 0.363636 ...$ $\frac{5}{13} = 0.384615384615 ...$ In each case, there is a predictable pattern of recurring digits.

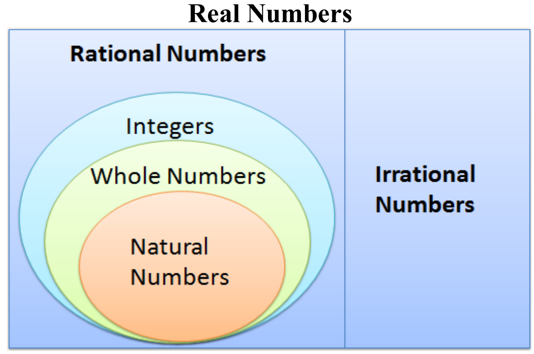
Irrational numbers do exist on the number line. In fact, between any two numbers, say 1 and 2, on the number line, there is an infinite number of irrational numbers.

Real numbers

If we were to combine the set of rational and irrational numbers we would obtain the set of real numbers. Note that $Q \cup Q' = R$, where Q is the set of Rational numbers Q', the set of Irrational Numbers and R is the set of Real Numbers. All the numbers that we have met so far are members of the set of real numbers, R.

The Venn Diagram below shows the relationship between these sets. The universal set represents the set of Real numbers, R.

Real Numbers



Odd and even numbers

Even numbers are natural numbers that are divisible by 2. The set of even numbers is an infinite set.
{2, 4, 6, 8, …}

Odd numbers are natural numbers that are not divisible by 2. The set of odd numbers is an infinite set.
{1, 3, 5, 7,…}

Prime and composite numbers

A prime number has only two distinct factors, itself and one. For this reason, 1 is not considered prime, since it has only one factor. The first fifteen prime numbers are:
{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47}.

A composite number has at least one other factor besides itself and one, that is, at least three factors. The set of composite numbers is infinite. The first fifteen composite numbers are
{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25}.

Multiples and factors

We can generate the set of multiples of a number by multiplying the number by 1, 2, 3, and so on. In the table below, multiples of the numbers 2, 5 and 12 are shown. Each multiple is obtained by multiplying the number first by 1, then by 2, then by 3 and so on.

Number	Multiples
2	2, 4, 6, 8, 10, 12, 14, …
5	5, 10, 15, 20, 25, 30, …
12	12, 24, 36, 48, 60, 72,…

The factors of a number are the whole number divisors of that number. Every number, except 1, has at least two factors.
The table below lists the factors of some numbers. Note that 5 is a prime number while 12 and 27 are composite numbers.

Number	Factors
5	1, 5
12	1, 2, 3, 4, 6, 12
27	1, 3, 9, 27

Lowest Common Multiple (LCM)

The LCM of a set of numbers is the lowest multiple that is common to all the numbers of the set.

To find the LCM of two or more numbers, we select any one of the numbers and list its multiples, until we arrive at a common multiple.

To find the LCM of 10 and 15, we first list the multiples of each number as shown below.

Multiples of 10: {10, 20, 30, 40, 50, 60, 70, 80,…}
Multiples of 15: {15, 30, 45, 60, 75, 90, 105,…}

Notice that the common multiples are {30, 60, 90}
The Lowest Common Multiple is 30.

Example 1

Determine the LCM of:
(i) 3 and 6
(ii) 8 and 12
(iii) 6 and 7

Solution

(i) Multiples of 3: {3, 6, 9, 12, 15, 18,…}
Multiples of 6: {6, 12, 18, 24, 30,…}
The LCM is 6.

(ii) Multiples of 8: {8, 16, 24, 32, 40, 48, 56, …}
Multiples of 12: {12, 24, 36, 48, 60, 72,…}
The LCM is 24.

(iii) Multiples of 6: { 6, 12, 18, 24, 30, 36, 42, 48…}
Multiples of 7: {7, 14, 21, 28, 35, 42, 49,…}
The LCM is 42.

When finding the LCM of a set of numbers we must note the following:

1. If one number in the set is a multiple of the others, then the LCM is simply the largest number in the set. For example, the LCM of 3, 6 and 9 is 9.
2. If there are no common factors among the set of numbers, then the LCM is the product of the numbers in the set. For example, the LCM of 3, 5 and 7 is 105.
3. If there are factors common to members of the set of the numbers, then we can choose to use the above method of listing the multiples.

Highest common factor (H.C.F.)

The Highest Common Factor or HCF of a set of numbers is the highest number that is a factor of all the numbers of the set.

The HCF of a set of numbers can be found by using the following steps sequentially:

1. List the factors of each number in the set
2. Identify the common factors
3. Select the highest among the common factors.

Example 2

Determine the HCF of each of the following number sets
(i) 8 and 12
(ii) 15, 25 and 40
(iii) 24, 36 and 60

Solution

(i) Listing all the factors of 8 and 12, we have
Factors of 8: {1, 2, 4, 8}
Factors of 12: {1, 2, 3, 4, 6, 12}
The common factors are 1, 2 and 4.
The highest factor is 4
The HCF of the set of numbers is 4.

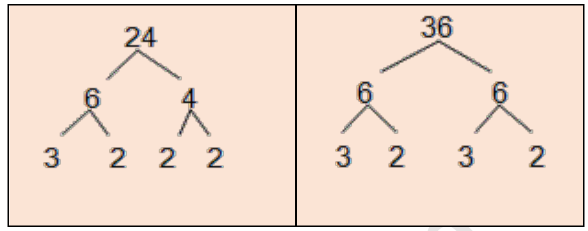
(ii) Listing all the factors of 15, 25 and 40, we have
Factors of 15 : {1, 3, 5, 15}
Factors of 25: {1, 5, 25}
Factors of 40: {1, 2, 4, 5, 8, 10, 20, 40}
The common factors are 1 and 5.
The HCF of the set of numbers is 5.

(iii) Listing all the factors of 24, 36 and 60, we have
Factors of 24 :{1, 2, 3, 4, 6, 8, 12, 24}
Factors of 36: {1, 2, 3, 4, 6, 9, 12, 18, 36}
Factors of 60:{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30}
The common factors are1, 2, 3. 4, 6, and 12.
The HCF of the set of numbers is 12.

LCM and HCF by prime factorisation

The above methods are very effective when computing LCM and HCF of small numbers. However, for larger numbers, it becomes tedious and the method of prime factorisation is preferred.

In using these methods we must express the number as a product of its prime factors. We may use a factor tree to perform this task. The following illustrates how to obtain the prime factors using a factor tree. The numbers in the lowest branch of the tree are prime numbers and represent the prime factors of the number.



To find the LCM of 24 and 36, express each number as a product of its prime factors.

$24 = 2 \times 2 \times 2 \times 3$
$36 = 2 \times 2 \times 3 \times 3$

By observation, the lowest common multiple of 24 and 36 must comprise the product of a set of multiples of 2 and 3.
The highest power of 2 contained in any number is 2^3, seen in 24.
The highest power of 3 contained in any number is 3^2, seen in 36.
Hence, the LCM of 24 and 36 is $2 \times 2 \times 2 \times 3 \times 3 = 72$

Note that:
72 is a multiple of 24 because it comprises the factors $2 \times 2 \times 2 \times 3$.
72 is a multiple of 36 because it comprises the factors $2 \times 2 \times 3 \times 3$.

To find the HCF of 24 and 36, we identify common factors of these numbers and multiply them.

$$24 = \boxed{2} \times \boxed{2} \times 2 \times \boxed{3}$$
$$36 = \boxed{2} \times \boxed{2} \times 3 \times \boxed{3}$$

The common factors are: $2 \times 2 \times 3$
The HCF is 12

Note that $2 \times 2 \times 3 = 12$ is a factor of 24 and also of 36.

Example 2

Express each of the numbers 60, 75 and 90 as a product of prime factors. Hence, calculate the
(i) LCM
(ii) HCF

Solution

(i) Express these numbers as a product of prime factors:

$60 = 2 \times 2 \times 3 \times 5$

$75 = 3 \times 5 \times 5$

$90 = 2 \times 3 \times 3 \times 5$

The LCM, if expressed as a product of prime factors, will comprise a combination of 2, 3 and 5. To obtain it, we must select the highest power of each factor.

The highest power of 2 contained in any number is 2^2, seen in 60.

The highest power of 3 contained in any number is 3^2, seen in 90.

The highest power of 5 contained in any number is 5^2, seen in 75.

The LCM is therefore $2^2 \times 3^2 \times 5^2 = 2 \times 2 \times 3 \times 3 \times 5 \times 5 = 900$

(ii) To obtain the HCF we examine their prime factors.

$60 = 2 \times 2 \times 3 \times 5$

$75 = 3 \times 5 \times 5$

$90 = 2 \times 3 \times 3 \times 5$

We can now easily identify the common factors and multiply them. The common factors are 3 and 5.

Hence the HCF of 60, 75 and 90 is $3 \times 5 = 15$.

Example 3

Determine the LCM and HCF of 24, 36 and 60.

Solution

Expressing each number as a product of prime factors:

$24 = 2 \times 2 \times 2 \times 3$

$36 = 2 \times 2 \times 3 \times 3$

$60 = 2 \times 2 \times 3 \times 5$

The LCM must comprise a product of factors of 2, 3 and 5.

The LCM is $2 \times 2 \times 2 \times 3 \times 3 \times 5 = 360$

The HCF must comprise the product of all the common factors.

The HCF is $2 \times 2 \times 3 = 12$

Solving problems involving HCF and LCM

In solving word problems on LCM and HCF, we try to figure if the answer involves finding multiples or factors. We must remember that multiples are associated with LCM, while factors are associated with HCF. Consider the following problem:

Two lighthouses flash their lights every 20 seconds and 30 seconds respectively. Given that they flashed together at 7:00 pm, when will they next flash together?

The first lighthouse flashes its lights at intervals of 20 seconds. Hence, the lights flash at multiples of 20 seconds.

The first lighthouse flashes at: 20, 40, 60, 80, 100, 120, …seconds

Similarly,

The second lighthouse flashes at 30, 60, 90, 120, …seconds

To determine when they will flash together, we need to select the multiples that are common. We can see that both lighthouses flash together after 60 seconds, after 120 seconds and so on. However, at 60 seconds, both lights flashed together for the first time. So, the lights will next flash together at 60 seconds or 1 minute after 7:00 pm which is at 7:01 pm.

Now, consider the following problem:

Three pieces of strings, of lengths, 240 cm, 318 cm and 426 cm are to be cut into equal lengths. What is the greatest possible length of each piece?

We wish to find a common length that can be obtained when each string is divided evenly. For example, the 240 cm string can be cut evenly into 30 cm pieces. But, it is not possible to cut the other lengths into 30 cm pieces because 30 is not a factor of either 318 or 426.

We are therefore interested in finding a common factor of all three numbers and selecting the highest common factor to obtain the greatest length that each piece of cut string can have.

Using the method of prime factorisation, we express each number as a product of prime factors. Next, we look for common factors - these are 2 and 3.

$240 = 2 \times 2 \times 2 \times 2 \times 3 \times 5$

$318 = 2 \times 3 \times 53$

$426 = 2 \times 3 \times 71$

The HCF is therefore $2 \times 3 = 6$.

The greatest possible length that each piece of string can have is 6 cm.

Number sequences

A number sequence is an ordered list of numbers in which there is a recognisable pattern. The sequence is predictable in that successive terms can be derived from some definite rule. The nature of the rule determines the type of sequence.

Arithmetic sequence

If a sequence of values follows a pattern of **adding** a fixed amount from one term to the next term in the sequence, it is referred to as an arithmetic sequence. The number that is added to each term is constant, that is, it is always the same. This constant value is called the common difference and may take any value: positive, negative or even a fraction.

An increasing sequence: 1, 4, 7, 10, 13, 16, 19, 22, 25, ...

In this sequence, the difference between any two consecutive terms is 3 and so we say that the sequence has a common difference of 3. Since the difference is positive, this is an increasing sequence.

A decreasing sequence: 25, 23, 21, 19, 17, 15, ...

In this sequence, the difference between any two consecutive terms is always -2 and so we say that the sequence has a common difference of -2. Since the difference is negative, this is a decreasing sequence.

Geometric sequence

A sequence in which each term is obtained by **multiplying** the term before by some constant value is called a geometric sequence. This constant value is called the common ratio and may take any value - positive, negative or even fractional.

An increasing sequence: 2, 4, 8, 16, 32, 64, 128, ...
This sequence is obtained by multiplying any term by 2 to get the next one and so we say that this sequence has a common ratio of 2.

A decreasing sequence: 2187, 729, 243, 81, 27, 9, ...
This sequence is obtained by dividing any term by 3 to get the next one and so we say that this sequence has a common ratio of $\frac{1}{3}$. Recall that dividing by 3 is equivalent to multiplying by $\frac{1}{3}$.

Other number sequences

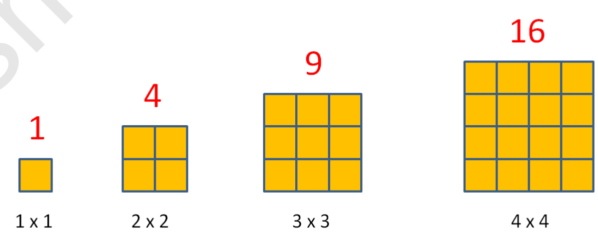
The above two types of number sequences have their consecutive terms separated by either a common difference (arithmetic) or a common ratio (geometric). These are other sequences that do not follow this pattern.

Square numbers

Numbers that are perfect squares are called square numbers. They have an exact square root. If we square the first ten positive numbers we shall get the first ten square numbers. These are:

{1, 4, 9, 16, 25, 36, 49, 64, 81, 100}

Square numbers form a square pattern when arranged as shown below. There is a recognisable pattern in this sequence and one can obtain any term by squaring its sequence number.

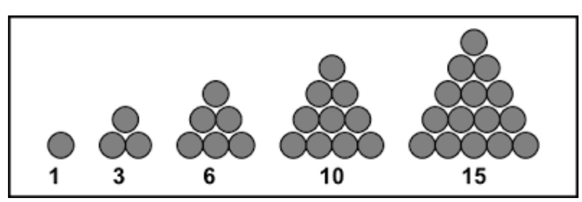


Triangular numbers

Triangular numbers are numbers that can be represented by triangular arrays or configurations, as shown below. The first ten triangular numbers are:

{1, 3, 6, 10, 15, 21, 28, 36, 45, 54}

The diagram below shows how the sequence of triangular numbers can be generated. The starting number is one, similar to the square numbers.



There is a recognisable pattern in this sequence and one can obtain any term by the rule $\frac{n}{2}(n+1)$, where n is the sequence number.

Example 4

Calculate the next term in the number sequence:
1, 2.5, 4, 5.5, 6, __, __, …

Solution

This is an additive sequence, increasing by 1.5.
The next two terms are:
$6 + 1.5 = 7.5$ and $7.5 + 1.5 = 9$

Example 5

Calculate the next term in the number sequence:
5000, 2000, 800, 320, __, __, …

Solution

This is a multiplicative sequence, decreasing by a ratio of $\frac{2}{5}$ or 0.4. The next two terms are:
$320 \times 0.4 = 128$ and $128 \times 0.4 = 51.2$

Example 6

Calculate the next term in the number sequence:
2, 4, 12, 48, …

Solution

Although this is an increasing sequence, there is no fixed difference or fixed ratio. However, there is a pattern:

$2 \times 2 = 4$ and $4 \times 3 = 12$ and $12 \times 4 = 48$
Each term is obtained by multiplying the previous one by 2, then 3 then 4, then 5, then 6 and so on.

The next two terms are:
$48 \times 5 = 240$ and $240 \times 6 = 1440$

Properties of operations

When performing the four arithmetic operations on numbers or algebraic symbols, we make use of certain properties that help us to simplify computational tasks. These properties are stated in the following section.

The commutative law

The commutative law says that when we add or multiply numbers, the order in which the operation is performed does not affect the eventual result. For example,

$2 + 3 = 3 + 2 = 5$ **OR** $2 \times 3 = 3 \times 2 = 6$

The 2 and the 3 can be switched in **addition** and the result is the same.
The 2 and the 3 can be switched in **multiplication** and the result is the same.

This property holds for the operations of addition and multiplication. For any real number, the law is stated as follows:

The commutative law

$$a + b = b + a$$
$$a \times b = b \times a$$

The Commutative Law does **NOT** hold for the operations of subtraction and division. In other words, when we switch the order the result is not the same.
We can test this as follows:
$9 - 4 = 5$ but $4 - 9 = -5$. Hence, $9 - 4 \neq 4 - 9$.

Also,
$12 \div 3 = 4$, but $3 \div 12 = \frac{1}{4}$. Hence, $12 \div 3 \neq 3 \div 12$

The associative law

This law is used when performing operations involving at least three numbers. It states that grouping the numbers in different ways for addition or for multiplication does not affect the result.

The different groupings can be illustrated by inserting brackets or parentheses within the expression – it does not matter how the groups are formed, the answer remains the same.

This law is illustrated for addition as follows:

$$(5 + 2) + 1 = 5 + (2 + 1)$$
$$7 + 1 = 5 + 3$$
$$8=8$$

This law is illustrated for multiplication as follows:

$$(5 \times 4) \times 3 = 5 \times (4 \times 3)$$
$$20 \times 3 = 5 \times 12$$
$$60 = 60$$

Even though the parentheses moved, the order of the numbers remained the same but the value of the expression was not altered.
For any real number, the law is stated as follows:

The associative law

$a + (b + c) = (a + b) + c,$

$a(bc) = (ab)c$

The Associative Law does **NOT** hold for the operation of subtraction. We can test this as follows:
$(9 - 4) - 3 = 5 - 3 = 2$, but

$9 - (4 - 3) = 9 - 1 = 8$

Hence, for subtraction, regrouping the numbers does affect the result.

Also, the Associative Law does **NOT** hold for the operation of division. We can test this as follows:
$(12 \div 3) \div 2 = 2$ but

$12 \div (3 \div 2) = 8$

Hence, for division, regrouping the numbers does affect the result.

The distributive law

This law is very useful both in arithmetic and algebra when removing or inserting brackets in expressions that involve two or more terms. The following example illustrates the law. In removing the brackets, we can distribute multiplication over addition, so that

$$5 \times (7 + 3) = (5 \times 7) + (5 \times 3)$$
$$5 \times 10 = 35 + 15$$
$$50 = 50$$

We can distribute multiplication over subtraction.

$$5 \times (7 - 3) = (5 \times 7) - (5 \times 3)$$
$$5 \times 4 = 35 - 15$$
$$20 = 20$$

We can distribute division over addition, so that

$$\frac{(12 + 4)}{2} = \frac{12}{2} + \frac{4}{2}$$
$$\frac{16}{2} = 6 + 2$$
$$8 = 8$$

We can distribute division over subtraction.

$$\frac{(12 - 4)}{2} = \frac{12}{2} - \frac{4}{2}$$
$$\frac{8}{2} = 6 - 2$$
$$4 = 4$$

For any real number, the law is stated as follows:

The distributive law

$a(b + c) = ab + ac$

$a(b - c) = ab - ac$

$\frac{b+c}{a} = \frac{b}{a} + \frac{c}{a}$

$\frac{b-c}{a} = \frac{b}{a} - \frac{c}{a}$

Example 7

Use the distributive law to calculate the value of (i) 6×204
(ii) $26 \times 3 - 24 \times 3$.

Solution

(i) $6 \times 204 = 6(200 + 4)$
$= 6 \times 200 + 6 \times 4$
$= 1\,200 + 24 = 1\,224$

(ii) $26 \times 3 - 24 \times 3 = 3 \times (26 - 24)$
$= 3 \times 2 = 6$

Closure

A set is said to be closed under an operation if the following is true:
When two elements of the set are combined under the operation, the result is also a member of the set.

For example, the set of whole numbers is closed under addition. This is so because if we add two whole numbers we will always get a whole number, for example, $5 + 7 = 12$, $19 + 23 = 42$, and so on.

However, if we subtract two whole numbers we do not always get a whole number, for example
$15 - 25 = -10$
Since -10 is not a member of the set of whole numbers, we say that the set of whole numbers is not closed under the operation of subtraction.

Even though it is true that we can subtract two whole numbers and sometimes get a whole number, for example, $6 - 2 = 4$, this does not mean that set of whole numbers is closed under the operation of subtraction. This is because if we can find one example when the result does not belong to the set, it is sufficient to say that closure is not a property of the set under the operation.

Example 8

Determine if the set of integers is closed under the following operations
(i) multiplication
(ii) division

Solution

(i) We take any two integers and perform the operation of multiplication.

$$5 \times 6 = 30$$
$$8 \times -2 = -16$$

It is easy to predict that the result will always be an integer.
We conclude that the set of integers is closed under the operation of multiplication.

(ii) When we divide two integers, we sometimes obtain an integer but there are many instances when we will get a non-integer result. For example,

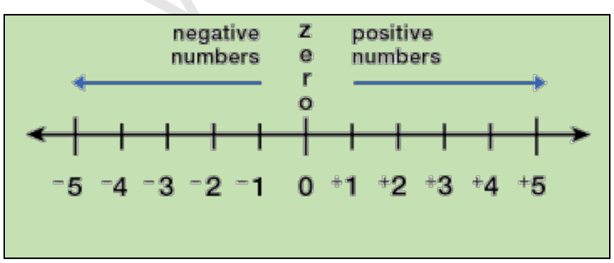
$$2 \div 3 = \frac{2}{3}$$

Clearly, if we find one example where the result is not an integer it is safe to say that the set of integers is not closed under the operation of division.

Identity elements and inverses

Addition

If we consider the vertical line passing through zero on the number line, as a line of symmetry, then each positive number can be considered as having an image or additive inverse. This is simply its negative counterpart. The same can be said for each negative number as having a positive counterpart. Hence, the additive inverse of 3 is –3, and conversely, the additive inverse of $^{-}3$ is 3. All numbers, even when written in fractional or decimal forms have inverses.



To form the additive inverse of a number, we simply change its direction. A positive number will become negative and a negative number will become positive. In both cases, the magnitude of the number remains unchanged.

The following table (columns 1 and 2) illustrates a few examples of the additive inverse of some numbers. When we add a number to its additive inverse the result is always zero (column 3),

Zero is the identity element for the operation of addition.

Number	Additive Inverse	Number + Additive Inverse
5	–5	$+5 + (-5) = 0$
$^{-}7$	7	$-7 + (+7) = 0$
$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{2}{3} + \left(-\frac{2}{3}\right) = 0$
a	$-a$	$a + -a = 0$

Multiplication

To form the multiplicative inverse of a number we invert the number. The inverted number is simply the reciprocal of the number.

The following table (columns 1 and 2) gives examples of the multiplicative inverse of some numbers. When we multiply a number by its multiplicative inverse the result is always one (column 3).

One is the identity element for the operation of multiplication.

Number	Multiplicative Inverse	Number × Multiplicative Inverse
5	$\frac{1}{5}$	$5 \times \frac{1}{5} = 1$
$\frac{3}{4}$	$\frac{4}{3}$	$\frac{3}{4} \times \frac{4}{3} = 1$
–3	$-\frac{1}{3}$	$-3 \times -\frac{1}{3} = 1$
a	$\frac{1}{a}$	$a \times \frac{1}{a} = 1$

Number Bases

So far in our study of numbers, we have only encountered one number base and this was base 10. In base 10, we specify the place value of a digit in powers of ten. The number 1 438 can be represented in expanded notation using the place values of each digit.

$1\ 438 = 1 \times 1000 + 4 \times 100 + 3 \times 10 + 8 \times 1$

We can express the place values as powers of ten.

$1\ 438 = 1 \times 10^3 + 4 \times 10^2 + 3 \times 10^1 + 8 \times 10^0$

When we write the numbers in base ten, we do not include the base as we do for other bases. The number 1 438 in base ten is written as $1\ 438_{10}$. We usually omit the base because it is the common base of our number system and there is no need for emphasis.

Place value in other bases other than base ten

In the base ten number system, there are 10 digits – $\{0,1,2,3,4,5,6,7,8,9\}$ and the place values of the digits in a number is expressed in powers of 10, starting with 10^0 and increasing to 10^1, then 10^2 and so on. Our place value system is called a denary or base ten system. When we change our group's size to any number instead of 10, we create new base systems. We will examine a few of the popular ones.

In the base 8 number system, there are 8 digits – $\{0,1,2,3,4,5,6,7\}$ and the place value of the digits will now be expressed in powers of 8, starting with 8^0 and increasing to 8^1, then 8^2 and so on. This new system is called an octal or **octonary** system.

In the base 5 number system, there are 5 digits – $\{0,1,2,3,4,5\}$ and the place value of the digits will now be expressed in powers of 5, starting with 5^0 and increasing to 5^1, then 5^2 and so on. This new system is called a **quinary** system.

In the base 4 number system, there are 4 digits – $\{0, 1, 2, 3\}$ and the place value of the digits will now be expressed in powers of 4, starting with 4^0 and increasing to 4^1, then 4^2 and so on. This new system is called a **quartenary** system.

In base two, there are only two digits – $\{0, 1\}$ and the place value of the digits will are expressed in powers of 2, starting with 2^0 and increasing to 2^1, then 2^2 and so on. This new system is called a **binary** system.

Expanded notation

We can now compare the base ten number system with other number systems using expanded notation.

$$1234_{10} = 1\times10^3 + 2\times10^2 + 3\times10^1 + 4\times10^0$$
$$1234_8 = 1\times8^3 + 2\times8^2 + 3\times8^1 + 4\times8^0$$
$$1234_5 = 1\times5^3 + 2\times5^2 + 3\times5^1 + 4\times5^0$$
$$1230_4 = 1\times4^3 + 2\times4^2 + 3\times4^1 + 0\times4^0$$
$$1011_2 = 1\times2^3 + 0\times2^2 + 1\times2^1 + 1\times2^0$$

Place Value and Value

The place value of a digit in a numeral refers to the value of its position in the number. For example, in the base 10 number, 3 789.25, the place value of the digit 7 is hundreds, and the place value of the digit 2 is tenths. Therefore, place value refers to the location of the digit in the number and is independent of the magnitude or value of the digit.

The value of a digit refers to its actual value. The value of the digit 7 in the number 3 789.25 is seven hundred (700), and the value of the digit 2 is two tenths $(\frac{2}{10})$. Hence, the value takes into consideration both its location and magnitude

Recently, the term face value has emerged. This merely represents the magnitude of the digit without taking into consideration its position. The face value of 7 in the number 3 789.25 is 7 and the face value of the digit 2 is 2.

Example 9

State the (i) the place value and (ii) the value of the digit 4 in the number 47_8.

Solution

We set up a base 8 place value chart and place the digits in their respective positions as shown below.

8^1	8^0
(8)	(1)
4	7

The place value of the digit 4 is 8.
The value of the digit 4 is $4 \times 8 = 32$

Example 11

State the value of the digit 3 in the number 302_4.

Solution

Setting up the place value chart, we have:

4^2	4^1	4^0
(16)	(4)	(1)
3	0	2

The value of the digit 3
$= (3 \times 4^2)$
$= 3 \times 16 = 48$

Example 12

State the place value of the underlined digit in the number $1\underline{1}010_2$.

Solution

We set up a base 2 place value chart and place the digits in their respective positions as shown below.

2^4	2^3	2^2	2^1	2^0
(16)	(8)	(4)	(2)	(1)
1	1	0	1	0_2

The underlined digit is the second digit.
The place value is 8.

Example 13

State the value of the digit 2 in the number 324_5.

Solution

Setting up the place value chart, we have:

5^2	5^1	5^0
(25)	(5)	(1)
3	2	4_5

The digit 2 is the second digit in the number.
Its value is $2 \times 5 = 10$.