FAS-PASS Maths

1. COMPUTATION

FRACTIONS

Representing fractions

We can represent a fraction using different forms.

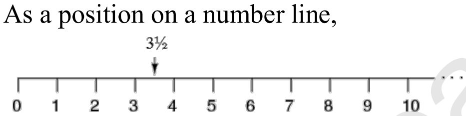
1. As part of a whole, for example, the fraction $\frac{3}{8}$ is really 3 parts of a whole divided into 8 equal parts. The top number, 3, is called the numerator and the bottom number, 8, is called the denominator.

$\frac{3}{8}$ → Numerator → Denominator

2. As part of a set,

$\frac{5}{7}$

3. As a position on a number line,



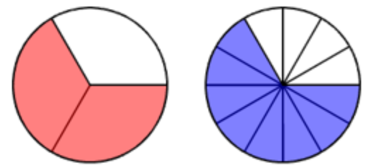
4. As a division of two whole numbers,

$\frac{3}{4}=3\div 4$ $\quad\frac{2}{7}=2\div 7$ $\quad\frac{12}{5}=12\div 5$

5. As a comparison between two quantities (ratio). For example, if the ratio of girls to boys in a class is 2:3 (for every 2 girls there are 3 boys), we may say that the number of girls is $\frac{2}{3}$ the number of boys or the number of boys is $\frac{3}{2}$ the number of girls.

Equivalent fractions

Two fractions are **equivalent** if they represent the same quantity or value but come from different *families* (fraction families have the same denominator). For example, the fractions $\frac{2}{3}$ and $\frac{8}{12}$ are equivalent since both represent the same portion of one whole, although they come from different families.



$\frac{2}{3}$ $\quad\frac{8}{12}$

Note that the value of a fraction is not changed if both the numerator and denominator are multiplied or divided by the same non-zero number.

$$\frac{2}{3}=\frac{2\times 4}{3\times 4}=\frac{8}{12} \qquad \frac{8}{12}=\frac{8\div 4}{12\div 4}=\frac{2}{3}$$

When division is used, this process is called reducing the fraction to its lowest terms.

Mixed numbers and improper fractions

A **proper fraction** is part of a whole. In this case, the numerator is of a lesser value than the denominator.

An **improper fraction** is more than a whole. The numerator, in this case, is of a greater value than the denominator.

Improper fractions can be written as a combination of whole numbers and fractions. In this form, they are called **mixed numbers**.

If both the numerator and the denominator are equal, the fraction is actually a whole or one. For example, $\frac{2}{2}, \frac{3}{3}$ and $\frac{5}{5}$ are all equivalent to1.

Example 1

Convert $3\frac{5}{6}$ to an improper fraction.

Solution

$$3\frac{5}{6}=3+\frac{5}{6}$$

$$=\frac{18}{6}+\frac{5}{6}=\frac{23}{6} \quad \text{3 wholes = 18 sixths}$$

So $3\frac{5}{6}=\frac{23}{6}$ as an improper fraction

Example 2

Convert $\frac{19}{8}$ to a mixed number.

Solution

$\frac{19}{8} = \frac{8}{8} + \frac{8}{8} + \frac{3}{8}$

So, $\frac{19}{8} = 2\frac{3}{8}$ as a mixed number

Addition and subtraction of fractions

Fractions can only be added or subtracted if their denominators are the same. When this is so, we simply add or subtract the numerators.

Same denominators

$a)\ \frac{4}{9}+\frac{3}{9}=\frac{7}{9}$ $\qquad$ $b)\ \frac{7}{12}-\frac{5}{12}=\frac{2}{12}=\frac{1}{6}$

$c)\ \frac{3}{5}+\frac{4}{5}+\frac{1}{5}=\frac{8}{5}=1\frac{3}{5}$ $\qquad$ $d)\ \frac{11}{15}-\frac{4}{15}=\frac{7}{15}$

Related denominators

If one denominator is a multiple of the other, then we must make change one so that both fractions have the same denominator.

a) $\frac{1}{3}+\frac{1}{6} = \frac{2}{6}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2}$ since $\frac{1}{3}=\frac{1\times 2}{3\times 2}=\frac{2}{6}$

b) $1\frac{1}{4}-\frac{5}{12}=\frac{5}{4}-\frac{5}{12}=\frac{15}{12}-\frac{5}{12}=\frac{10}{12}=\frac{5}{6}$ since $\frac{5}{4}=\frac{15}{12}$

Different denominators

If the denominators are unrelated, we must change both so that the denominators are the same.

Example 3

a) $\frac{2}{7}+\frac{1}{2}$ $\qquad$ b) $\frac{4}{5}-\frac{1}{4}$

Solution

We must convert both fractions to a common denominator. We chose 14 because it is the lowest common denominator (LCM) of 7 and 2.
Expressing both fractions in fourteenths, we have:

$\frac{2}{7}=\frac{4}{14} \qquad \frac{1}{2}=\frac{7}{14}$

a)

$$\frac{2}{7}+\frac{1}{2} = \frac{4}{14}+\frac{7}{14} = \frac{11}{14}$$

OR

$$\frac{2}{7}+\frac{1}{2} = \frac{4+7}{14} = \frac{11}{14}$$

b)

$$\frac{4}{5}-\frac{1}{4} = \frac{16}{20}-\frac{5}{20} = \frac{11}{20}$$

OR

$$\frac{4}{5}-\frac{1}{4} = \frac{16-5}{20} = \frac{11}{20}$$

Example 4

a) $3\frac{1}{3}-1\frac{1}{7}$ $\qquad$ b) $4\frac{1}{4}-2\frac{1}{3}$

Solution

a)

$$3\frac{1}{3}-1\frac{1}{7} = 2\frac{7-3}{21} = 2\frac{4}{21}$$

b)

$$4\frac{1}{4}-2\frac{1}{3} = 3\frac{5}{4}-2\frac{1}{3} = 1\frac{15-4}{12} = 1\frac{11}{12}$$

Multiplication of fractions

Earlier in this chapter, we saw that any fraction of the form $\frac{a}{b} = a \div b$. For example, $\frac{1}{2} = 1 \div 2$. Hence, multiplying by $\frac{1}{2}$ is the same as multiplying by 1 and dividing by 2.

$$3\times\frac{1}{2}=\frac{3\times 1}{2}=\frac{3}{2}=1\frac{1}{2}$$

$$\frac{3}{7}\times\frac{1}{2}=\frac{3\times 1}{7\times 2}=\frac{3}{14}$$

Example 5

a) $\frac{3}{4}\times\frac{5}{7}$ b) $\frac{5}{9}\times\frac{4}{11}$

Solution

a) $\frac{3}{4}\times\frac{5}{7}=\frac{3\times5}{4\times7}=\frac{15}{28}$

b) $\frac{5}{9}\times\frac{4}{11}=\frac{5\times4}{9\times11}=\frac{20}{99}$

Example 6

$1\frac{1}{8}\times3\frac{2}{3}$

Solution

Mixed numbers must be expressed as improper fractions when computing.

$1\frac{1}{8}\times3\frac{2}{3}=\frac{9}{8}\times\frac{11}{3}=\frac{99}{24}=4\frac{3}{24}=4\frac{1}{8}$

Division of fractions

In order to perform the operation of division on fractions, we must apply the inverse property, which connects multiplication and division.

$15\div3=15\times\frac{1}{3}$ $\frac{2}{3}\div5=\frac{2}{3}\times\frac{1}{5}=\frac{2}{15}$

Using the inverse property, we note that dividing by any number is equivalent to multiplying by its multiplicative inverse. When our divisors are fractions, the law still applies.

$30\div\frac{3}{5}=30\times\frac{5}{3}=50$ The multiplicative inverse of $\frac{5}{3}$ is $\frac{3}{5}$,

$75\div\frac{5}{9}=75\times\frac{9}{5}=135$ The multiplicative inverse of $\frac{5}{9}$ is $\frac{9}{5}$.

Example 7

a) $\frac{5}{7}\div\frac{3}{4}$ b) $3\frac{1}{5}\div1\frac{2}{3}$

Solution

a) $\frac{5}{7}\div\frac{3}{4}=\frac{5}{7}\times\frac{4}{3}=\frac{20}{21}$

b) $3\frac{1}{5}\div1\frac{2}{3}=\frac{16}{5}\div\frac{5}{3}=\frac{16}{5}\times\frac{3}{5}=\frac{48}{25}=1\frac{23}{25}$

Mixed operations involving fractions

When we are required to perform computations on fractions, involving more than one operation, we simplify operations within the brackets first.

Example 8

Calculate the exact value of $\left(\frac{19}{5}\div\frac{6}{5}\right)-1\frac{2}{3}$.

Solution

$=\left(\frac{19}{5}\div\frac{6}{5}\right)-1\frac{2}{3}$

$=\left(\frac{19}{5}\times\frac{5}{6}\right)-1\frac{2}{3}$

$=\frac{19}{6}-1\frac{2}{3}=3\frac{1}{6}-1\frac{2}{3}$

$=2\frac{7}{6}-1\frac{4}{6}=1\frac{3}{6}$

$=1\frac{1}{2}$

Example 9

Calculate the exact value of $3\frac{1}{2}+\left(\frac{15}{7}\times\frac{4}{5}\right)$.

Solution

$3\frac{1}{2}+\left(\frac{15}{7}\times\frac{4}{5}\right)=3\frac{1}{2}+\frac{12}{7}=3\frac{1}{2}+1\frac{5}{7}$

$=3\frac{7}{14}+1\frac{10}{14}=5\frac{3}{14}$

Example 10

Calculate the exact value of $\frac{2\frac{2}{3}+1\frac{3}{5}}{4\frac{2}{3}}$.

Solution

$\frac{2\frac{2}{3}+1\frac{3}{5}}{4\frac{2}{3}}=\frac{2\frac{10}{15}+1\frac{9}{15}}{4\frac{2}{3}}$

$=\frac{4\frac{4}{15}}{4\frac{2}{3}}=\frac{64}{15}\div\frac{14}{3}$

$=\frac{64}{15}\times\frac{3}{14}$

$=\frac{32}{35}$

DECIMALS

Decimals are convenient forms of expressing fractions without a numerator or denominator. They are also known as base ten fractions, since they have denominators that are multiples of ten, such as10, 100, 1000 etc.

Expanded Notation

The place value of decimal numbers follows the same pattern as whole numbers, decreasing in powers of ten as one moves to the right of the decimal point.

$$1.024=(1\times 1)+\left(0\times\frac{1}{10}\right)+\left(2\times\frac{1}{100}\right)+\left(4\times\frac{1}{1000}\right)$$

$$34.56=(3\times 10)+(4\times 1)+\left(5\times\frac{1}{10}\right)+\left(6\times\frac{1}{100}\right)$$

Expressing decimals as common fractions

Since decimals are base ten fractions, we can express them as common fractions first by writing them in expanded notation. We then add the base ten fractions to obtain a single fraction.

Example 11

Express as a fraction in its lowest terms
a) 0.58 b) 0.175 c) 4.08

Solution

a) $0.58=\frac{5}{10}+\frac{8}{100}=\frac{58}{100}=\frac{29}{50}$

b) $0.175=\frac{1}{10}+\frac{7}{100}+\frac{5}{1000}=\frac{175}{1000}=\frac{35}{200}=\frac{7}{40}$

c) $4.08=4+\frac{0}{10}+\frac{8}{100}=4\frac{8}{100}=4\frac{2}{25}$

Expressing common fractions as decimals

Common fractions can have denominators of any value except zero. If their denominators are easily expressed as a power of ten, then they are in a form that allows ready conversion to base ten fractions.

Example 12

Express as decimals: $\frac{61}{100},\frac{2}{5},\frac{3}{8}$.

Solution

$\frac{61}{100}=\frac{6}{10}+\frac{1}{100}=0.61$ $\frac{2}{5}=\frac{4}{10}=0.4$

$\frac{3}{8}=\frac{375}{1000}=0.375$

In the examples above, the decimal equivalent is an exact value of the fraction. We refer to these as ***terminating*** decimals. Not all fractions can be expressed as terminating decimals.

Example 13

Express as decimals: a) $\frac{2}{3}$ b) $\frac{5}{7}$

Solution

a) It is not possible to express the denominator of these fractions as powers of ten. So, we must use another strategy.

Recall, $\frac{2}{3}=2\div 3$. Hence, $\frac{2}{3}=0.66666...=0.\dot{6}$

A single dot, written over the 6 indicates that the digit 6 is being repeated indefinitely (ad infinitum).

b) We can use the division meaning of fractions to obtain:

$\frac{5}{7}=5\div 7$

$=0.714285\ 714285\ 714285...=0.\dot{7}1428\dot{5}$

When more than two digits recur, the dots are placed on the first and last digits in the 'string'.

In the above examples, the decimal equivalents are not the exact values of the fractions, although they are all close to the exact values. We refer to these as ***recurring*** decimals.

Mixed operations involving decimals

In performing operations on decimals where more than one operation is involved, we simplify operations within the brackets first.

Roots and Powers

In performing mixed operations on numbers, we are sometimes required to evaluate square roots, cube roots and numbers with powers. These computations can be performed with a calculator but it is important to interpret the meaning of roots and powers.

A number raised to the power 2 is the same as multiplying the number by itself twice or squaring the number. For example,

$15^2 = 15 \times 15$ $\quad$ $2.3^2 = 2.3 \times 2.3$

A number raised to the power 3 is the same as multiplying the number by itself three times or cubing the number. For example,

$15^3 = 15 \times 15 \times 15$ $\quad$ $2.3^3 = 2.3 \times 2.3 \times 2.3$

The square root of a number is that number when multiplied by itself gives the number. For example,

$$4 \times 4 = 16 \quad \text{Hence, } \sqrt{16} = 4$$
$$1.2 \times 1.2 = 1.44 \quad \text{Hence, } \sqrt{1.44} = 1.2$$

The cube root of a number is that number when multiplied by itself three times gives the number. For example,

$$4 \times 4 \times 4 = 64 \quad \text{Hence, } \sqrt[3]{64} = 4$$
$$1.2 \times 1.2 \times 1.2 = 1.728 \quad \text{Hence, } \sqrt[3]{1.728} = 1.2$$

Note that for cube roots we must insert a 3 at the left of the root sign. If we were interested in the fourth root we would insert a 4 in the same position. For square roots only, it is not necessary to insert the 2.

Example 14

Calculate the exact value of
$8.64\,(3.27-1.12) + (1.2)^2$

Solution

$8.64\,(3.27-1.12) + (1.2)^2$
$= 8.64\,(2.15) + 1.44$
$= 18.576 + 1.44$
$= 20.016$

Example 15

Calculate the exact value of
$\dfrac{2.34+1.08}{0.65}$.

Solution

Calculate the exact value of
$$\frac{2.34+1.08}{0.65} = \frac{3.42}{0.65} = 5.26$$

Example 16

Calculate the exact value of
$\dfrac{79.38}{6.3\times 4.2} + \sqrt{12.96}$.

Solution

$$\frac{79.38}{6.3\times 4.2} + \sqrt{12.96}$$
$$= \frac{79.38}{26.46} + 3.6$$
$$= 3 + 3.6$$
$$= 6.6$$

APPROXIMATIONS

When working with very large and very small numbers, we may wish to round off the number to a given number of significant figures, decimal places, or write the number using scientific notation (standard form). Some general rules to follow in performing approximations are:

1. Ensure that the place value of the digits in the new number remains unchanged.
2. Decide which is the target digit – if we are approximating to the nearest ten, the target digit is in the tens position.
3. Round up when the digit on the immediate right of the target digit is 5, 6, 7, 8 or 9. Round down when it is 0, 1, 2, 3 or 4.

In rounding 5 362 to the nearest hundred, the target digit is 3. The digit to the immediate right of 3 is 6 and is the deciding digit. Since 6 belongs to the set {5, 6, 7, 8, 9} we round up to 5 340.

Decimal Places

When approximating a number to a given number of decimal places, we use the decimal point as the starting point and count to the right of the point.

15.6574, correct to three decimal places is 15.657
0.0756, correct to two decimal places is 0.08
1.2365, correct to one decimal place is 1.2

Example 17

Simplify

$$(4.14 \div 5.75) + (1.62)^2$$

stating your answer correct to two decimal places.

Solution

$(4.14 \div 5.75) + (1.62)^2$
$= 0.72 + 2.6244$
$= 3.3444$
$= 3.34$ (correct to 2 decimal places)

Standard forms (scientific notation)

In writing a number in standard form or scientific notation, we represent it using index notation, $A \times 10^n$, where n is an integer and $1 \leq A < 10$

For example, to write the number 256 in standard form, we place a decimal point so that its value lies between 1 and 10. The value of A will now be 2.56. However, this is not our original number and we must multiply it by 100 or 10^2 to obtain its original value.
Hence, $256 = 2.56 \times 10^2$, which is in standard form.

In the examples below, note that when A is larger than the original number, we must divide by powers of 10 to obtain the original value and so the power of 10 is negative in such cases.

$67 = 6.7 \times 10^1$
$458 = 4.58 \times 10^2$
$0.00568 = 5.68 \times 10^{-3}$
$0.0124 = 1.24 \times 10^{-2}$

Example 18

Simplify

$$(12.8)^2 - (30 \div 0.375)$$

stating your answer in standard form.

Solution

$(12.8)^2 - (30 \div 0.375)$
$= 163.84 - 80$
$= 83.84$
$= 8.38 \times 10$ (in standard form)

Significant figures

The first non-zero digit in a number, reading from left to right is the first significant figure. It is the digit with the largest place value and so gives the best indication of the size of the number.

Large Numbers

6 843, correct to three significant figures is 6 840. (Round down)
6 843, correct to two significant figures is 6 800.
6 843, correct to one significant figure is 7 000.

Small Numbers

0.01728, correct to three significant figures is 0.0173.
0.01728, correct to two significant figures is 0.017.
0.01728, correct to one significant figure is 0.02.

Example 19

Simplify
$(2.56 + 0.65) + 0.451^2$, giving your answer correct to 3 significant figures.

Solution

$(2.56 + 0.65) + 0.451^2$
$= 3.21 + 0.203401$
$= 3.413401$
$= 3.41$ (correct to 3 significant figures)