FAS-PASS
Maths

CSEC MATHEMATICS JANUARY 2023 PAPER 2

SECTION I

Answer ALL questions.

All working MUST be clearly shown.

1. (a) (i) By rounding each number in the expression below to **one significant figure**, estimate the value of

$$\frac{\sqrt{108}}{19.72+5.296}$$

SOLUTION:

Required to estimate: The value of $\frac{\sqrt{108}}{19.72+5.296}$ by rounding each number to one significant figure.

Solution:

Written to one significant figure $108=100$

$19.72=20$

and $5.296=5$

Hence, $\frac{\sqrt{108}}{19.72+5.296}\approx\frac{\sqrt{100}}{20+5}$

Taking the positive root of 100 to be 10, the computation reduces to

$$\approx\frac{10}{25}$$

$$\approx\frac{2}{5}$$

(ii) Find the EXACT value of

$$3\frac{3}{8}\div\left(\frac{5}{12}+\frac{1}{3}\right)$$

Give your answer as a mixed number in its **simplest** form.

SOLUTION:

Required to find: The exact value of $3\frac{3}{8}\div\left(\frac{5}{12}+\frac{1}{3}\right)$ expressing the answer as a mixed number in its simplest form.

Solution: Working the part within the brackets first we get

$$\begin{aligned}\frac{5}{12}+\frac{1}{3}&=\frac{5}{12}+\frac{4}{12}\\&=\frac{9}{12}\\&=\frac{3}{4}\end{aligned}$$

Hence,
$$\begin{aligned}3\frac{3}{8}\div\left(\frac{5}{12}+\frac{1}{3}\right)&=\frac{27}{8}\div\frac{3}{4}\\&=\frac{\cancel{27}^{9}}{{}_{2}\cancel{8}}\times\frac{\cancel{4}^{1}}{\cancel{3}_{1}}\\&=\frac{9}{2}\\&=4\frac{1}{2}\text{(as a mixed number in exact form)}\end{aligned}$$

(b) Due to the COVID-19 pandemic the number of available seats in a hall was reduced from 125 to 93. Calculate the percentage decrease in the number of available seats.

SOLUTION:
Data: Due to the COVID-19 pandemic, the number of seats available in a hall was reduced from 125 to 93.
Required to calculate: The percentage decrease in the number of available seats.
Calculation:
The decrease in the number of seats $=125-93$
$=32$

Hence, the percentage decrease in the number of seats

$$\begin{aligned}&=\frac{\text{Decrease}}{\text{Original number of seats}}\times 100\\&=\frac{32}{125}\times 100\\&=25.6\%\end{aligned}$$

(c) Mica invests a certain amount of money in a bank that pays compound interest at a rate of 2.5% per annum. At the end of 2 years, the value of her investment is \$7 564.50.

Calculate the amount Mica invests.

$$\left[\begin{array}{l}\text{Compound interest}: A = P\left(1+\frac{r}{100}\right)^n, \text{ where } A = \text{total amount after } n \text{ years,}\\ P = \text{principal or original value,}\\ r = \text{rate of interest per annum, } n = \text{number of years the money is invested}\end{array}\right]$$

SOLUTION:
Data: Mica invests a certain amount of money in a bank that pays compound interest at a rate of 2.5% per annum, which amounts to \$7 564.50 after 2 years.
Required to calculate: The amount of money Mica invested.
Calculation:
According to the formula in the data, we substitute to get

$$7\ 564.50 = P\left(1+\frac{2.5}{100}\right)^2$$

$$7\ 564.50 = P(1.025)^2$$

$$P = \frac{7\ 564.50}{(1.025)^2}$$

$$P = \frac{7\ 564.50}{1.025\times 1.025}$$

$$= 7\ 200$$

Hence, the amount of money Mica invested, *P*, is \$7 200.00.

2. (a) Simplify

(i) $\left(x^3\right)^2$

SOLUTION:
Required to simplify: $\left(x^3\right)^2$
Solution:
Recall $(a^m)^n = a^{mn}$
So,

$$\left(x^3\right)^2 = x^{3\times 2}$$

$$= x^6$$

(ii) $y^8 \div y^{-5}$

SOLUTION:
Required to simplify: $y^8 \div y^{-5}$

Solution:

Recall $a^m \div a^n = a^{m-n}$

So, $y^8 \div y^{-5} = y^{8-(-5)}$
$= y^{13}$

(b) Factorise:

(i) **a)** $xy - y^2$

SOLUTION:

Required to factorise: $xy - y^2$

Solution:

$$xy - y^2 = x\underline{\underline{y}} - y.\underline{\underline{y}}$$
$$= y(x-y)$$

b) $x^2 - y^2$

SOLUTION:

Required to factorise: $x^2 - y^2$

Solution:

$$x^2 - y^2 = (x)^2 - (y)^2$$

This is in the form of 'a difference of two squares.'

Hence, $x^2 - y^2 = (x-y)(x+y)$

(ii) Hence, simplify the expression

$$\frac{xy - y^2}{x^2 - y^2}$$

SOLUTION:

Required to simplify: $\dfrac{xy - y^2}{x^2 - y^2}$

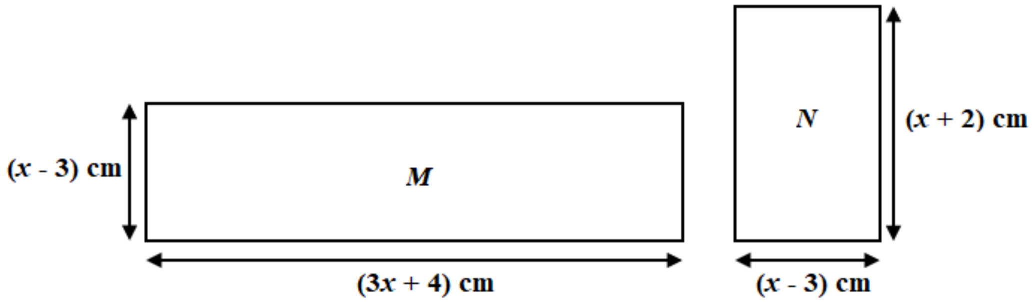
Solution:

Recall the factorisations in (i) a) and b), we obtain

$$\frac{xy - y^2}{x^2 - y^2} = \frac{y\cancel{(x-y)}}{\cancel{(x-y)}(x+y)}$$
$$= \frac{y}{x+y}$$

(in its simplest form)

(c) The diagram below shows 2 rectangles, *M* and *N*, with their dimensions expressed in terms of x.



Given that the difference between the areas of the two rectangles is 64 cm^2, show that $x^2-2x-35=0$.

SOLUTION:
Data: Diagrams showing two rectangles *M* and *N* whose difference in area is 64 cm^2.
Required to show: $x^2-2x-35=0$
Proof:
Area of a rectangle = Length × Width
Area of $M=(x-3)(3x+4)\text{ cm}^2$
Area of $N=(x+2)(x-3)\text{ cm}^2$

Both rectangles have the same width. However, the length of *M*, $(3x+4)$ is greater than the length of *N*, $(x+2)$. So, *M* has a greater length than *N* and consequently a greater area.

Hence, the positive difference in the area will be: Area of *M* – Area of *N*
So, $(x-3)(3x+4)-(x+2)(x-3)=64$

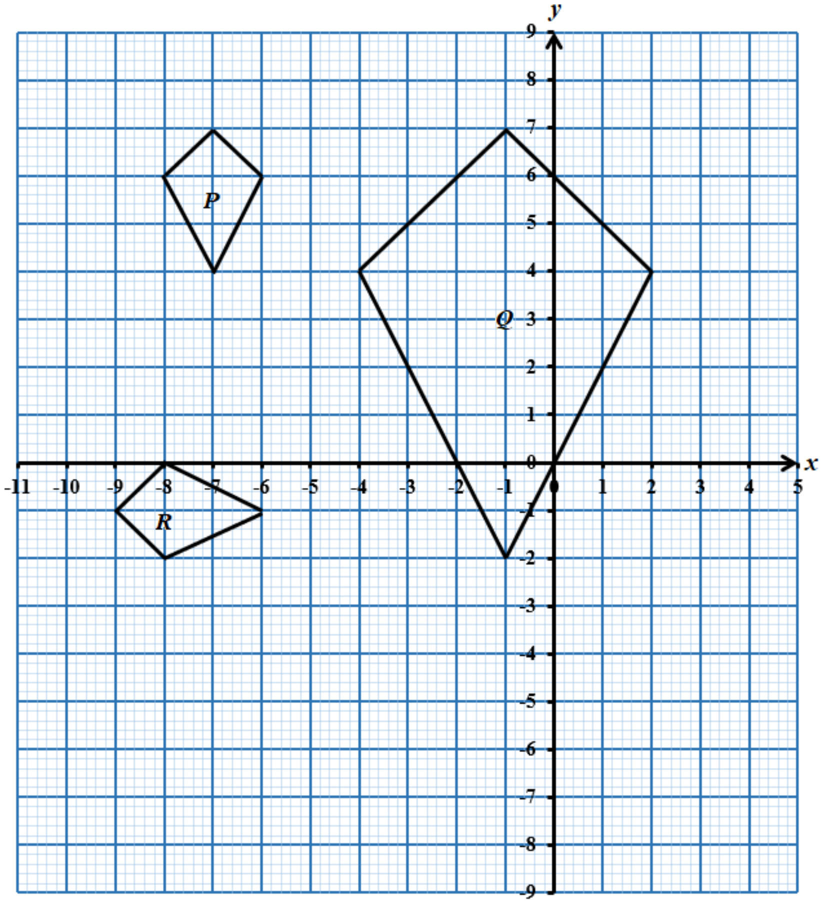
$$(3x^2-9x+4x-12)-(x^2+2x-3x-6)-64=0$$
$$(3x^2-5x-12)-(x^2-x-6)-64=0$$
$$2x^2-4x-6-64=0$$
$$2x^2-4x-70=0$$

$(\div 2)$

$$x^2-2x-35=0$$

Q.E.D.

3. The following diagram shows 3 quadrilaterals, *P*, *Q* and *R* on a square grid. *Q* and *R* are the images of *P* after it underwent 2 different transformations.



(a) On the grid, draw the image of quadrilateral *P* after a

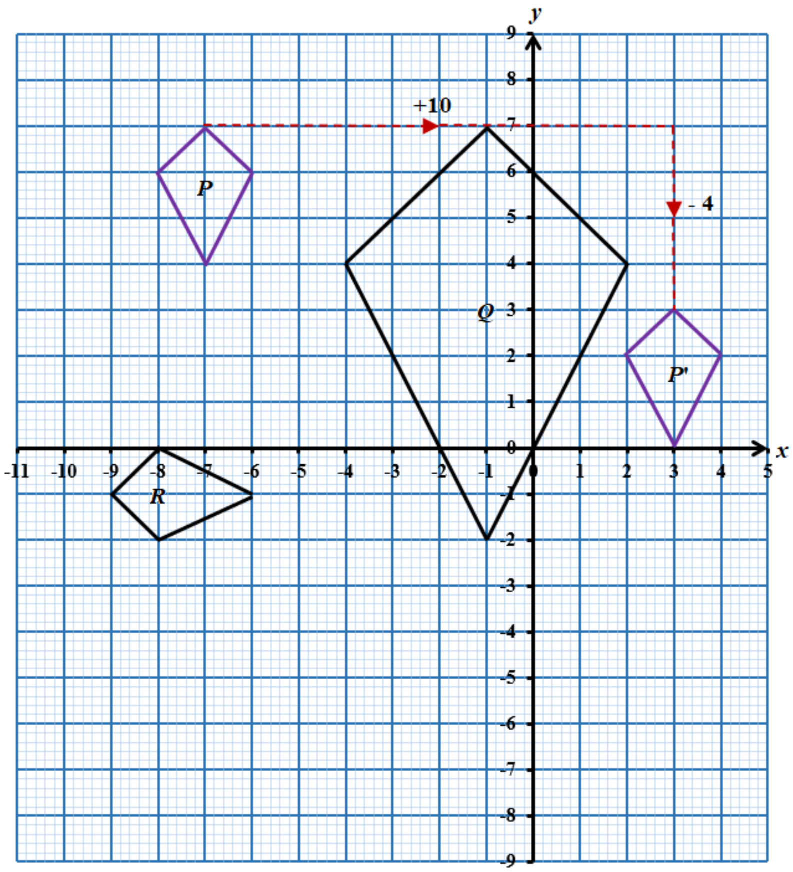
(i) translation by the vector $\begin{pmatrix} 10 \\ -4 \end{pmatrix}$. Label this image P'.

SOLUTION:
Data: Diagram showing three quadrilaterals *P*, *Q* and *R* on a square grid, where *Q* and *R* are the images of *P* after it underwent two different transformations.

Required to draw: P', the image of *P* after a translation by vector $\begin{pmatrix} 10 \\ -4 \end{pmatrix}$.

Solution:

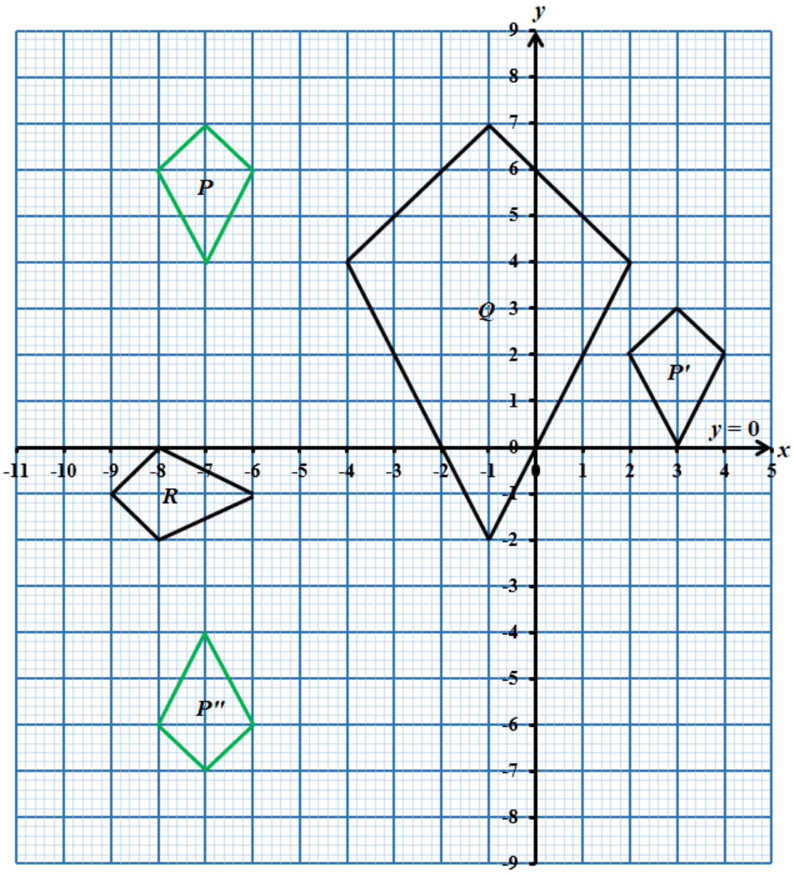
The translation vector $\begin{pmatrix} 10 \\ -4 \end{pmatrix}$ represents a movement or shift of 10 units horizontally to the right followed by 4 units vertically downward. (Note: The reverse order would produce the same resulting image)



Each vertex of *P* is shifted by the translation $\begin{pmatrix} 10 \\ -4 \end{pmatrix}$ and connected to form the image P'.

(ii) reflection in the line $y = 0$. Label this image P''.

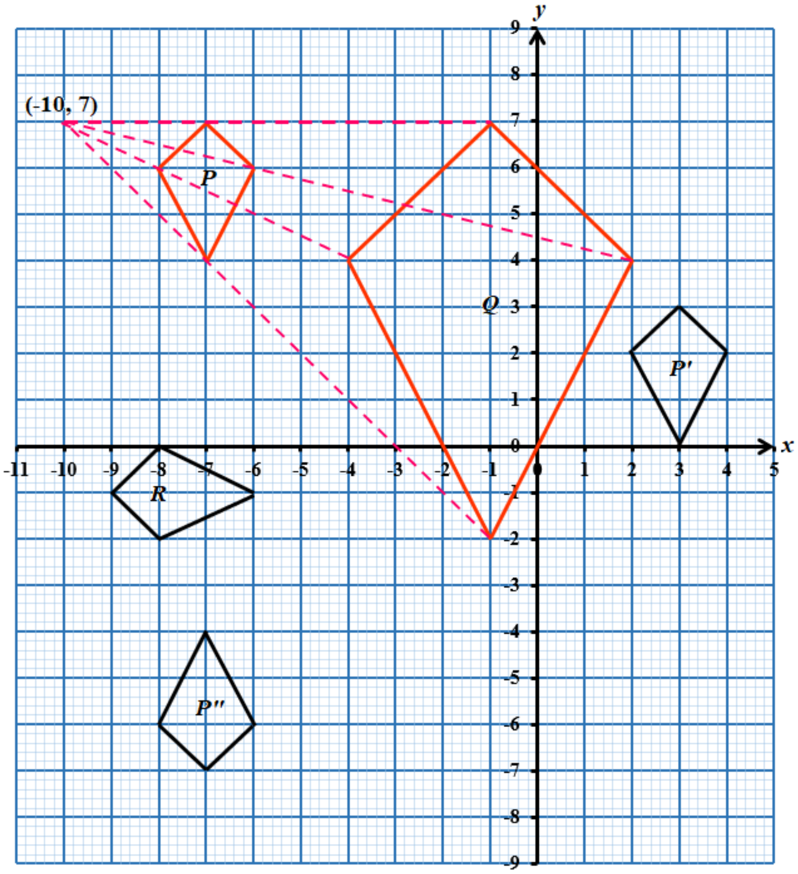
SOLUTION:
Required to draw: P'', the image of *P* after a reflection in the line $y = 0$, which is the *x*-axis.
Solution:
The image, P'', is the same perpendicular distance on the opposite side of the reflection line as the object, *P*. We obtain the image by checking the squares on either side of the reflection line. Note the image is laterally inverted or flipped, with respect to the object.



(b) Describe fully a single transformation that maps Quadrilateral *P* onto

(i) Quadrilateral *Q*

SOLUTION:
Required to describe: A single transformation that maps Quadrilateral *P* onto Quadrilateral *Q*.
Solution:
The image, *Q,* is similar (equiangular) to object *P*. The difference in size indicates that the transformation is an enlargement. We need to determine the centre of enlargement and the scale factor of the enlargement.



Straight lines are drawn from the vertices of *Q* (image) to the corresponding vertices of *P* (object) and produced backwards. They all meet at $(-10, 7)$ and which will be the centre of enlargement.

We choose any (Image length) ÷ (The corresponding object length) to obtain the scale factor.

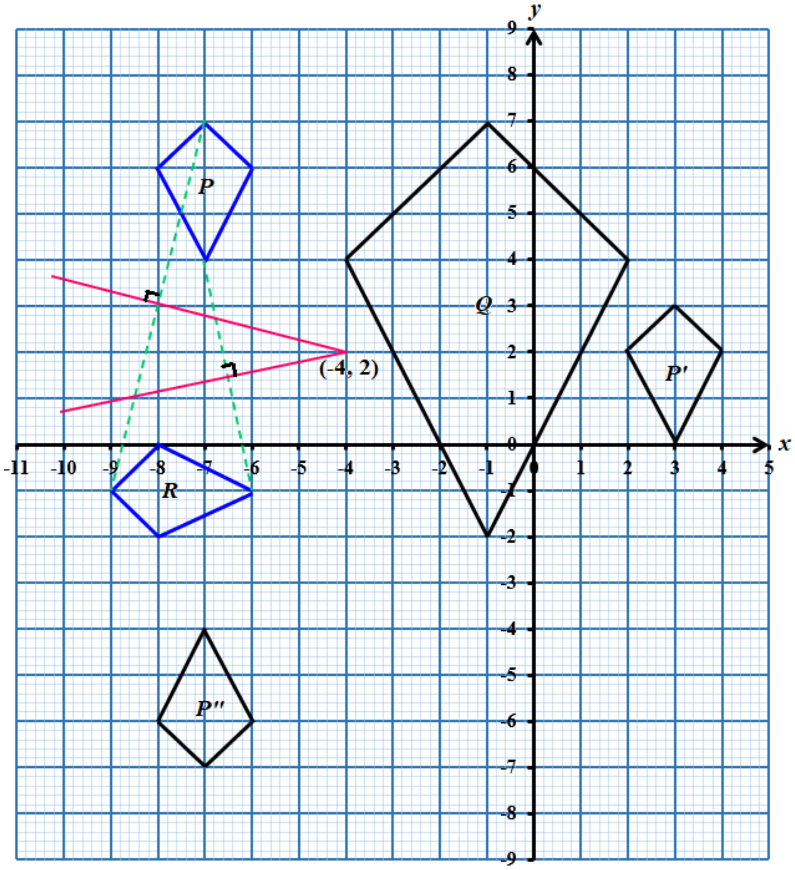
$$\frac{\text{Length of the leading diagonal of } Q}{\text{Length of the leading diagonal of } P} = \frac{9}{3}$$

$$= 3$$

Hence, *P* is mapped onto *Q* by an enlargement with centre $(-10, 7)$ and scale factor 3.

(ii) Quadrilateral *R*

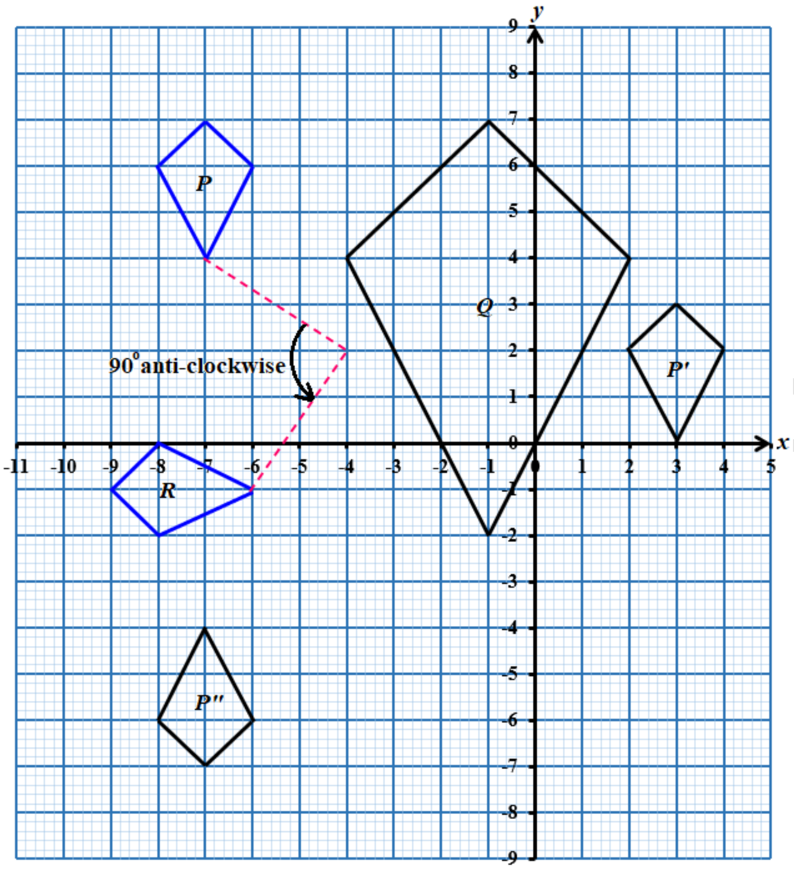
SOLUTION:
Required to describe: A single transformation that maps Quadrilateral *P* onto Quadrilateral *R*
Solution:



P and *R* are congruent and *R* is re-oriented with respect to *P*. Hence, the transformation that maps *P* onto *R* is a rotation.

Locating the centre of rotation:

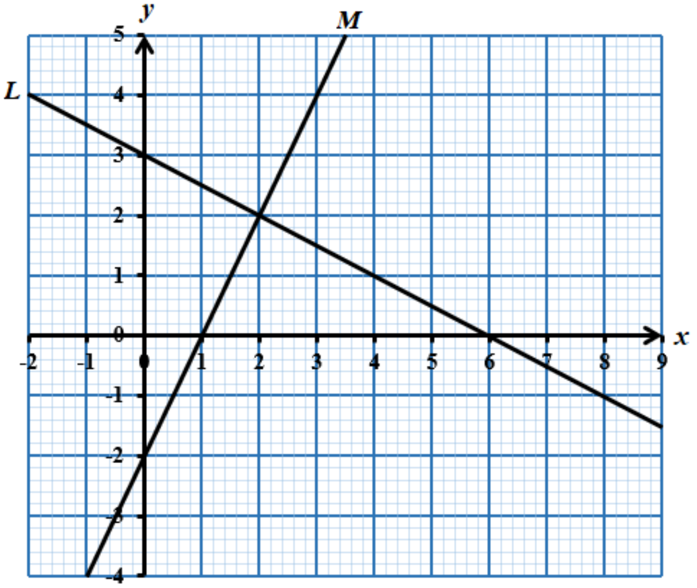
From the object, *P*, we join two object points to their corresponding image points on *R* and construct their perpendicular bisectors. The perpendicular bisectors meet at $(-4, 2)$.
(If this was done with a third set of object-image points, the perpendicular bisector would have passed through the same point (-4, 2). Hence, a third perpendicular bisector is not necessary.)



By observing the angle made from the object to the centre of rotation and them to the image, we can deduce the size of the angle and the direction of the rotation.
The angle of rotation is 90° anti-clockwise.

Hence, P is mapped onto R by a rotation of 90° anti-clockwise about the point $(-4, 2)$.

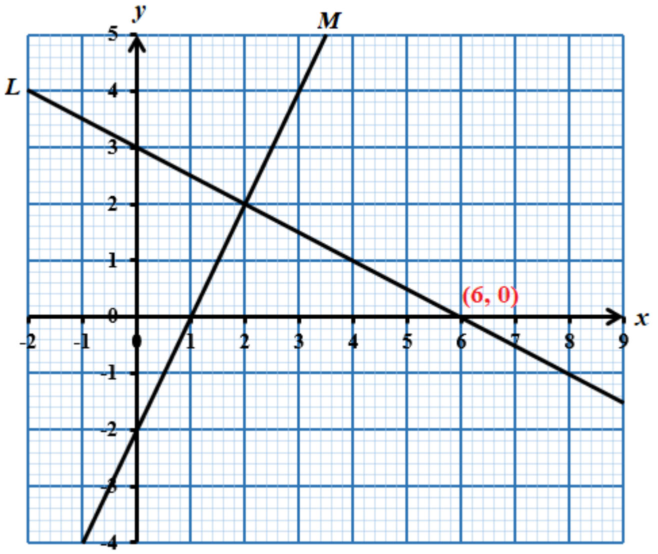
4. Lines L and M are drawn on the square grid below.



(a) Write down the coordinates of the

(i) x-intercept of the Line L

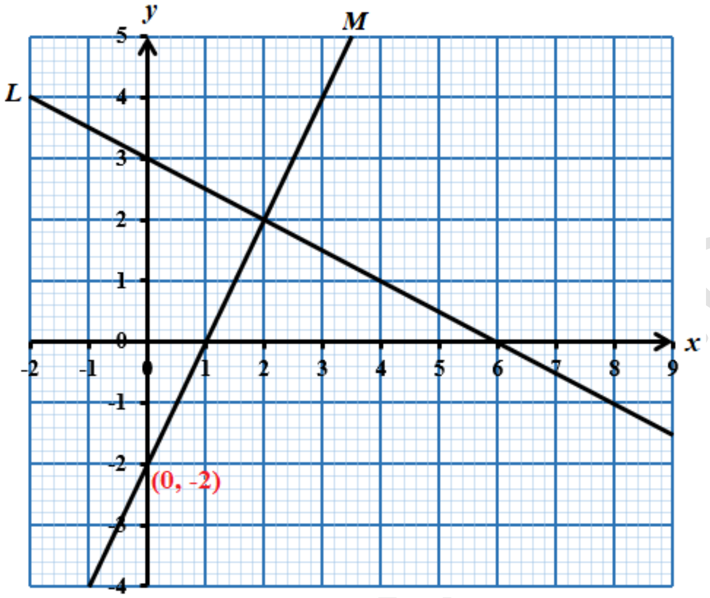
SOLUTION:
Data: Diagram showing two straight lines L and M on a square grid.
Required to write: The coordinates of the x-intercept of Line L
Solution:



By a read-off, the coordinates of the x–intercept of L is $(6, 0)$.

(ii) *y*-intercept of Line *M*

SOLUTION:
Required to write: The coordinates of the *y*-intercept of line *M*
Solution:



By a read-off, the coordinates of the *y*–intercept of Line *M* is $(0, -2)$.

(b) The equation of Line *L* is $x+2y-6=0$. Find the value of *k* given that the point $(9, k)$ lies on Line *L*.

SOLUTION:
Data: Line *L* has equation $x+2y-6=0$ and the point $(9, k)$ lies on it.
Required to find: The value of *k*
Solution:
The equation of *L* is $x+2y-6=0$.
$(9, k)$ lies on *L*.
So, $y = k$ when $x = 9$

$$\therefore 9+2(k)-6=0$$

$$2k=-3$$

$$k=-1\frac{1}{2}$$

(c) Find the equation of Line *M* in the form $y = mx + c$.

SOLUTION:
Required to find: The equation of Line *M* in the form $y = mx + c$.
Solution:
Choosing any two points on the line *M*: $(0, -2)$ and $(2, 2)$

Gradient of *M* $= \dfrac{2-(-2)}{2-0} = 2$

The general equation of a straight line is $y = mx + c$ where *m* is the gradient and *c* is the intercept on the *y* or vertical axis.
M has a gradient of 2 and cuts the *y*-axis at –2. So we can write down the equation of *M*.

$\therefore$The equation of Line *M* is $y = 2x - 2$ and which is of the form $y = mx + c$, where $m = 2$ and $c = -2$.

(d) Show by calculation, that Line *L* and Line *M* are perpendicular.

SOLUTION:
Required to show: The Line *L* and Line *M* are perpendicular by calculation.
Proof:
Equation of Line *L* is $x + 2y - 6 = 0$
We re-arrange the equation of *L* to express it in the form $y = mx + c$

$$2y = -x + 6$$

$y = -\frac{1}{2}x + 3$ is of the form $y = mx + c$ where

$m = -\frac{1}{2}$ is the gradient of *L*.

Gradient of the line *M* = 2

$$\text{Gradient of } L \times \text{Gradient of } M = -\frac{1}{2} \times 2$$
$$= -1$$

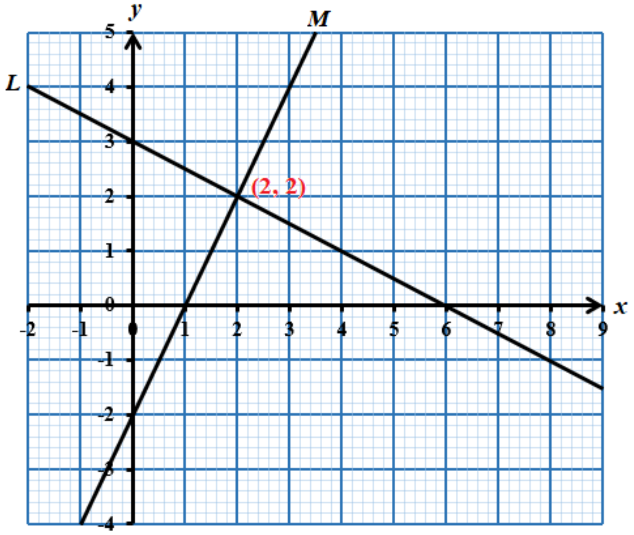
Hence, *L* and *M* are perpendicular to each other (the product of the gradients of perpendicular lines is equal to -1).

Q.E.D

(e) Line *L* and Line *M* represent the graph of a pair of simultaneous equations. Using the graph, write down the solution to the pair of simultaneous equations.

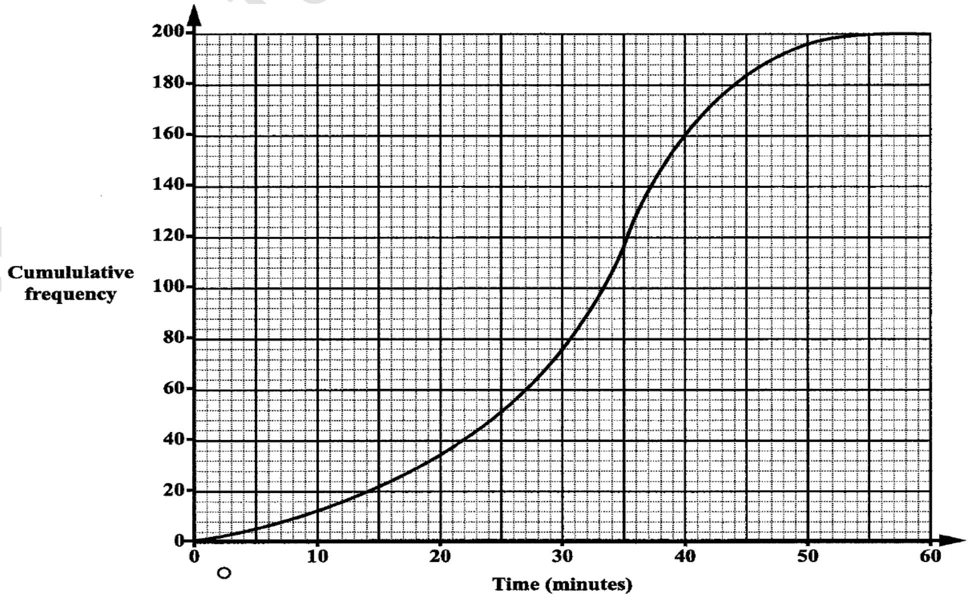
SOLUTION:

Data: Line *L* and Line *M* represent the graph of a pair of simultaneous equations.
Required to write: The solution to the pair of simultaneous equations using the graph given.
Solution:
The solution of the pair of simultaneous equations is the point of intersection when both graphs are drawn on the same pair of axes.



In this case, the graphs of *L* and *M* intersect at $(2,\ 2)$.
Hence, $x=2$ (the *x*-coordinate) and $y=2$ (the *y*-coordinate) f the point of intersection..

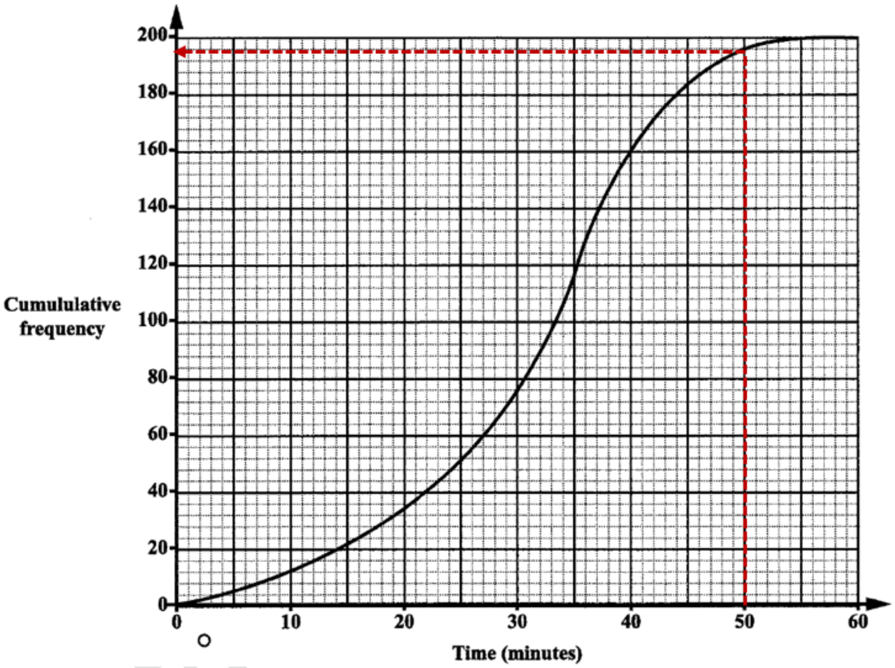
5. The cumulative frequency curve below shows information about the times taken by 200 students to solve a Mathematics Olympiad problem.



(a) Using the cumulative frequency curve shown above, find an estimate for the

(i) number of students who took more than 50 minutes to solve the problem

SOLUTION:
Data: Cumulative frequency curve showing the times taken by 200 students to solve a Mathematics Olympiad problem.
Required to estimate: The number of students who took more than 50 minutes to solve the problem
Solution:



We draw a vertical at 50 minutes to meet the curve. At the point where the vertical meets the curve, we draw a horizontal.
The number of students who took 50 minutes or less to solve the problem = 196 (read off)
Hence, the number who took more than 50 minutes $= 200 - 196$

$$= 4 \text{ students}$$

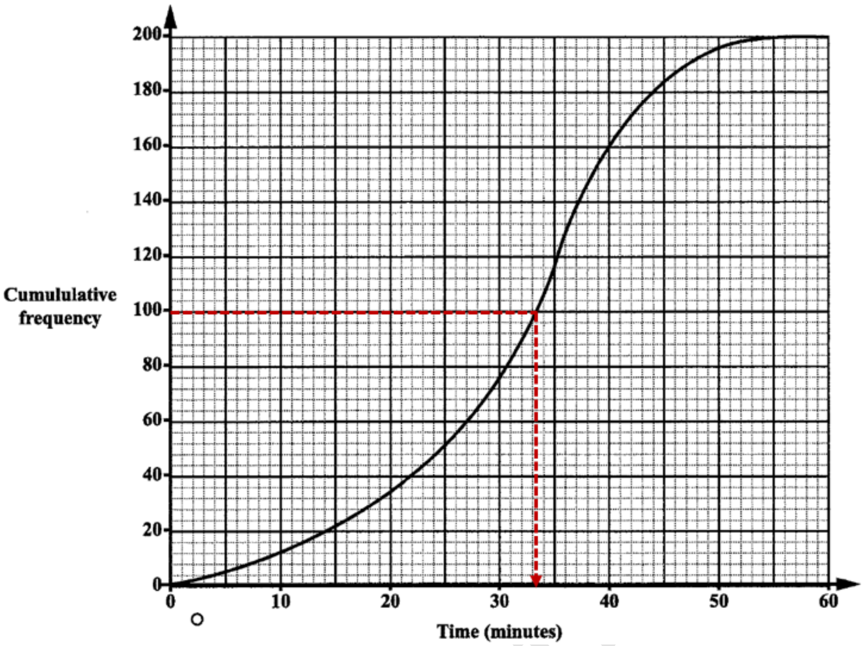
(ii) median time taken to solve the problem

SOLUTION:
Required to estimate: The median time taken to solve the problem.
Solution:

$$\text{Position of median } = \frac{1}{2}n^{\text{th}} \text{ term(where } n \text{ is the total number of values)}$$

$$= \frac{1}{2}(200)^{\text{th}} \text{ term}$$

$$= 100^{\text{th}} \text{ term}$$



We draw the horizontal from 100 to meet the curve. From the point where the horizontal meets the curve, we drop a vertical to obtain the median time.

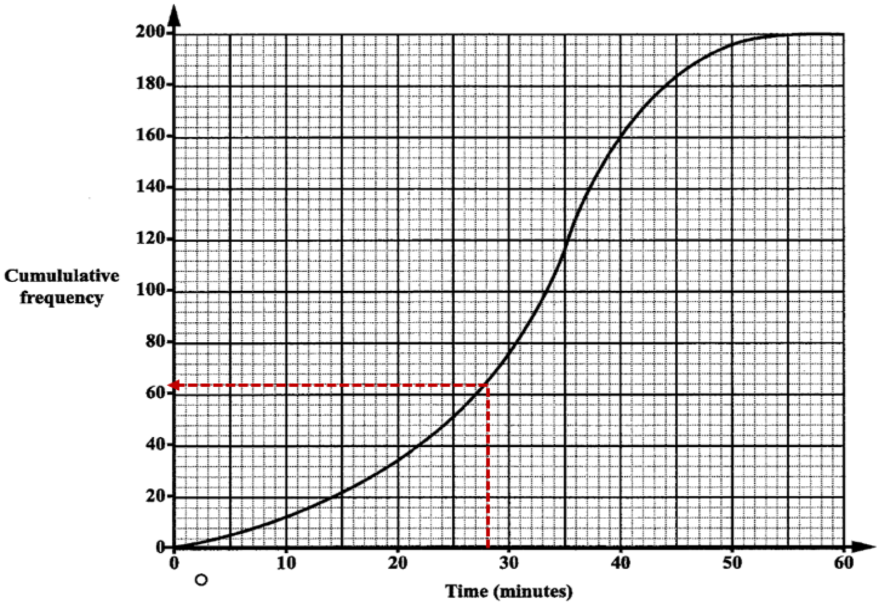
Median time taken to solve the problem $= 33.5$ minutes

(iii) probability that a student chosen at random takes at most 28 minutes to solve the problem.

SOLUTION:

Required to estimate: The probability that a student chosen at random spent at most 28 minutes to solve the problem.

Solution:



We draw the vertical from $t = 28$ to meet the curve. From the point where the vertical meets the curve, we draw the horizontal to find the number of students who took at most 28 minutes (which is 28 minutes or less).

Number of students taking at most 28 minutes $= 64$ students (read off)

$$P(\text{students take at most 28 minutes})$$

$$= \frac{\text{Number of students taking at most 28 minutes to solve the problem}}{\text{Total number of students}}$$

$$= \frac{64}{200}$$

$$= \frac{8}{25}$$

(b) **(i)** Using the cumulative frequency curve, complete the table below.

Time (minutes)	**Midpoint (x)**	**Frequency (f)**	**Frequency × Midpoint (fx)**
1 – 10	5.5	12	66
11 – 20	15.5		
21 – 30	25.5	42	1071
31 – 40	35.5	84	2982
41 – 50	45.5		
51 – 60	55.5	4	222

SOLUTION:

Data: Incomplete table showing time (minutes), midpoint (x), frequency (f) and frequency $\times$midpoint (fx)

Required to complete: The table by using the cumulative frequency curve

Solution:

We first complete the **Frequency** Column

The total number of students in the sample = 200

The number of students scoring less than 40 = 196 (Read-off from curve)

The number of students scoring less than 50 = 160 (Read-off from curve)

Therefore the number of students scoring between 41-50 = 196 – 160 = 36

To calculate the number of students scoring between 11-20, we can add all the frequencies and subtract this total from 200.

The number of students scoring between 11-20
= 200 – (12+42+84+36+4)
= 22
[Note this method eliminates errors due to scale reading]

We now complete the **Frequency** ×**Midpoint** Column as shown below

Time (minutes)	**Midpoint (x)**	**Frequency (f)**	**Frequency × Midpoint (fx)**
1 – 10	5.5	12	66
11 – 20	15.5	**22**	**22 x 15.5 = 341**
21 – 30	25.5	42	1071
31 – 40	35.5	84	2982
41 – 50	45.5	**36**	**36 x 45.5 = 1638**
51 – 60	55.5	4	222

(ii) Use the information in the completed table above to calculate an estimate of the average time taken by the students to solve the problem.

SOLUTION:
Required to calculate: An estimate of the average time taken by students to solve the problem, using the completed table.
Calculation:

The values of the frequency for the classes 11-20 and 41-50 are read off from the graph

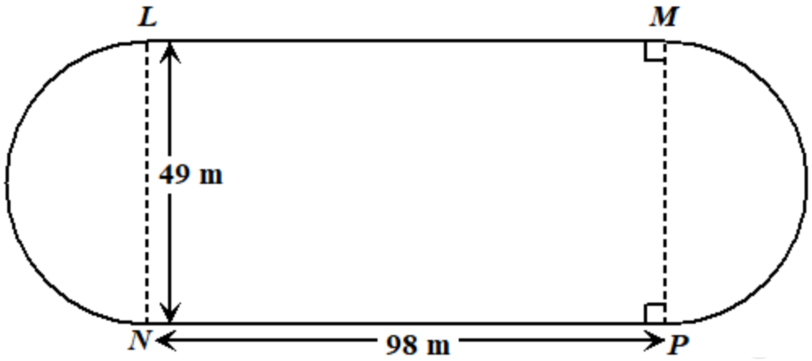
$$\text{Average time} = \frac{\sum fx}{\sum f}$$

= (66 + 341+1071+ 2982+1638+222) ÷ (12+22+42+84+36+4)

$$= \frac{6320}{200}$$

$$= 31.6 \text{ minutes}$$

6. The diagram below shows a scaled drawing of a running track. It consists of a rectangle and two semi-circles with diameters *LN* and *MP*. $MP = LN = 49$m and $LM = NP = 95$m.



(a) **(i)** Show that the TOTAL length of the running track is 350 m.

SOLUTION:
Data: Diagram showing a scaled drawing of a running track consisting of a rectangle and two semi-circles with diameters *LN* and *MP* where $MP = LN = 49$m and $LM = NP = 98\ m$.
Required to show: The total length of the running track is 350 m
Proof:
The semi-circular ends of the track are calculated using the formula:
Perimeter of a semi-circle

$$= \frac{1}{2}\ Circumference\ of\ circle$$
$$= \frac{1}{2}\ Diameter\ \times \pi$$

$$\begin{aligned}\text{Length of track } &= 98 + 98 + \text{Length of the two semi} - \text{circular arcs}\\ &= 98 + 98 + \frac{1}{2}\left(49 \times \frac{22}{7}\right) + \frac{1}{2}\left(49 \times \frac{22}{7}\right)\\ &= \left(98 + 98 + 77 + 77\right)\text{ m}\\ &= 350\text{ m}\end{aligned}$$

Q.E.D.

(ii) Nathan walks at a constant rate of 1.4 m/s. Calculate the time it will take him to walk 7 laps around the track.

SOLUTION:
Data: Nathan walks at a constant rate of 1.4 m/s.
Required to calculate: The time it takes Nathan to walk 7 laps around the track.
Calculation:

7 laps around the track is a total distance of (350 × 7) m

$$\text{Time taken} = \frac{\text{Total distance covered}}{\text{Speed}}$$

$$= \frac{350 \times 7}{1.4}\ \text{s}$$

$$= 1750\,\text{s}$$

$$= 29 \text{ minutes } 10 \text{ seconds}$$

(b) Tafari runs one lap of the track in 68 seconds.

(i) Determine the number of laps Tafari can complete in one hour, running at the same speed.

SOLUTION:
Data: Tafari runs one lap of the track in 68 seconds.
Required to determine: The number of laps Tafari can complete in one hour, running at the same speed.
Solution:
Tafari completes 350 m in 68 seconds.

In 1 second Tafari will complete $\frac{350}{68}$ m

In 1 hour he will complete $\frac{350}{68} \times 60 \times 60$m

$$\text{Number of laps} = \frac{350 \times 60 \times 60}{68 \times 350}$$

$= 52.94$ laps (correct to 2 decimal places)

OR we can say Tafari will complete 52 laps since the 53rd will not be complete in the time of 1 hour.

(ii) Nathan completes running one lap of the track every 72 seconds. Tafari and Nathan start running at the same time from point *L* on the track. Each completed a number of laps on the track. Calculate the LEAST number of laps that each will complete before they are both at point *L* again at the same time.

SOLUTION:
Data: Nathan completes running one lap of the track every 72 seconds. Tafari and Nathan start running at the same time from point *L* on the track. Each completed a number of laps on the track.
Required to calculate: The LEAST number of laps that each will complete before they are both at point *L* again at the same time.
Calculation:
Nathan takes 72 seconds and Tafari takes 68 seconds to complete a lap.

Hence, they will meet for the first time after t seconds where t is the L.C.M. of 72 and 68.
$72 = 2 \times 2 \times 2 \times 3 \times 3$
$68 = 2 \times 2 \times 17$
LCM of 72 and 68 $= 2 \times 2 \times 2 \times 3 \times 3 \times 17 = 1\ 224$

$t = 1\ 224$ seconds

In 1224 seconds, Nathan will complete $\frac{1224}{72} = 17$ laps

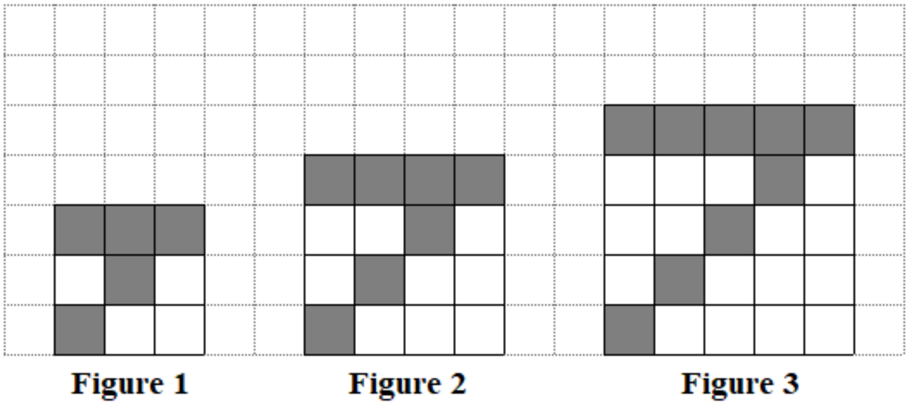
In 1224 seconds, Tafari will complete $\frac{1224}{68} = 18$laps

$\therefore$Tafari completes 18 laps and Nathan completes 17 laps.

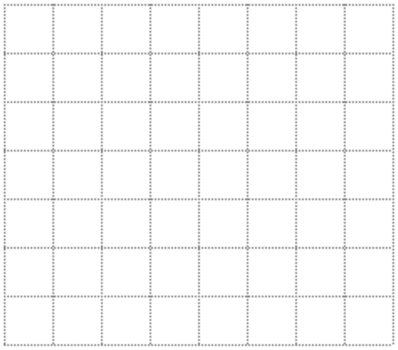
Alternative Method:
When the slower runner, Nathan, reaches the point L for the first time, the faster runner, Tafari will have passed L and be 4 seconds ahead of Nathan. After Nathan completes the second lap, Tafari will be 4×2second ahead of Nathan and past the point L.

Tafari takes 68 seconds to complete a lap.
$\therefore$Tafari will complete an entire lap ahead of Nathan and meet Nathan at L after Nathan has completed $\frac{68}{4} = 17$ laps.
$\therefore$Tafari and Nathan meet for the first time after setting off at L together when Tafari completes 17+1 = 18 laps and Nathan completes 17 laps.

7. The grid below shows the first 3 figures in a sequence. Each figure is made using a set of small squares of unit length that are both coloured (shaded) and white (unshaded).

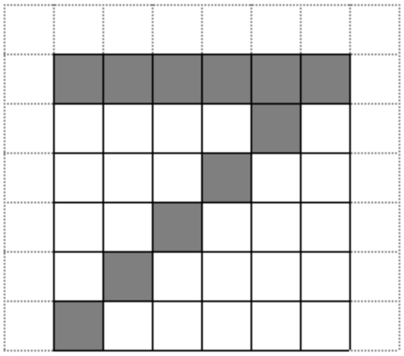


(a) In the space provided below, draw Figure 4 of the sequence.



SOLUTION:
Data: Diagrams showing the first 3 figures in a sequence where each figure is made using a set of small squares of unit length that are **both** coloured (shaded) and white (unshaded).
Required to draw: The fourth figure in the sequence.
Solution:
Figure 1 is drawn on a 3 by 3 square, Figure 2 is drawn on a 4 by 4 square and Figure 3 is drawn on a 5 by 5 square.

Therefore, figure 4 will be drawn on a 6 by 6 block of squares within the given grid. This is shown in the figure below.



(b) The number of coloured squares, *C*, the total number of squares, *T* and the perimeter of the figure, *P*, follow a pattern. Study the patterns in the table below and answer the questions that follow.

Complete Rows (i), (ii) and (iii) in the table below.

	Figure Number (F)	**Number of Coloured Squares (C)**	**Perimeter of Figure (P)**	**Total Number of Squares (T)**
	1	5	12	$(1+2)^2=9$
	2	7	16	$(2+2)^2=16$
	3	9	20	$(3+2)^2=25$
	⋮	⋮	⋮	⋮
(i)	11	________	52	________
	⋮	⋮	⋮	⋮
(ii)	________	49	________	$(23+2)^2=625$
	⋮	⋮	⋮	⋮
(iii)	n	________	________	________

SOLUTION:

Data: Incomplete table showing patterns followed by the number of coloured squares, C, the total number of squares, T and the perimeter of the figure, P.

Required to complete: The table given

Solution:

Let us consider column 4 of the table-The number of squares, T.

From observation, the total number of coloured squares, T, appears to be the square of 2 added to the figure number, F.

That is, $T=(F+2)^2$

Let us observe

When $F=1$ $\quad T=(1+2)^2=9$ (true)

When $F=2$ $\quad T=(2+2)^2=16$ (true)

When $F=3$ $\quad T=(3+2)^2=25$ (true)

$\therefore$When $F=n$, $T=(n+2)^2$ and which completes the fourth column of (iii).

Let us consider column 2-the number of coloured squares, *C*.

Method 1

$C = 5, 7, 9, \ldots$which increases by 2
Hence, $C = 2F + k$, where k is a constant
When $C = 5, F = 1$

$$5 = 2(1) + k$$

and we obtain $k = 3$

When $C = 7, F = 2$

$$7 = 2(2) + 3$$

So, we confirm $k = 3$
$\therefore C = 2F + 3$

Hence, when $F = n$, $C = 2n + 3$ and which completes the second column of (iii)

Method 2

F	1	2	3	n
C	5	7	9	**?**

When $F = 1, C = 5 + 0 = 5$
When $F = 2, C = 5 + 2 = 7$
When $F = 3, C = 5 + 2 + 2 = 9$

Note that the number of twos added is one less than F

$$C = 5 + (F - 1) \times 2 = 7$$
$$C = 5 + 2F - 2$$
$$C = 2F + 3$$

Hence, when $F = n$, $C = 2n + 3$

Let us consider column 3-The perimeter of the figure, *P*.

Method 1

$P = 12, 16, 20, \ldots$which increases by 4
Hence, $P = 4F + a$, where a is a constant.
When $P = 12, F = 1,$

$$12 = 4(1) + a$$

and we obtain $a = 8$

When $P = 16, F = 2,$

$$16 = 4(2) + 8$$

So, we confirm $a = 8$

Hence, when $F = n$, $P = 4n + 8$
This completes the third column of (iii)

Method 2

F	1	2	3	n
P	12	16	20	**?**

When $F = 1, C = 12 + 0 = 5$
When $F = 2, C = 12 + 4 = 16$
When $F = 3, C = 12 + 4 + 4 = 20$

Note that the number of fours added is one less than F

$$C = 12 + (F - 1) \times 4 = 7$$
$$C = 12 + 4F - 4$$
$$C = 4F + 8$$

Hence, when $F = n$, $P = 4n + 8$

Method 3

By inspection, we can let $P = 2C + 2$
When $C = 5$ $P = 12 = 2(5) + 2$
When $C = 7$ $P = 16 = 2(7) + 2$
When $C = 9$ $P = 20 = 2(9) + 2$

$$\begin{aligned}\therefore P &= 2C+2\\ &= 2(2n+3)+2\\ &= 4n+8\end{aligned}$$

From the completed third row, (iii) we can calculate the missing figures in (i) and (ii) as shown to complete the given incomplete table.
The completed table now looks like:

	Figure Number (*F*)	**Number of Coloured Squares (*C*)**	**Perimeter of Figure (*P*)**	**Total Number of Squares (*T*)**
	1	5	12	$(1+2)^2=9$
	2	7	16	$(2+2)^2=16$
	3	9	20	$(3+2)^2=25$
	⋮	⋮	⋮	⋮
(i)	11	$2(11)+3=25$	52	$(11+2)^2=169$
	⋮	⋮	⋮	⋮
(ii)	23	49	$4(23)+8$ $=100$	$(23+2)^2=625$
	⋮	⋮	⋮	⋮
(iii)	n	$2n+3$	$4n+8$	$(n+2)^2$

(c) How many white squares are in Figure 11

SOLUTION:

Required to find: The number of white squares in Figure 11

Solution:

In Figure 11, the number of squares will be $(11+2)^2 = 13\times 13$
$= 169$

The number of coloured squares $= 2(11)+3 = 25$

$\therefore$The number of white squares $= 169-25$
$= 144$

SECTION II

Answer ALL questions.

ALGEBRA, RELATIONS, FUNCTIONS AND GRAPHS

8. The functions *f*, *g* and *h* are defined as follows:

$$f(x)=4x-1,\ g(x)=x^2-5 \text{ and } h(x)=3^x$$

(a) Find:

(i) $g(x-2)$ in its simplest form

SOLUTION:
Data: $f(x)=4x-1,\ g(x)=x^2-5$ and $h(x)=3^x$
Required to find: $g(x-2)$ in its simplest form
Solution:

$$g(x-2)=(x-2)^2-5$$
$$=x^2-4x+4-5$$
$$=x^2-4x-1$$

(in its simplest form)

(ii) $f^{-1}(11)$

SOLUTION:
Required to find: $f^{-1}(11)$
Solution:
Let $y=4x-1$
Making x the subject

$$\therefore 4x=1+y$$
$$x=\frac{1+y}{4}$$

We replace y by x to get:

$$f^{-1}(x)=\frac{1+x}{4}$$
$$\therefore f^{-1}(11)=\frac{1+11}{4}$$
$$=3$$

Alternatively, we wish to find the input for an output of 11. Let the input be $x=a$.

Therefore,

$$f(a)=11$$
$$4a-1=11$$
$$4a=12$$
$$a=3$$
$$\therefore f^{-1}(11)=3$$

(b) Determine the value of $hh(1)$.

SOLUTION:
Required to determine: $hh(1)$. This can also be written as $h^2\ (1)$
Solution:

$$h(x) = 3^x$$
$$\therefore h(1) = 3^1$$
$$= 3$$
$$hh(1) = h(3)$$
$$= 3^3$$
$$= 3 \times 3 \times 3$$
$$= 27$$

(c) The function *f* is defined as follows:

$$f : x \to x^2 - x - 2$$

Complete the table below and plot the graph for the function $f(x) = x^2 - x - 2$ on the grid that follows.

(Use a scale of 2 cm to represent 1 unit on both axes.)

x	–2	–1	0	1	2	3
f(x)		0		–2	0	4

SOLUTION:
Data: $f : x \to x^2 - x - 2$ together with an incomplete table of values.
Required to complete: The table of values given for *f* and plot the graph of *f*.
Solution:

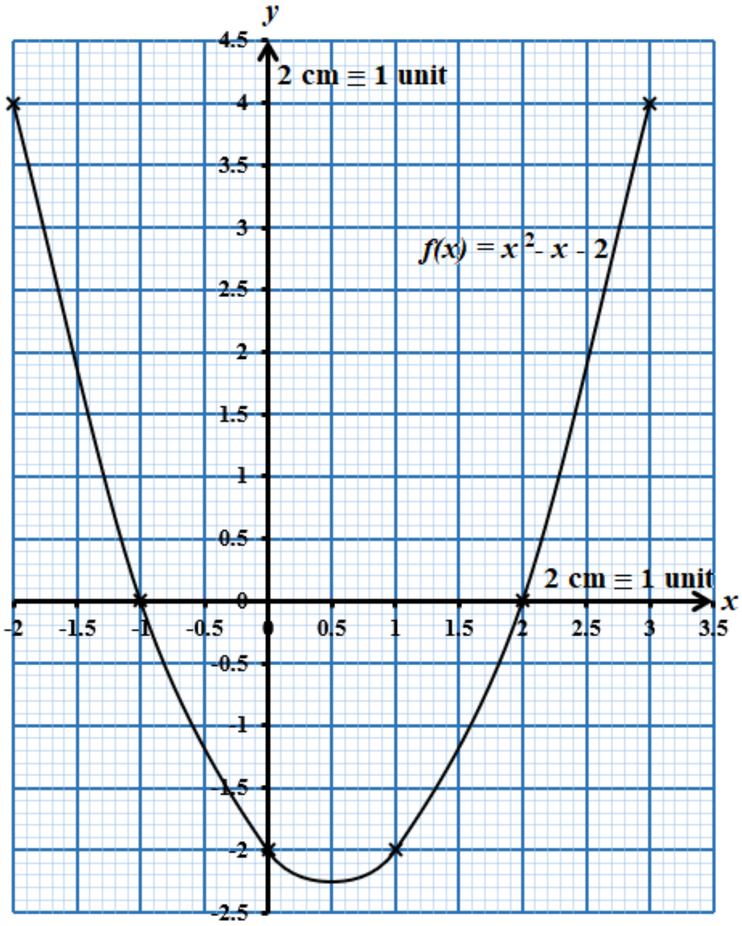
$$f(-2) = (-2)^2 - (-2) - 2$$
$$= 4$$

$$f(0) = (0)^2 - (0) - 2$$
$$= -2$$

The completed table now looks like:

x	–2	–1	0	1	2	3
f(x)	4	0	–2	–2	0	4

Using the scales given we plot each point on the table and join by a smooth curve.



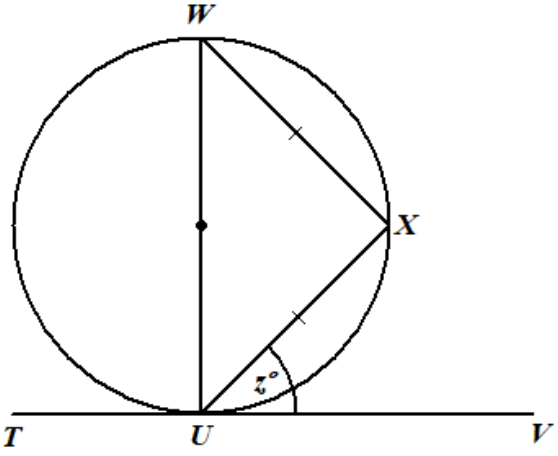
As a point of interest, the minimum point will occur at $x = \frac{-(-1)}{2(1)} = \frac{1}{2}$

When $x = \frac{1}{2}$, $y = (\frac{1}{2})^2 - (\frac{1}{2}) - 2 = -2\frac{1}{4}$

The minimum point on the graph will be $(\frac{1}{2}, -2\frac{1}{4})$

GEOMETRY AND TRIGONOMETRY

9. (a) *W*, *X* and *U* are points on the circumference of a circle. *TV* is a tangent to the circle at *U*. *UW* is a diameter of the circle and triangle *WXU* is isosceles.



Using appropriate theorems, state THREE reasons that explain why the measure of angle *z* is 45°.

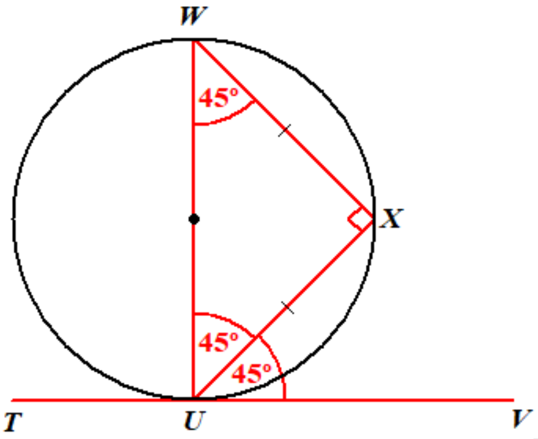
SOLUTION:

Data: Diagram showing a circle where *W*, *X* and *U* are three points along its circumference. *TV* is a tangent to the circle at *U*. *UW* is a diameter of the circle and triangle *WXU* is isosceles. $XUV = z^0$

Required to state: Three reasons, using appropriate theorems that explain why $z = 45°$.

Solution:

Reason 1:



$W\hat{X}U = 90°$ (Angle in a semi-circle is a right angle.)

ΔWXU is isosceles and the base angles of an isosceles triangle are equal.

$\therefore W\widehat{U}X = U\widehat{W}X$

$\therefore$Each base angle $= \frac{1}{2}(180^0 - 90^0) = 45^0$

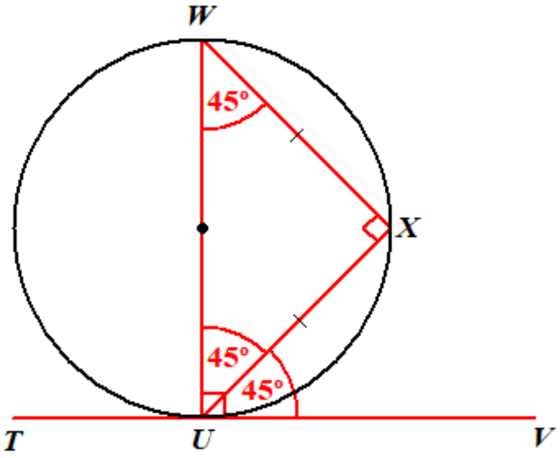
Hence, $U\hat{W}X = 45°$

Now, $X\widehat{U}V = z^0$ (Given)

And, $X\widehat{U}V = U\widehat{W}X$ [The angle made by a tangent to a circle (*UV*)and a chord (*UX*) at the point of contact (*U*) is equal to the angle in the alternate segment, (*UWX*)].

$\therefore z = U\widehat{W}X = 45^0$

Reason 2:

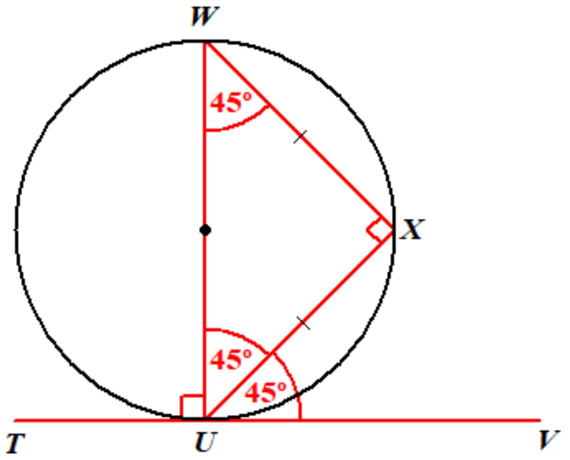


$W\hat{U}V = 90°$ [The angle (*WUV*) made by a tangent to a circle (*UV*) and a radius (from the centre to *U*) at the point of contact (*U*) is a right angle.]

$W\hat{U}X = 45°$ (Base angles in an isosceles triangle are equal.)

$$z = W\hat{U}V - W\hat{U}X$$
$$= 90° - 45°$$
$$= 45°$$

Reason 3:

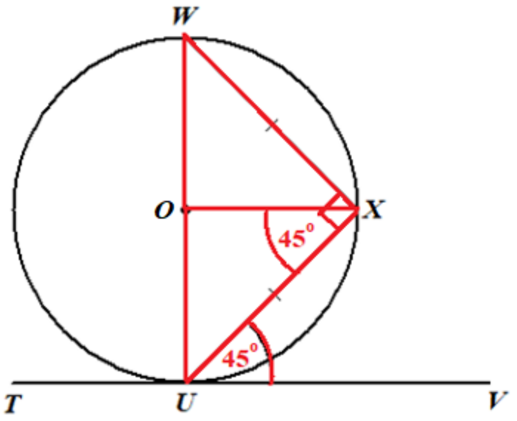


TUV is a straight line and $T\hat{U}W = 90°$.
(The angle made by a tangent to a circle and a radius at the point of contact is a right angle.)

$W\hat{U}X = 45°$ (Base angles in an isosceles triangle are equal.)

$$\therefore z = 180° - (90° + 45°)$$
$$= 45°$$

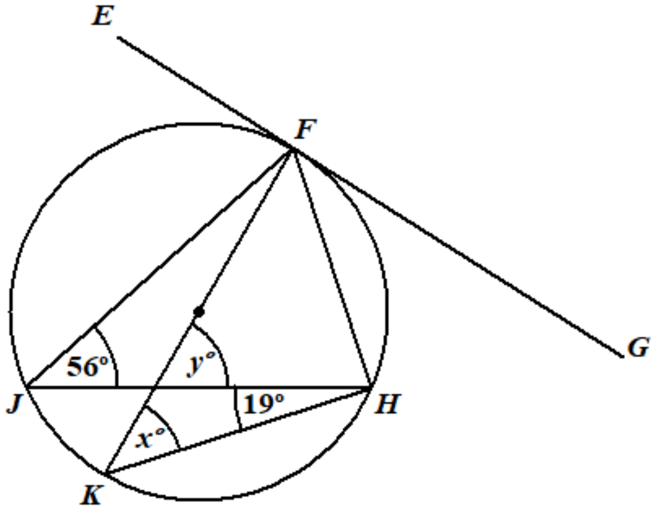
We'll add a 4th reason for the benefit of our keen and interested readers



If we drew a horizontal from X to the centre of the circle, say O, the triangles OXW and OXU will be congruent. Hence, the angle OXU will be ½ (90^0) = 45^0.
And $W\hat{O}X = X\hat{O}U = 90^0$

$\therefore$ The lines *OX* and *TUV* are parallel
Angle *z* will also be 45^0 (alternate angle to angle *OXU*)

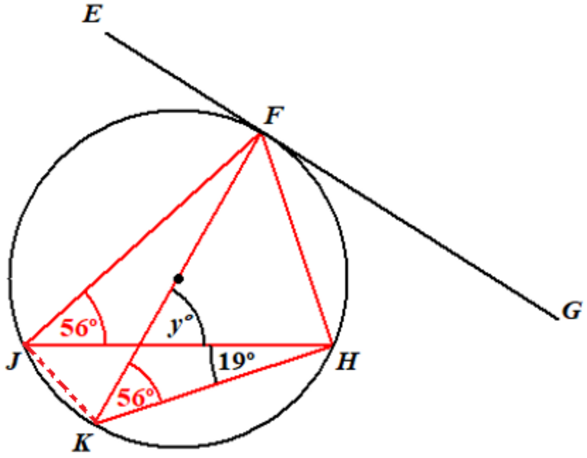
(b) The diagram below shows a circle with diameter *KF*. Line *EFG* is a tangent to the circle at *F*. The points *F*, *H*, *J* and *K* lie on the circumference of the circle.



By showing EACH step in your work, where appropriate, find the value for EACH of the following angles:

(i) Angle *x*

SOLUTION:
Data: Diagram showing a circle with diameter *KF*. Line *EFG* is a tangent to the circle at *F*. The points *F*, *H*, *J* and *K* lie on the circumference of the circle.
Required to find: *x*
Solution:



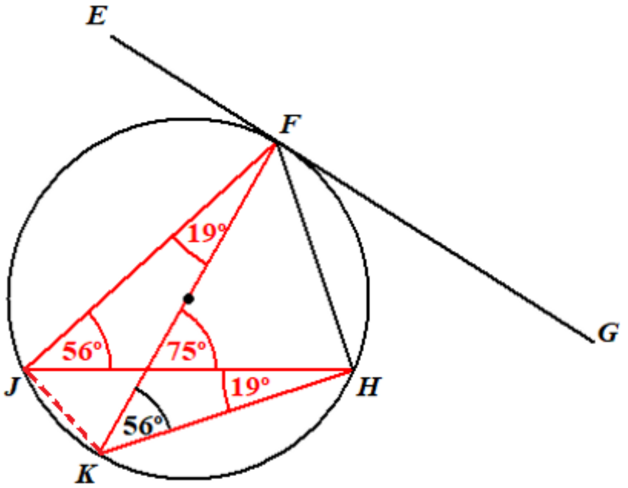
The chord *FH* subtends *x* and $F\hat{J}H$ on the circumference.

Since $F\hat{J}H = 56^\circ$ (given),

$x = 56^\circ$ [The angles subtended by a chord (*FH*) at the circumference of a circle and standing on the arc (x and $F\hat{J}H$) are equal.]

(ii) Angle y

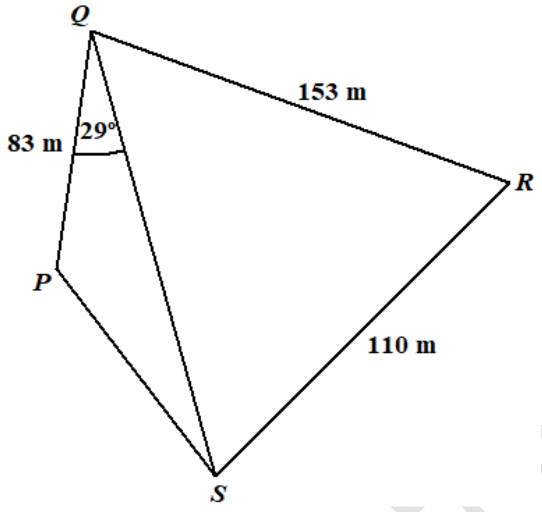
SOLUTION:
Required to find: y
Solution:



$J\hat{H}K = J\hat{F}K = 19^\circ$
(The angles subtended by a chord (*JK*) at the circumference of a circle and standing on the arc are equal.)

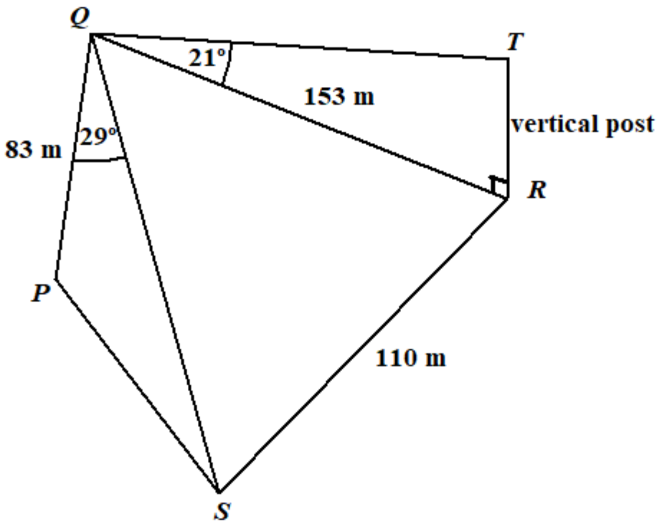
$y = 56^\circ + 19^\circ$
$= 75^\circ$
(The exterior angle of a triangle is equal to the sum of the interior opposite angles.)

(c) The diagram below shows 4 points, *P*, *Q*, *R* and *S* on level ground, where pillars will be placed to mark the outline for a foundation.

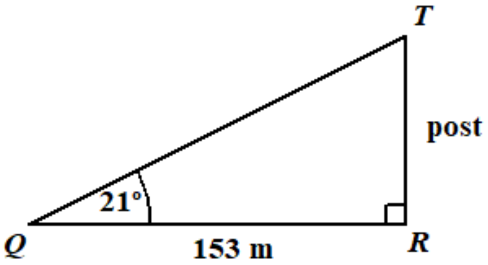


(i) There is a vertical post, *RT*, at *R*. From *Q*, the angle of elevation of the top of the post, *T*, is 21°. Find the height of the post.

SOLUTION:
Data: Diagram showing 4 points, *P*, *Q*, *R* and *S* on level ground, where pillars will be placed to mark the outline for a foundation. There is a vertical post, *RT*, at *R*. From *Q*, the angle of elevation of the top of the post, *T*, is 21°.
Required to find: The height of the post
Solution:



The height of the post is TR



$$\frac{TR}{153} = \tan 21°$$

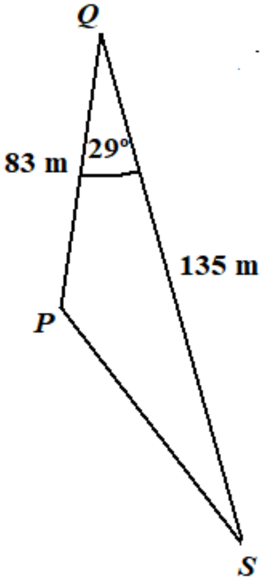
$$\therefore TR = 153 \tan 21°$$

$$= 58.73\underset{=}{1} \text{ m}$$

$$= 58.73 \text{ m (correct to 2 decimal places)}$$

(ii) Given that the length *QS* is 135 m, calculate the perimeter of the foundation *PQRS*.

SOLUTION:
Data: $QS = 135\text{m}$
Required to calculate: The perimeter of the foundation, *PQRS*
Calculation:
Consider ΔQPS:



Applying the cosine law to ΔQPS

$$PS^2 = (83)^2 + (135)^2 - 2(83)(135)\cos 29°$$
$$= 6889 + 18225 - 19600.23$$
$$= 5513.77$$
$$PS = 74.254$$

$\therefore$Perimeter of the foundation $= (74.254 + 110 + 153 + 83)$ m
$= 420.254$ m
$= 420.25$ m (correct to 2 decimal places)

VECTORS AND MATRICES

10. (a) The matrices *Q*, *R* and *S* are as follows:

$$Q = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix},\ R = \begin{pmatrix} 1 & 6 \\ -5 & 4 \end{pmatrix},\ S = \begin{pmatrix} 2 & 7 \\ 4 & -1 \\ -8 & 9 \end{pmatrix}$$

(i) Explain why the matrix product *QS* is NOT possible.

SOLUTION:

Data: $Q = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix},\ R = \begin{pmatrix} 1 & 6 \\ -5 & 4 \end{pmatrix},\ S = \begin{pmatrix} 2 & 7 \\ 4 & -1 \\ -8 & 9 \end{pmatrix}$

Required to explain: Why the product *QS* is not possible

Solution:

$$\begin{array}{ccc} Q & \times & S \\ 2\times \underbrace{2 \quad\quad 3}_{\neq}\times 2 \end{array}$$

The number of columns of *Q* is not equal to the number of rows of *S*. Hence, matrix multiplication, *QS*, is not possible. The matrix product *QS* is said to be non-conformable to multiplication.

(ii) State the order of the matrix product *SR*.

SOLUTION:

Required to state: The order of the matrix product *SR*.

Solution:

$$\begin{array}{ccc} S & \times & R \end{array}$$
$$\underbrace{3\times\underbrace{2\qquad 2}_{=}\times 2}_{3\times 2}$$

The number of columns of S is equal to the number of rows of R. This product SR is conformable to multiplication and the order of the matrix product SR is 3×2.

(iii) Calculate the matrix product QR.

SOLUTION:
Required to calculate: The matrix product QR
Calculation:

$$\begin{array}{ccc} Q & \times & R \end{array}$$
$$\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ -5 & 4 \end{pmatrix}=\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$$

This is a (2×2) matrix by a (2×2) and the result will be a (2×2)

$$\begin{aligned} e_{11} &= (2\times 1)+(-1\times -5) \\ &= 2+5 \\ &= 7 \end{aligned}$$

$$\begin{aligned} e_{12} &= (2\times 6)+(-1\times 4) \\ &= 12-4 \\ &= 8 \end{aligned}$$

$$\begin{aligned} e_{21} &= (4\times 1)+(3\times -5) \\ &= 4-15 \\ &= -11 \end{aligned}$$

$$\begin{aligned} e_{22} &= (4\times 6)+(3\times 4) \\ &= 24+12 \\ &= 36 \end{aligned}$$

$$\therefore QR=\begin{pmatrix} 7 & 8 \\ -11 & 36 \end{pmatrix}$$

(b) Given that $A = \begin{pmatrix} 4 & -1 \\ -7 & x \end{pmatrix}$, determine the value of x when $|A| = 5$.

SOLUTION:

Data: $A = \begin{pmatrix} 4 & -1 \\ -7 & x \end{pmatrix}$ and $|A| = 5$

Required to determine: The value of x

Solution:

$$|A| = 5$$

$$\therefore (4 \times x) - (-1 \times -7) = 5$$

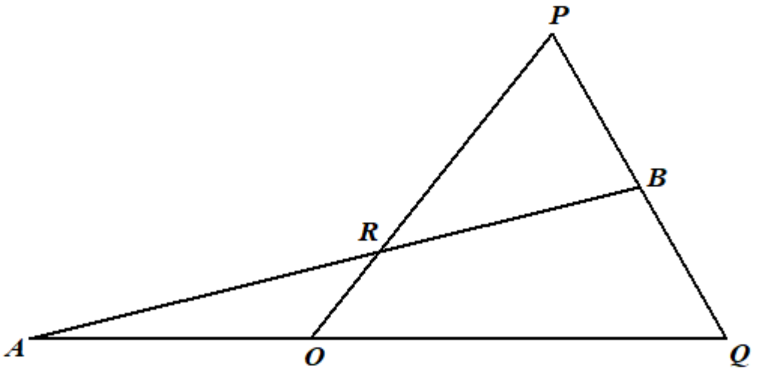
$$4x - 7 = 5$$

$$4x = 5 + 7$$

$(\div 4)$

$$x = 3$$

(c) In the diagram below, *OPQ* is a triangle. *ARB* and *AOQ* are straight lines.
B is the midpoint of *PQ*.
R is the midpoint of *AB*.
$OR : RP = 1:3$
$\overrightarrow{OP} = 4\mathbf{a}$ and $\overrightarrow{OQ} = 8\mathbf{b}$



Find, in terms of **a** and **b**, in its simplest form

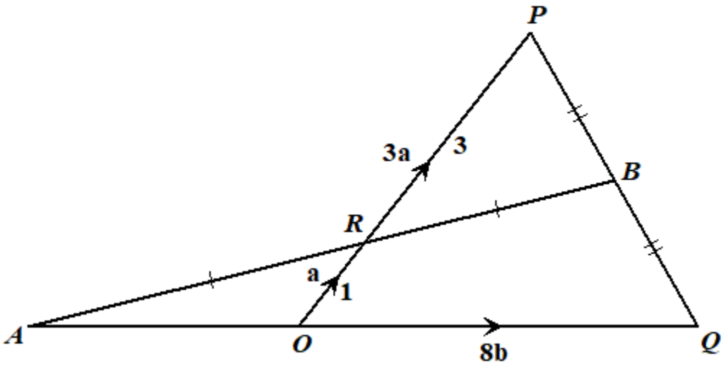
(i) $\overrightarrow{PQ}$

SOLUTION:

Data: Diagram showing triangle *OPQ*. *ARB* and *AOQ* are straight lines. *B* is the midpoint of *PQ*. *R* is the midpoint of *AB*. $OR : RP = 1:3$

$\overrightarrow{OP} = 4\mathbf{a}$ and $\overrightarrow{OQ} = 8\mathbf{b}$

Required to find: $\overrightarrow{PQ}$ in terms of **a** and **b**
Solution:



$$\overrightarrow{OP} = 4\mathbf{a}$$

$$\overrightarrow{OR} = \frac{1}{1+3} \times 4\mathbf{a}$$
$$= \mathbf{a}$$

$$\overrightarrow{RP} = \frac{3}{1+3} \times 4\mathbf{a}$$
$$= 3\mathbf{a}$$

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$
$$= -(4\mathbf{a}) + 8\mathbf{b}$$
$$= 4(2\mathbf{b} - \mathbf{a})$$

(ii) $\overrightarrow{PR}$

SOLUTION:
Required to find: $\overrightarrow{PR}$ in terms of **a** and **b**
Solution:

$$\overrightarrow{RP} = 3\mathbf{a}$$
$$\therefore \overrightarrow{PR} = -(3\mathbf{a})$$
$$= -3\mathbf{a}$$

(iii) $\overrightarrow{RB}$

SOLUTION:
Required to find: $\overrightarrow{RB}$ in terms of **a** and **b**

Solution:

$$\begin{aligned}\overrightarrow{RB} &= \overrightarrow{RP} + \overrightarrow{PB} \\ &= 3\mathbf{a} + \frac{1}{2}\overrightarrow{PQ} \\ &= 3\mathbf{a} + \frac{1}{2}\left(-4\mathbf{a} + 8\mathbf{b}\right) \\ &= 3\mathbf{a} - 2\mathbf{a} + 4\mathbf{b} \\ &= \mathbf{a} + 4\mathbf{b}\end{aligned}$$