FAS-PASS Maths

CSEC MATHEMATICS JANUARY 2020 PAPER 2

SECTION I

1. (a) Using a calculator, or otherwise, calculate the exact value of the following:

(i) $4\frac{1}{5} \times \frac{1}{3} - 1\frac{1}{4}$

SOLUTION:

$$\left(4\frac{1}{5} \times \frac{1}{3}\right) - 1\frac{1}{4}$$

We first evaluate: $4\frac{1}{5} \times \frac{1}{3}$

$$= \frac{21}{5} \times \frac{1}{3} = \frac{7}{5} = 1\frac{2}{5}$$

Now we have, $1\frac{2}{5} - 1\frac{1}{4}$. Since $1 - 1 = 0$, we now have:

$$\frac{2}{5} - \frac{1}{4} = \frac{8-5}{20}$$

$= \frac{3}{20}$ (in exact form)

(ii) $\dfrac{4.1 - 1.25^2}{0.005}$

SOLUTION:

$$\frac{4.1 - 1.25^2}{0.005}$$

$$\frac{4.1 - (1.25 \times 1.25)}{0.005}$$

Using the calculator,

$$= \frac{4.1 - 1.5625}{0.005}$$

$$= \frac{2.5375}{0.005}$$

= 507.5 (in exact form)

1(b) A stadium currently has a seating capacity of 15 400 seats.

(i) Calculate the number of people in the stadium when 75% of the seats are occupied.

SOLUTION:

75% of 15 400 seats

$=\frac{75}{100}\times 15400$

=11 550 persons

(ii) The stadium is to be renovated with a new seating capacity of 20 790 seats. After the renovation, what will be the percentage increase in the number of seats?

SOLUTION:

Increase in the number of seats

= 20 790 – 15 400

= 5 390

So, percentage increase $=\frac{Increase\ in\ the\ number\ of\ seats}{Original\ number\ of\ seats}\times 100\%$

$=\frac{5\ 390}{15\ 400}\times 100\%$

$=35\%$

1(c) A neon light flashes five times every 10 seconds. Show that the light flashes 43 200 times in one day.

SOLUTION:

1 day = 24 hours = 24 × 60 minutes = 24 × 60 × 60 seconds = 86 400 seconds

The light flashes five times in each 10-second interval.

The number of 10-second intervals in 86 400 seconds $=\frac{86\ 400}{10}=8640$

If there are 5 flashes in every 10 seconds, then the number of times the light flashes in one day $=8\ 640\times 5$

$=43\ 200$

Q.E.D.

2. (a) Factorise the following expressions completely.

(i) $5h^2 - 12hg$

SOLUTION:

$5h^2 - 12hg$

$= 5h.h - 12hg$

$= h(5h - 12g)$

(ii) $2x^2 - x - 6$

SOLUTION:

$2x^2 - x - 6$

$(2x + 3)(x - 2)$

2(b) Solve the equation

$r + 3 = 3(r - 5)$

SOLUTION:

$r + 3 = 3r - 15$

$r - 3r = -15 - 3$

$-2r \quad = -18$

$r \ = \ \frac{-18}{-2}$

$r = 9$

2(c) Make k the subject of the formula

$$2A = \pi k^2 + 3t$$

SOLUTION:

$$2A = \pi k^2 + 3t$$

$$\pi k^2 = 2A - 3t$$

$$k^2 = \frac{2A-3t}{\pi}$$

$$k = \sqrt{\frac{2A-3t}{\pi}}$$

2(d) A farmer plants two crops, potatoes and corn, on a ten-hectare piece of land. The number of hectares of corn planted, c, must be at least twice the number of hectares of potatoes, p.

Write two inequalities to represent the scenario above.

SOLUTION:

The area of the entire plot of land = 10 hectares

The number of hectares of corn = c

The number of hectares of potatoes = p

Clearly, the total area planted cannot exceed the area of the plot

$c + p$ $\ngtr$ 10

Hence, $c + p \leq 10$

The number of hectares of corn is at least twice the number of hectares of potatoes

c $\geq$ $2p$

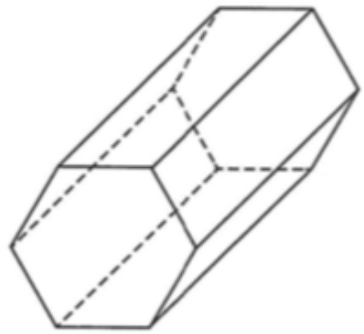
Hence, $c \geq 2p$

The two inequalities are:

$c + p \leq 10$…………(1)

$c \geq 2p$ …………….(2)

3 (a) The diagram below shows a hexagonal prism.



Complete the following statement.

The prism has

_______________ faces.

_______________ edges and

_______________ vertices.

SOLUTION:

Faces

The prism has 6 rectangular faces and 2 hexagonal faces.

Total number of faces = $6 + 2 = 8$

Edges

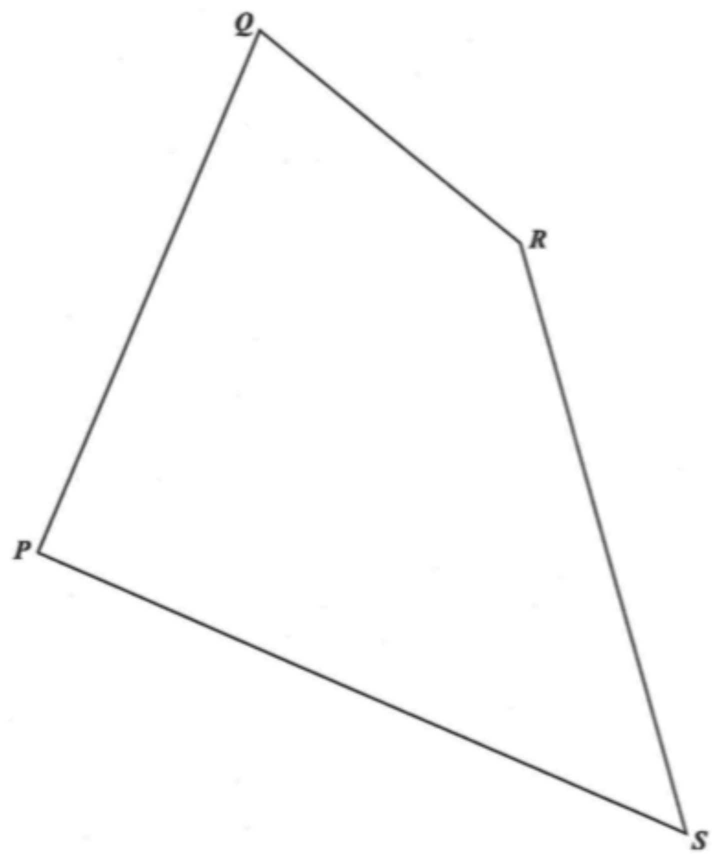
The prism has 6 edges bounding each of the two hexagonal faces and 6 edges bounding the rectangular faces.
Total number of edges = $(6 \times 2) + 6 = 18$

Vertices

The prism has 6 vertices on each of the two hexagonal face.

Total number of vertices $= 6 \times 2 = 12$

3(b) A Sports Club owns a field, PQRS, in the shape of a quadrilateral. A scale diagram of this field is shown below. (1 centimetre represents 10 metres)



In the following parts show all your construction lines.

The field is to be divided with a fence from P to the side RS so that different sports can be played at the same time.

Each point on the fence is the same distance from PQ as from PS

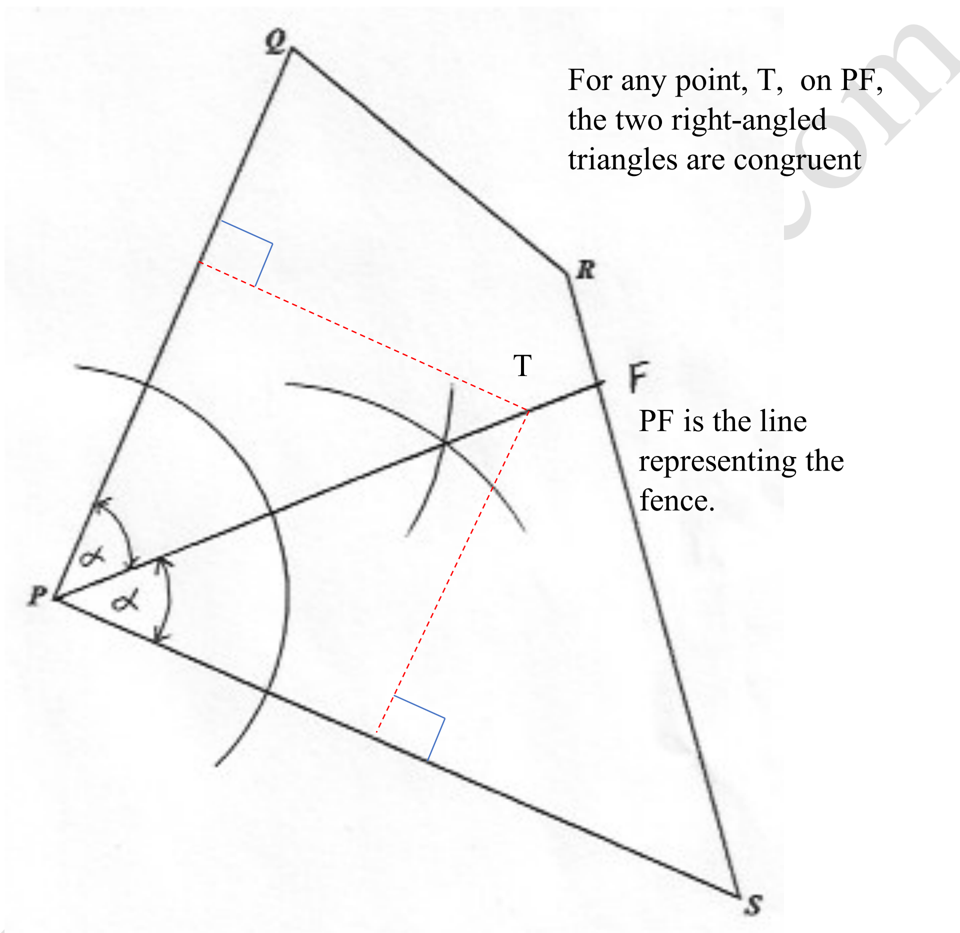
(i) Using a straight edge and compasses only, construct the line representing the fence.

(ii) Write down the length of the fence , in metres.

SOLUTION:

(i) Points equidistant from the line PQ and PS will lie on the line bisecting the angle QPS.

The bisector of the angle QPS meets RS at F.

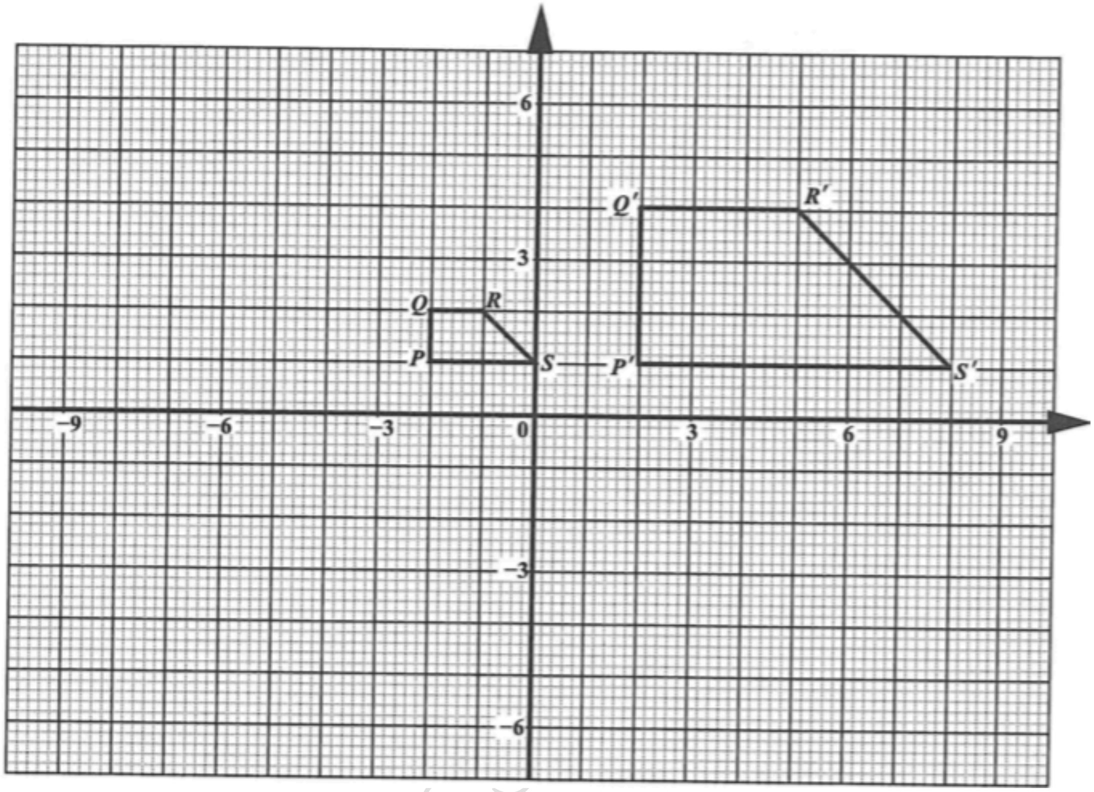


(ii) The line BF is 7.4 cm by measurement.

Since 1 centimetre represents 10 metres, the length of the fence is

$7.4 \times 10 = 74\ metres$

3 (c) A quadrilateral $PQRS$ and its image $P'Q'R'S'$ are shown on the grid below.



(i) Write down the mathematical name for the quadrilateral $PQRS$

(ii) $PQRS$ is mapped onto $P'Q'R'S'$ by an enlargement with scale factor, k, about the centre, $C(a, b)$. Using the diagram above, determine the values of a, b and k.

SOLUTION:

(i) The quadrilateral $PQRS$ has only one pair of parallel sides and is therefore a trapezium.

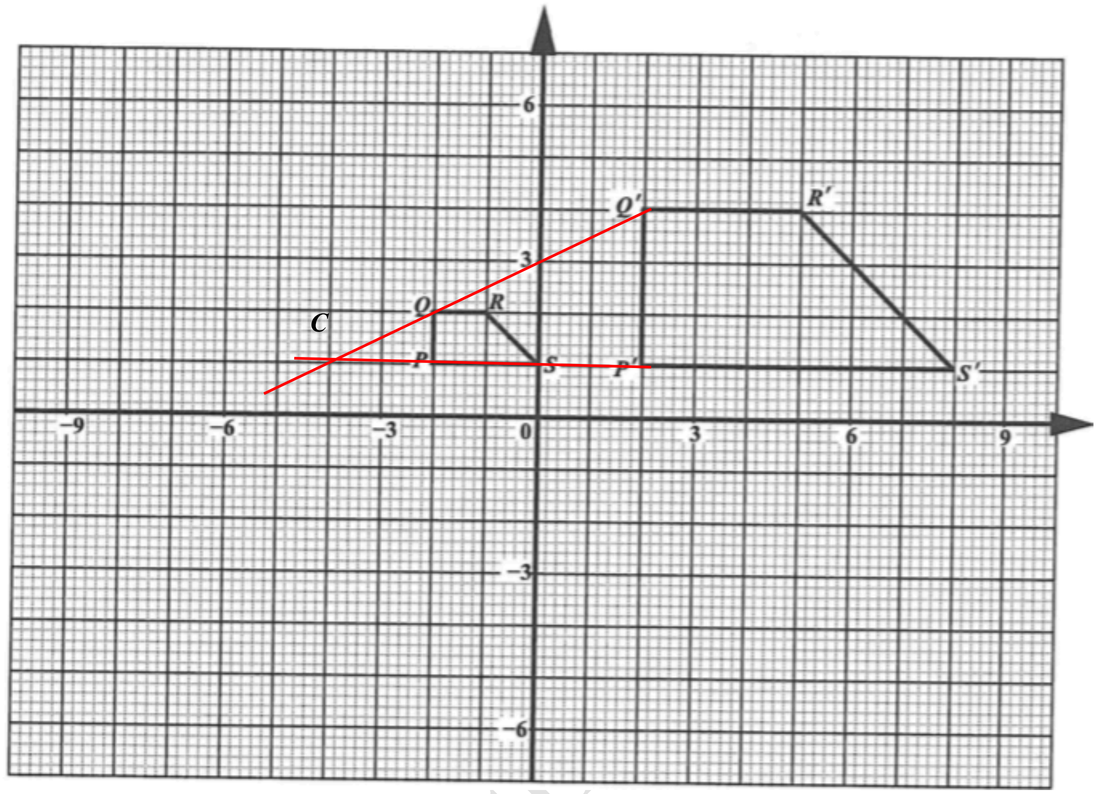
(ii) To locate the center of enlargement, join image points to their corresponding object points by straight lines. Produce them to meet at the center of enlargement.

We join PP' and QQ' and the point of intersection, $C(-4,1)$ is shown below. It is not necessary to join more than two such lines since all such lines are concurrent.

Hence, $a = -4$ and $b = 1$.

The scale factor of the enlargement is the ratio of image length to object length.

Hence, $k = \frac{P'Q'}{PQ} = \frac{3}{1} = 3$



4 (a) The function f is defined as

$$f(x) = \frac{2x + 7}{5}$$

(i) Find the value of $f(4) + f(-4)$.

SOLUTION:

$$f(4) = \frac{2(4) + 7}{5} = \frac{8 + 7}{5} = \frac{15}{5} = 3.$$

$$f(-4) = \frac{2(-4) + 7}{5} = \frac{-8 + 7}{5} = -\frac{1}{5}$$

$$\text{Hence, } f(4) + f(-4) = 3 + \left(-\frac{1}{5}\right) = 2\frac{4}{5}$$

(ii) a) Calculate the value of x for which $f(x) = 9$.

SOLUTION:

$f(x) = 9$

Therefore, $\frac{2x+7}{5} = 9$

$2x + 7 = 45$

$2x = 45 - 7$

$2x = 38$

$x = 19$

b) Hence, or otherwise, determine the value of $f^{-1}(9)$

SOLUTION:

From above, when $x = 19, f(x) = 9$.

This means that, for the function $f(x)$, an input of 19 produces an output of 9. We can also say that the function, f maps 19 onto 9.

Hence, the inverse function, $f^{-1}(x)$, will map 9 onto 19. So, $f^{-1}(19) = 9$.

Alternatively, we could find $f^{-1}(x)$, and substitute $x = 19$ in this inverse function.

Let. $y = \frac{2x+7}{5}$

Interchange x and y:

$x = \frac{2y+7}{5}$

$5x = 2y + 7$

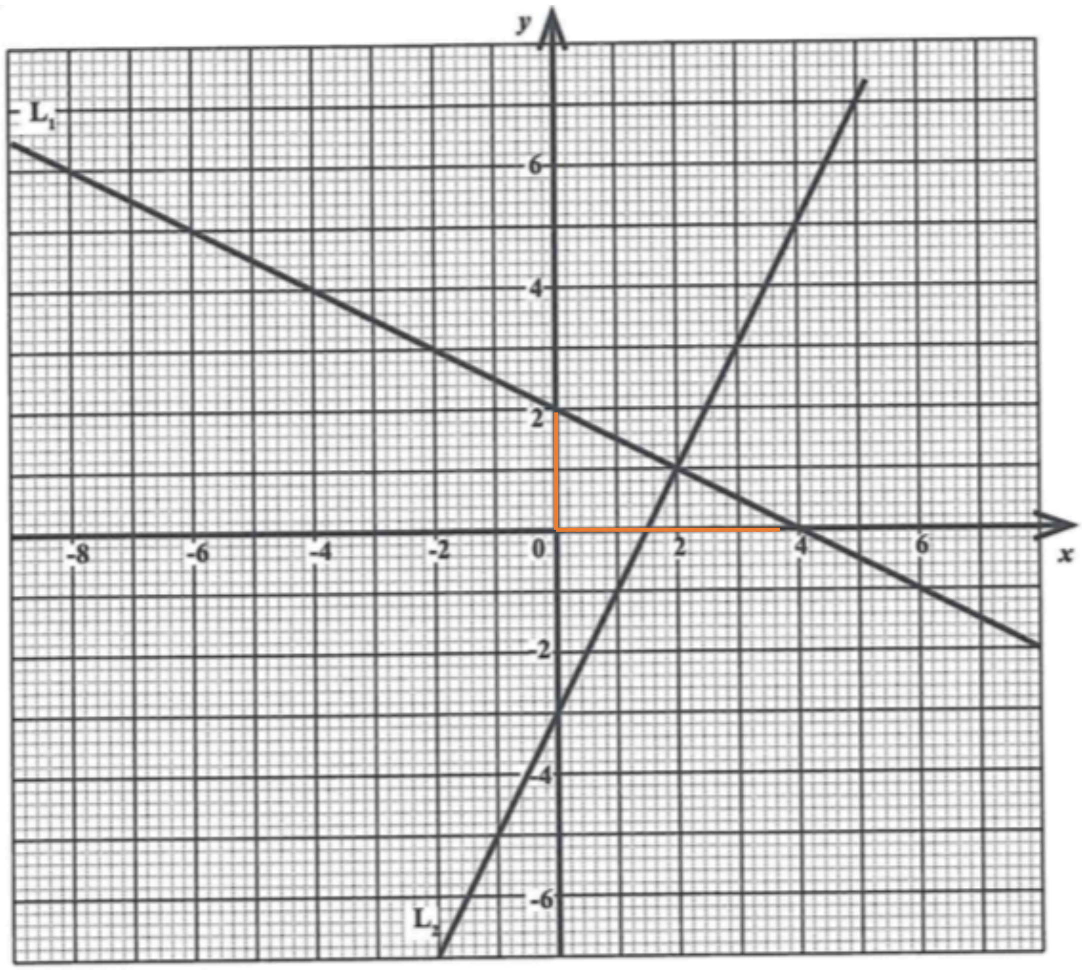
$2y = 5x - 7$

$y = \frac{5x-7}{2}$

$f^{-1}(x) = \frac{5x-7}{2}$

$f^{-1}(9) = \frac{5(9)-7}{2} = \frac{38}{2} = 19$

4(b) The graph below shows two straight lines, L_1 and L_2. L_1 intercepts the x and y axes at $(4, 0)$ and $(0, 2)$ respectively. L_2 intercepts the x and y axes at $(1.5, 0)$ and $(0, -3)$ respectively.



(i) Determine the equation of the line L_1.

SOLUTION:

The gradient of L_1 can be found by inspection as $\frac{2}{-4} = -\frac{1}{2}$

OR by using the points $(0,2)$ and $(4,0)$ and the gradient formula, $\frac{y_2 - y_1}{x_2 - x_1}$.

Gradient of $L_1 = \frac{0-2}{4-0} = -\frac{2}{4} = -\frac{1}{2}$

The general equation of any straight line is $y = mx + c$, where m is the gradient and c is the y-intercept. L_1 has a y-intercept of 2, so $c = 2$. Hence, the equation of L_1 is:

$$y = -\frac{1}{2}x + 2$$

OR

Using the point $(x_1, y_1) = (2,0)$ and $m = -\frac{1}{2}$, the equation of a line is given as:

$$\frac{y - y_1}{x - x_1} = m$$

$$\frac{y - 2}{x - 0} = -\frac{1}{2}$$

$$2(y - 2) = -x$$

$$2y - 4 = -x$$

$$2y = -x + 4$$

$$y = -\frac{1}{2}x + 2$$

(ii) What is the gradient of the line L_2, given that L_1 and L_2 are perpendicular?

SOLUTION:

The product of the gradients of perpendicular lines is -1.

$$\text{Gradient of } L_1 \times \text{ Gradient of } L_2 = -1$$

$$-\tfrac{1}{2} \times \text{Gradient of } L_2 = -1$$

$$\text{Gradient of } L_2 = -\frac{1}{-\frac{1}{2}}$$

$$\text{Gradient of } L_2 = 2$$

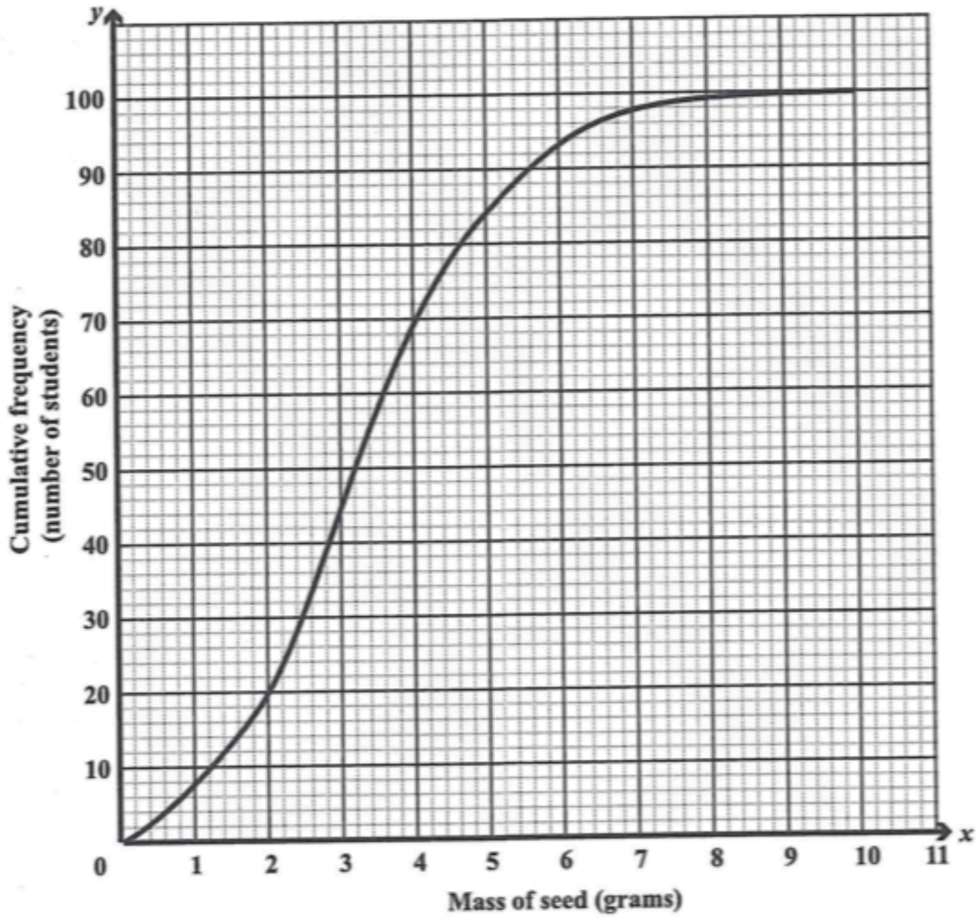
OR

By using the points $(1.5, 0)$ and $(0, -3)$,

Gradient of L_2:

$$= \frac{-3 - 0}{0 - 1.5} = \frac{-3}{-1.5} = 2$$

5. A group of 100 students estimated the mass, m (grams) of a seed. The cumulative frequency curve below shows the results.



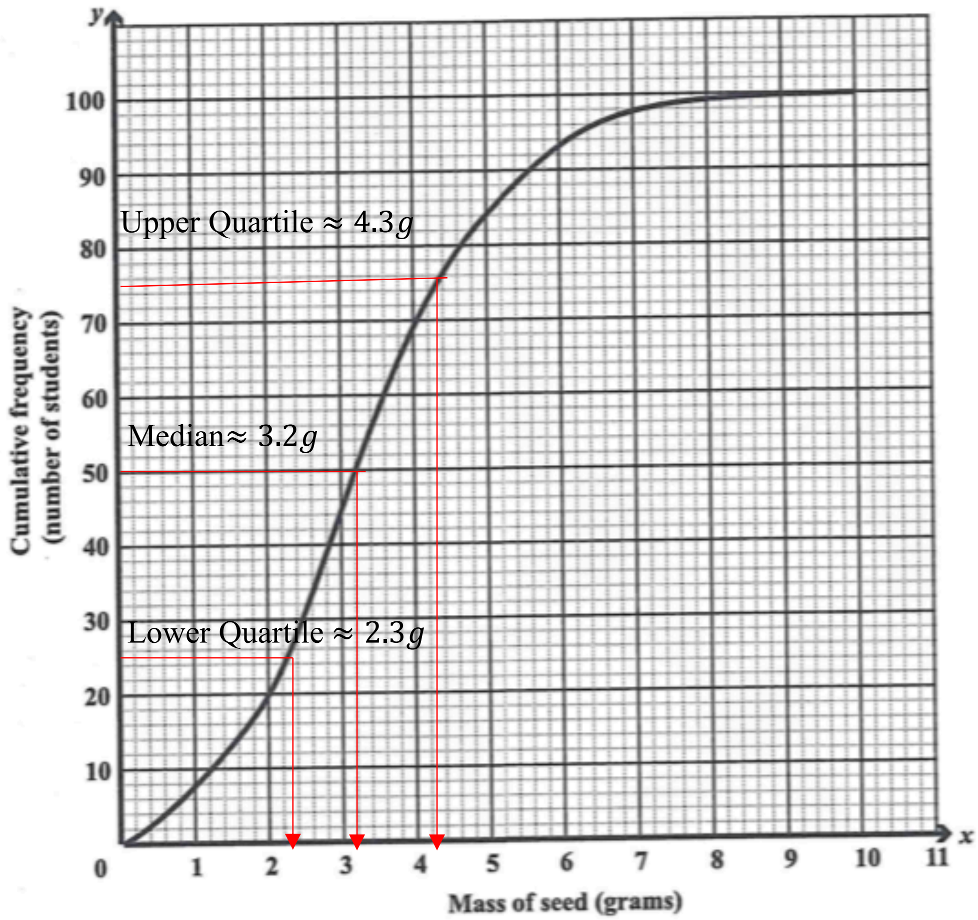
(i) (a) Using the cumulative frequency curve, estimate the
(i) median
(ii) upper quartile

SOLUTION:

(i) The median is the middle score in a distribution and will occur at the 50th percentile. The estimate of the median is approximately 3.2 grams.

(ii) The upper quartile is the 75th percentile. The estimate of the upper quartile approximately 4.3 grams.

[These values are obtained from the curve and shown on the diagram.]



(iii) Semi-interquartile range

To obtain the semi-interquartile range, we use the formula:

$\frac{Q_3 - Q_1}{2}$, where Q_3 is the upper quartile and Q_1 the lower quartile.

The lower quartile is at the 25th percentile and shown on the graph.

$$Q_1 \approx 2.3g \ \ and \ Q_3 \approx 4.3g$$

$$\frac{Q_3 - Q_1}{2} = \frac{4.3 - 2.3}{2} = \frac{2}{2} = 1$$

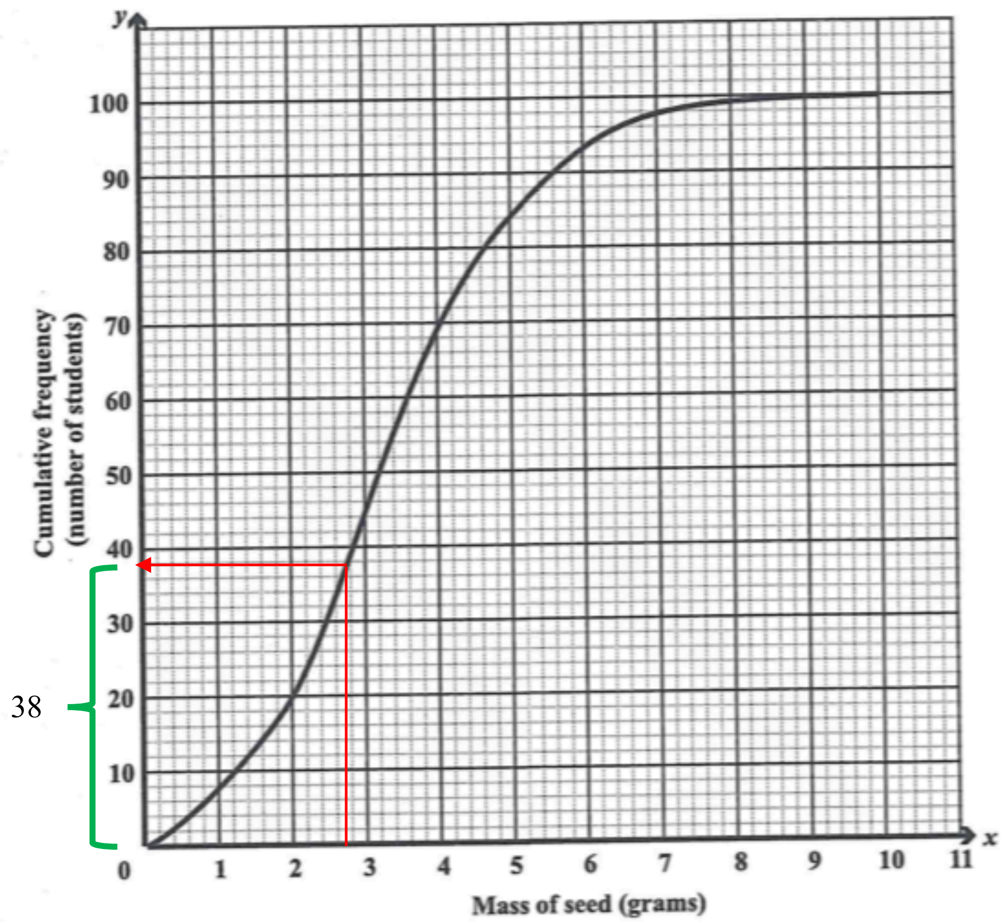
The semi-interquartile range is 1.

(iv) Number of students whose estimate is 2.8 grams or less.

SOLUTION:

The cumulative frequency that corresponds to a mass of 2.8 grams is 38.

Hence 38 students had estimates that were less than 2.8 grams.



(b) (i) Use the cumulative frequency curve given to complete the frequency table below.

Mass of Seed, m (grams)	Frequency
$0 < m \leq 2$	20
$2 < m \leq 4$	
$4 < m \leq 6$	
$6 < m \leq 8$	6
$8 < m \leq 10$	1

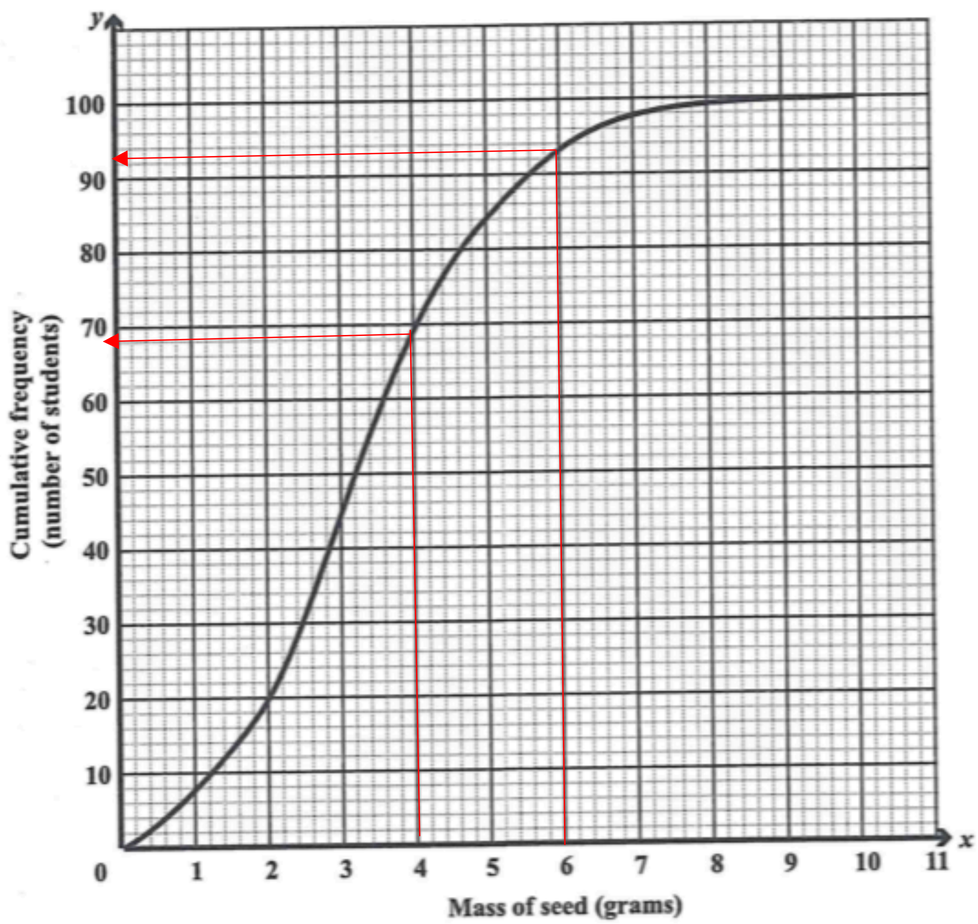
SOLUTION:

The cumulative frequency that corresponds to $m \leq 4$ is 68. Hence, 68 students had estimates that were less than or equal to 4 grams.

Since 20 students had estimates that were less than or equal to 2 grams, the number having estimates between 2 and 4 grams is $68 - 20 = 48$.

The cumulative frequency that corresponds to $m \leq 6$ is 93. Hence, 93 students had estimates that were less than or equal to 6 grams.

Since 68 students had estimates that were less than or equal to 4 grams, the number having estimates between 4 and 6 grams is $93 - 68 = 25$



The completed table is shown below:

Mass of Seed, m (grams)	Frequency
$0 < m \leq 2$	20
$2 < m \leq 4$	48
$4 < m \leq 6$	25
$6 < m \leq 8$	6
$8 < m \leq 10$	1

$$\sum f = 100$$

(b) (ii) A student is chosen at random. Find the probability that the student estimated the mass to be greater than 6 grams.

From the frequency table, the number of students who estimated the mass to be greater than 6 grams is $6 + 1 = 7$.

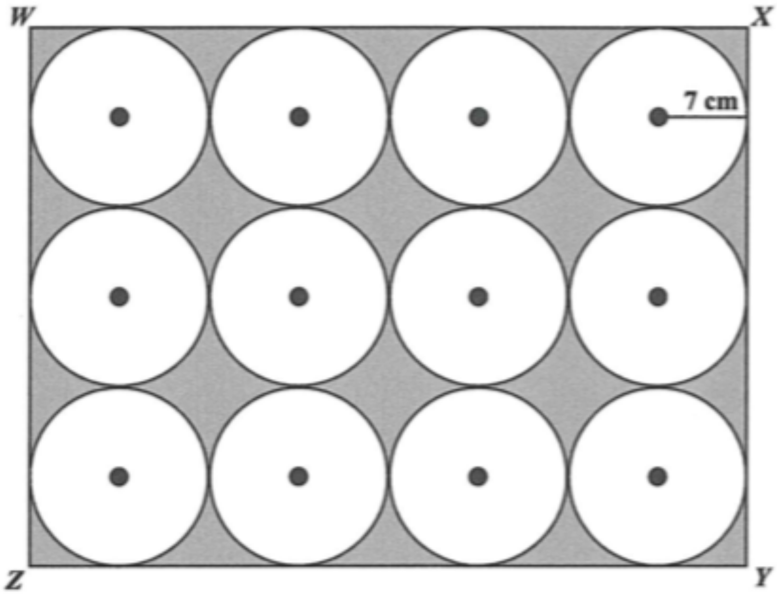
The total number of students is 100.

Probability (estimate is greater than 6 grams)

$$= \frac{Number\ of\ students\ who\ estimated\ the\ mass\ to\ be\ greater\ than\ 6\ g}{Total\ number\ of\ students}$$

$$= \frac{7}{100}$$

6. (a) The radius of EACH circle in the rectangle $WXYZ$ shown below is 7 cm. The circles fit exactly into the rectangle



(ii) Show that the area of the rectangle is 2 352 cm^2.

SOLUTION:

The diameter of the circle $= 7cm \times 2 = 14cm$

The length of the rectangle $= 14cm \times 4 = 56cm$

The width of the rectangle $= 14cm \times 3 = 42cm$

The area of the rectangle $= 56 \times 42cm^2$

$= 2\ 352cm^2$

(iii) Calculate the area of the shaded region.

SOLUTION:

Area of the 12 circles $= 12 \times \pi r^2 = 12 \times \frac{22}{7} \times 7^2$

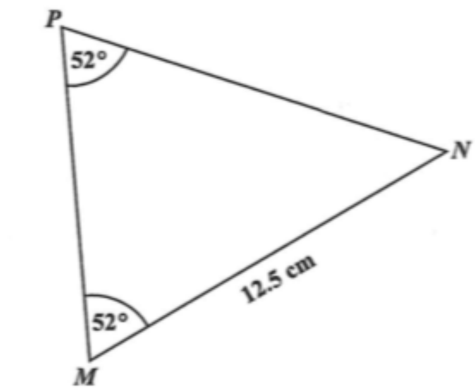
$= 12 \times \frac{22}{7} \times 7 \times 7 = 1\ 848cm^2$

Area of the shaded region = Area of the rectangle – Area of the 12 circles

$= 2\ 352 - 1\ 848\ cm^2$

$= 504cm^2$

6 (b) The diagram below, not drawn to scale, shows triangle *MNP* in which angle *MNP* = angle *PMN* = 52^0 and *MN* = 12.5 cm.



(i) State the type of triangle shown above.

SOLUTION:

Since angle *N* = $180^0 - 2(52)^0 \neq 52^0$, only two angles of triangle *MNP* are equal.

Hence, triangle *MNP* is isosceles.

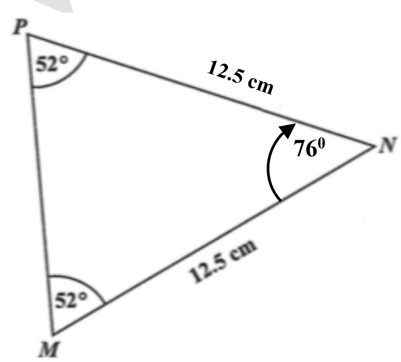
(ii) Determine the value of angle *PNM*.

SOLUTION:

Angle *PNM* = $180^0 - 2(52)^0 = 76^0$ (sum of angles in a triangle is 180^0)

(iii) Calculate the area of triangle *MNP*.

SOLUTION:



Area of triangle *MNP*:

$= \frac{1}{2} \times 12.5 \times 12.5 \times sin76^0$

$= \frac{1}{2} \times 12.5 \times 12.5 \times 0.970$

$= 75.80\ cm^2$

7. A sequence of figures is made up stars, using dots and sticks of different lengths. The first three figures in the sequence are shown below.

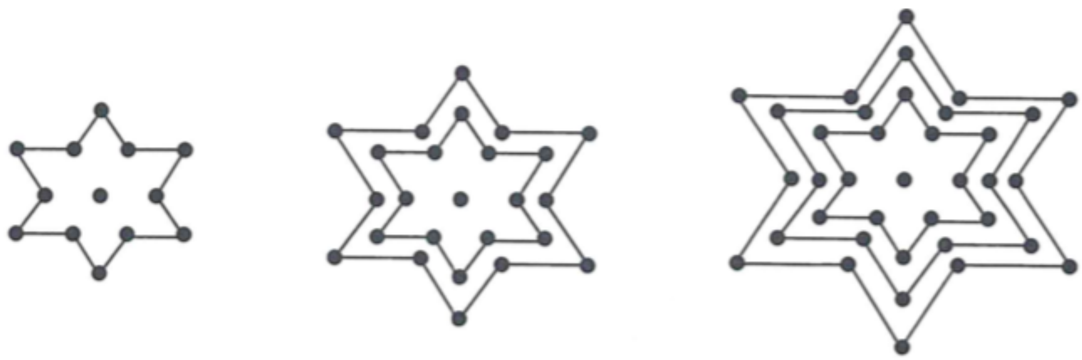


Figure 1 Figure 2 Figure 3

Study the pattern of numbers in each row of the table below. Each row relates to a figure in the sequence of figures stated above. Some rows have not been included in the table.

(a) Complete Rows (i), (ii) and (iii).

	Figure	Number of Sticks (S)	Number of Dots (D)	
	1	12	13	
	2	24	25	
	3	36	37	
	4	48	49	
(i)	5	________	________	(2 marks)
	⋮	⋮	⋮	
(ii)	________	156	________	(2 marks)
	⋮	⋮	⋮	
(iii)	n	________	________	(2 marks)

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SOLUTION:

Part (a)

(i) The number of sticks is 12 times the Figure Number. So, in Figure 5, there will be $5 \times 12 = 60$ sticks.

The number of dots is one more than the number of sticks. Since Figure 5 has 60 sticks it will have $(60 + 1) = 61$ dots

(ii) The Figure Number will be the number of dots divided by 12. If a figure has 156 sticks then, the Figure Number will be $(156 \div 12) = 13$

Hence, the number of dots for Figure 13 is $156 + 1 = 157$.

(iii) The number of sticks is always 12 times the Figure Number. So, the n^{th} figure will have $12n$ sticks.

The number of dots is always one more than the number of sticks. So, the n^{th} figure will have $[(12 \times n) + 1] = 12n + 1$ sticks.

The completed table is shown below:

	Figure	Number of Sticks (S)	Number of Dots (D)
	1	12	13
	2	24	25
	3	36	37
	4	48	49
(i)	5	60	61
	⋮	⋮	⋮
(ii)	13	156	157
	⋮	⋮	⋮
(iii)	n	$12n$	$12n + 1$

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(b) The sum of the number of dots in two consecutive figures is recorded. This information for the first three pairs of consecutive figures is shown in the table below.

Figure 1 and Figure 2	Figure 2 and Figure 3	Figure 3 and Figure 4
$13 + 25 = 38$	$25 + 37 = 62$	$37 + 49 = 86$

Determine the total number of dots in

(i) Figure 7 and 8
(ii) Figure n and Figure $(n + 1)$

SOLUTION:

Part (b)

(i) The total number of dots in Figures 7 and 8

The number of dots in Figure 7 $= 12(7) + 1 = 85$ $(n = 7)$

The number of dots in Figure 8 $= 12(8) + 1 = 97$ $(n = 8)$

Therefore the total number of dots in Figures 7 and 8 $= 85 + 97 = 182$

(iii) The total number of dots in Figure n and Figure $(n + 1)$

The number of dots in Figure n is $12n + 1$
The number of dots in Figure $n + 1 = 12(n + 1) + 1$

Therefore, the total number of dots in Figure n and Figure $(n + 1)$

$$= [12n + 1] + [12(n + 1) + 1]$$

$$= 12n + 1 + 12n + 12 + 1$$

$$= 24n + 14$$

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SECTION II

ALGEBRA, RELATIONS, FUNCTIONS AND GRAPHS

8. (a) Solve the pair of equations:

$y^2 + 2y + 11 = x$

$x = 5 - 3y$

SOLUTION:

Let

$y^2 + 2y + 11 = x$ Eq (1)

$x = 5 - 3y$ Eq (2)

Substitute x from Equation 2 in Equation 1:

$y^2 + 2y + 11 = 5 - 3y$

$y^2 + 2y + 11 - 5 + 3y = 0$

$y^2 + 5y + 6 = 0$

$(y + 3)(y + 2) = 0$

$y + 3 = 0 \ or \ \ y + 2 = 0$

$y = -3 \ or \ y = -2$

When $y = -3, x = 5 - 3(-3) = 5 + 9 = 14$

When $y = -2, x = 5 - 3(-2) = 5 + 6 = 11$

Solutions are: $x = 14 \ and \ y = -3 \ \ OR \ \ x = 11 \ \ and \ y = -2$

(b) The function f is defined as follows:

$$f(x) = 4x^2 - 8x - 2$$

(i) Express $f(x)$ in the form $a(x+h)^2 + k$, where a, h and k are constants.

SOLUTION:

$4x^2 - 8x - 2$

$= 4(x^2 - 2x) - 2$

$= 4[(x-1)^2 - 1] - 2$

$= 4(x-1)^2 - 4 - 2$

$= 4(x-1)^2 - 6$

OR

$4x^2 - 8x - 2$

$= 4(x^2 - 2x) - 2$

$= 4(x-1)^2 + ?$

$4(x-1)(x-1) = 4(x^2 - 2x + 1) = 4x^2 - 8x + 4$

The constant in the original expression is -2. Hence, $4 + ? = -2$ and $? = -6$

So, $4x^2 - 8x - 2 = 4(x-1)^2 - 6$

OR

Expanding and equating coefficients;

$a(x+h)^2 + k = a(x^2 + 2xh + h^2) + k$

$= ax^2 + 2xah + ah^2 + k$

$4x^2 - 8x - 2 = ax^2 + 2xah + ah^2 + k$

Equating coefficients:

$a = 4$	$2ah = -8$	$ah^2 + k = -2$
	$2(4) \times h = -8$	$4(-1)^2 + k = -2$
	$h = -1$	$k = -6$

So, $a(x+h)^2 + k = 4(x-1)^2 - 6$

(ii) State the minimum value of $f(x)$

$$f(x) = 4(x-1)^2 - 6$$

SOLUTION:

$4(x-1)^2 \geq 0$, for all values of x.

$f(x)_{min} = 0 - 6 = -6$

(iii) Determine the equation of the axis of symmetry.

The axis of symmetry passes through the minimum point.

This occurs when $x - 1 = 0$, or $x = 1$

OR

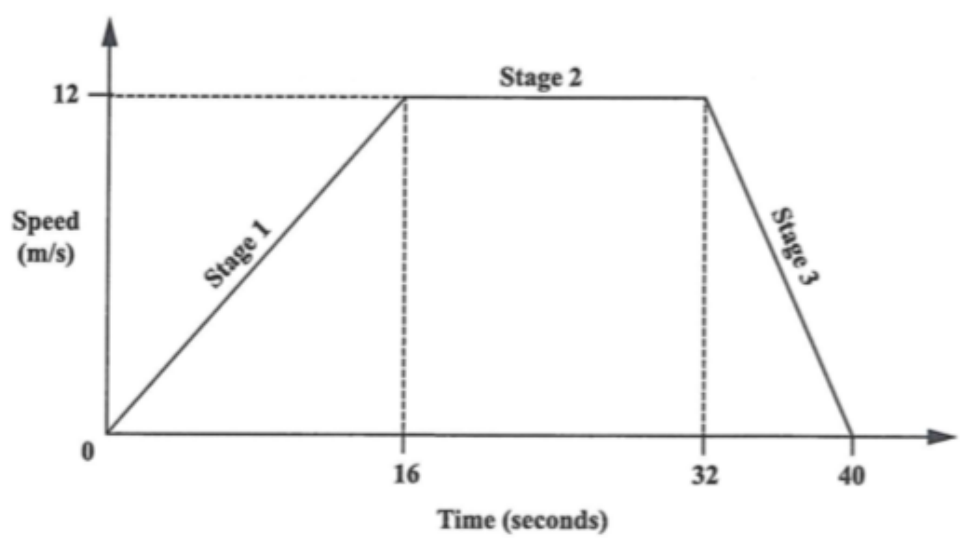
The equation of the axis of symmetry is $x = \frac{-b}{2a}$, for a quadratic of the form $f(x) = ax^2 + bx + c$

For the function, $f(x) = 4x^2 - 8x - 2$, $a = 4, b = -8$

$x = \frac{-(-8)}{2(4)} = 1$

The equation of the axis of symmetry is $x = 1$.

(c) The speed-time graph below, not drawn to scale, shows the three-stage journey of a car over a period of 40 seconds.



Determine the acceleration of the car for EACH of the following stages of the journey

Stage 2

Stage 3

SOLUTION:

Acceleration of the car in Stage 2 of the journey

Stage 2 of the journey is represented by a horizontal line. This indicates that the velocity is constant.

Since the velocity is not changing, the rate of change of velocity is zero.

Therefore the acceleration, $a = 0\ m/s^2$.

OR

The acceleration can be calculated by finding the gradient of the line. A horizontal line has a gradient of zero. By calculation,

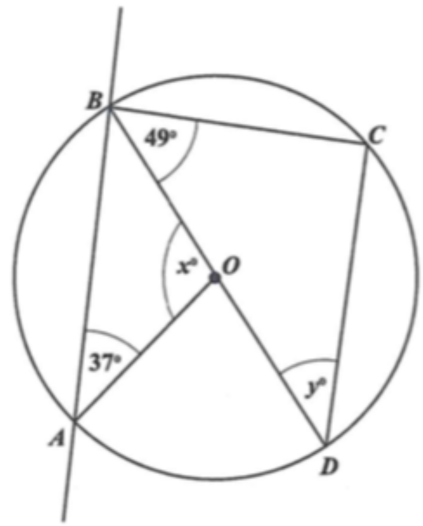
Gradient $= \frac{12-12}{32-16} = 0$.

Acceleration of the car in Stage 3 of the journey

Gradient $= \frac{12-0}{32-40} = \frac{12}{-8} = -\frac{3}{2}\ m/s^2$

The car is decelerating (slowing down). The acceleration is $-1.5\ m/s^2$ or the deceleration is $1.5\ m/s^2$.

9. (a) The circle shown below has center O and the points A, B, C and D lying on the circumference. A straight line passes through the points A and B. Angle $CBD = 49^0$ and angle $OAB = 37^0$.



(i) Write down the mathematical names of the straight lines BC and OA.

SOLUTION:

The straight line, BC, joins two points on the circle and it is therefore a chord of the circle.

The straight line OA joins the center of the circle to a point on the circle and is therefore a radius of the circle.

(ii) Determine the value of EACH of the following angles. Show detailed working where necessary and give a reason to support your answer.
a) x
b) y

SOLUTION:

Consider triangle OAB
$OA = OB$ (radii)

Hence, triangle OAB is isosceles.
Angle $OBA = 37^0$ (Base angles of an isosceles triangle are equal)

$x^0 = 180^0 - (37 + 37)^0$ (Sum of angles in a triangle is 180^0)

$x^0 = 180^0 - 74^0$
$x = 106$

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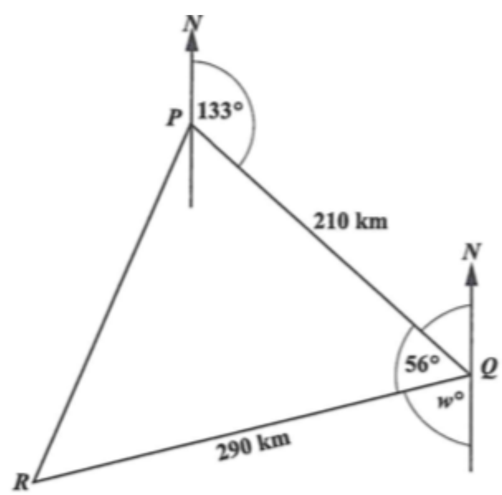
Consider triangle BCD

Angle $BCD = 90^0$ (Angle in a semi-circle is a right angle)

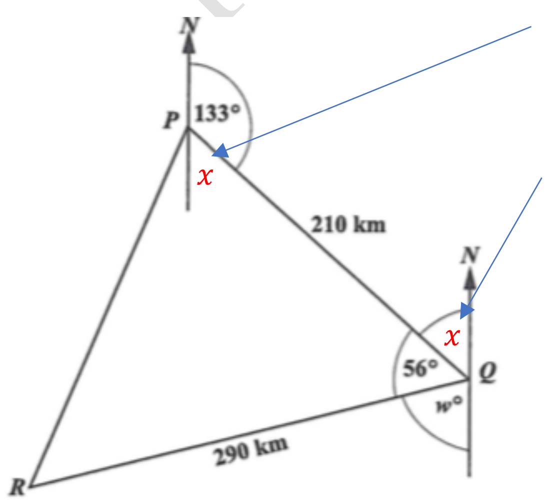
$y^0 = 180^0 - (90^0 + 49^0)$ (Sum of angles in a triangle is 180^0)
$y^0 = 180^0 - 139^0$
$y = 41$

(b) The diagram below, not drawn to scale, shows the route of a ship cruising from Palmcity (P) to Quayton (Q) and then to Rivertown (R). The bearing of Q from P is 133^0 and the angle PQR is 56^0.



(i) Calculate the value of the angle w.

SOLUTION:

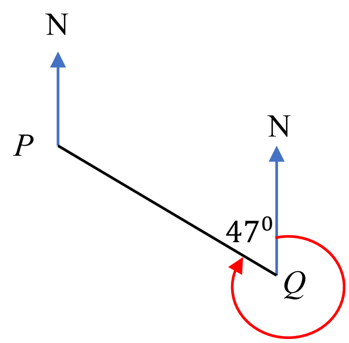


Let this angle be x.
$x = 180^0 - 133^0 = 47^0$
(Angles on a straight line)

The two North lines are parallel, hence, this angle is also equal to x (alternate angles).
Therefore,

$w^0 = 180^0 - (56 + 47)^0$
$w = 77$

(ii) Determine the bearing of P from Q.



Bearing of P from Q = 313^0

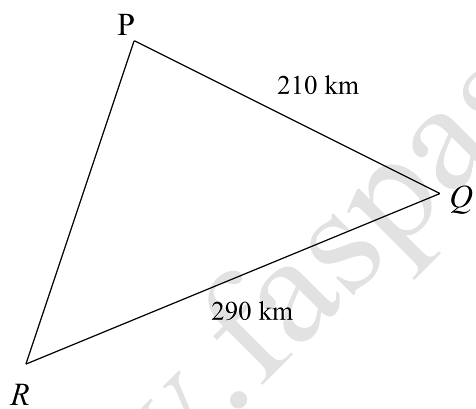
The angle that represents the bearing of P from Q is show in the diagram.

It is calculated as

$$360^0 - 47^0 = 313^0$$

(iii) Calculate the distance RP.

Applying the cosine rule to triangle PQR we have:



$$RP^2 = 210^2 + 290^2 - 2(210)(290)cos56^0$$

$$= 44\ 100 + 84\ 100 - 121\ 800 \times 0.559$$

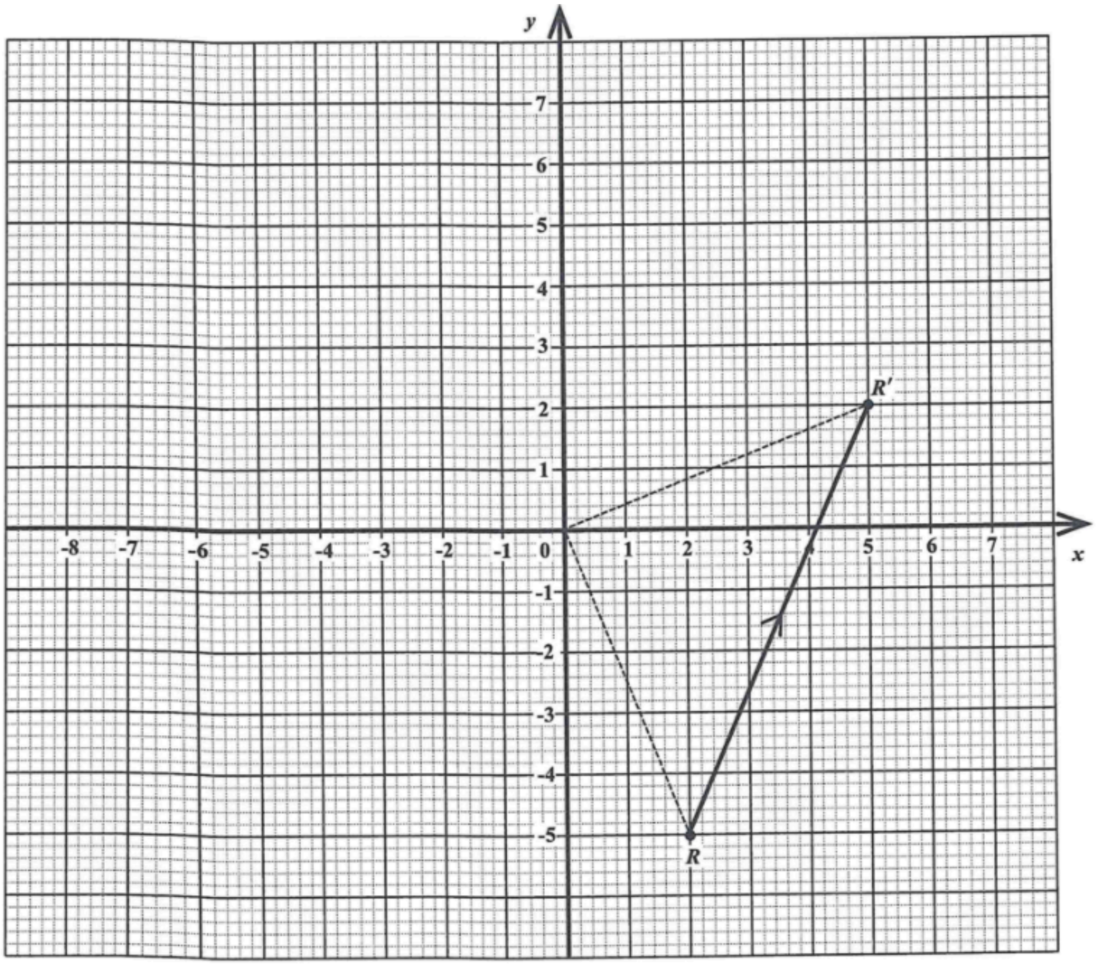
$$= 128\ 200 - 68\ 109.70$$

$$= 60\ 090.30$$

$$RP = \sqrt{60\ 090.30}$$

$$RP = 245.13\ km$$

10. (a) The transformation, $M = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$ maps the point R onto R' as shown in the diagram below.



(i) Determine the value of p and of q.

SOLUTION:

The transformation, $M = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$ maps $R(2, -5)$ onto $R'(5,2)$. Hence,

$$\underset{2\times 2}{\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} 2 \\ -5 \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} 5 \\ 2 \end{pmatrix}}$$

$$\begin{pmatrix} 0 \times 2 + p \times -5 \\ q \times 2 + 0 \times -5 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -5p \\ 2q \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

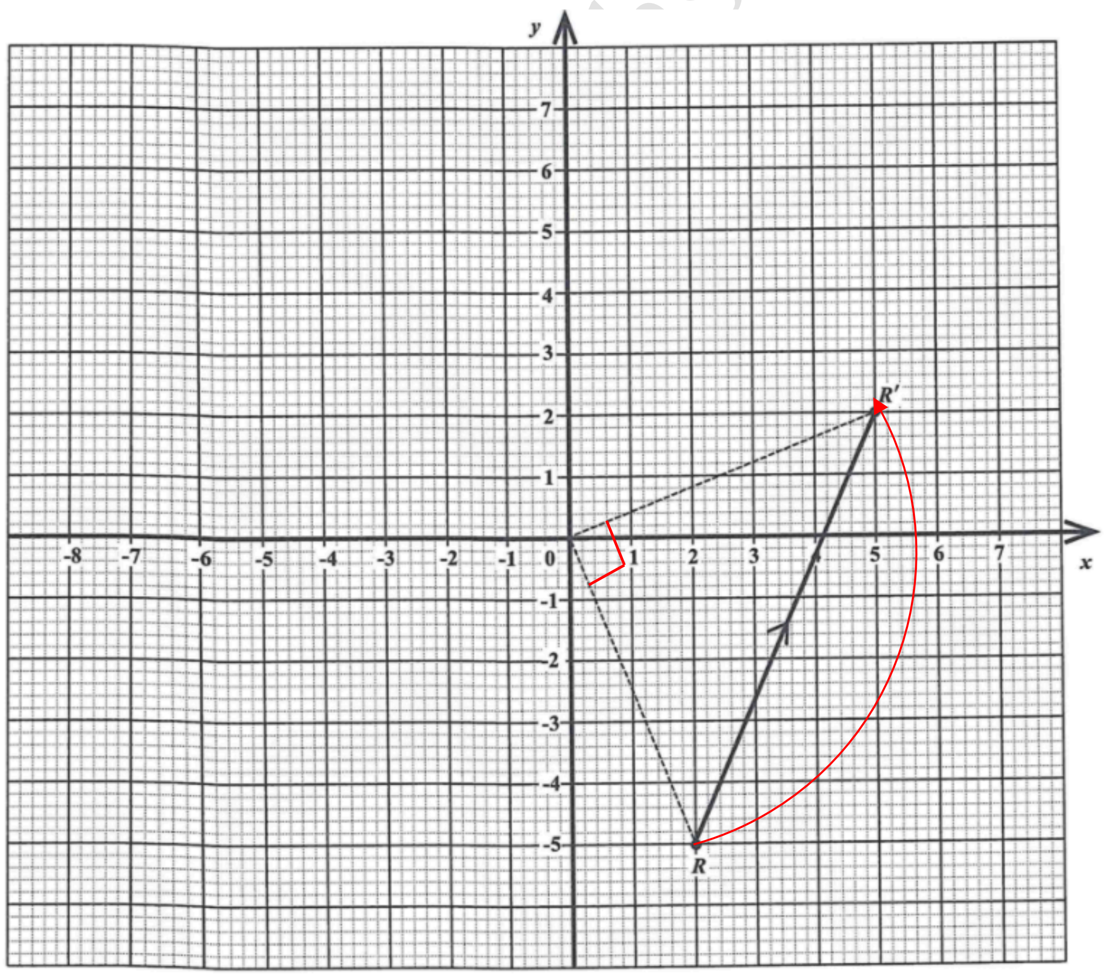
Equating corresponding elements, $-5p = 5$ and $2q = 2$

So, $p = -1$ and $q = 1$

(ii) Describe fully the transformation, *M*.

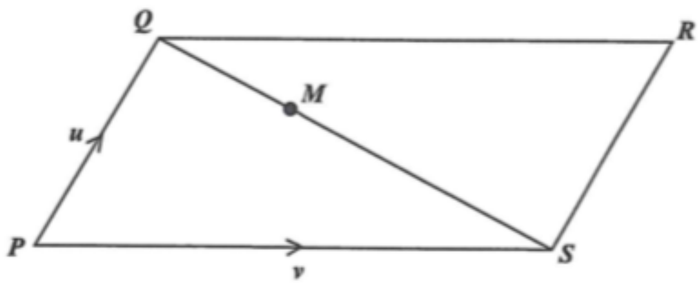
SOLUTION:

The transformation, $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, is an anticlockwise (or positive) rotation about the origin through 90^0.



(b) $PQRS$ is a parallelogram in which $\overrightarrow{PQ} = u$ and $\overrightarrow{PS} = v$.

M is a point on QS such that $QM: MS = 1:2$



(i) Write in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$ an expression for

a) $\overrightarrow{QS}$

SOLUTION:

$\overrightarrow{QS} = \overrightarrow{QP} + \overrightarrow{PS} = -(\boldsymbol{u}) + \boldsymbol{v}$

$\overrightarrow{QS} = -\boldsymbol{u} + \boldsymbol{v}$

b) $\overrightarrow{QM}$

SOLUTION:

$QM: MS = 1:2$

Therefore, $QM = \frac{1}{3}QS$

$\overrightarrow{QM} = \frac{1}{3}(-\boldsymbol{u} + \boldsymbol{v})$

$\overrightarrow{QM} = -\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v}$

(ii) Show that $\overrightarrow{MR} = \frac{1}{3}(\boldsymbol{u} + 2\boldsymbol{v})$

SOLUTION:

$\overrightarrow{MR} = \overrightarrow{MQ} + \overrightarrow{QR}$

$\overrightarrow{MR} = -\overrightarrow{QM} + \overrightarrow{QR}$

$= -\left(-\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v}\right) + \boldsymbol{v}$ $\qquad [\overrightarrow{QR} = \overrightarrow{PS} = \boldsymbol{v}]$

$= \frac{1}{3}\boldsymbol{u} - \frac{1}{3}\boldsymbol{v} + \boldsymbol{v}$

$= \frac{1}{3}\boldsymbol{u} + \frac{2}{3}\boldsymbol{v}$

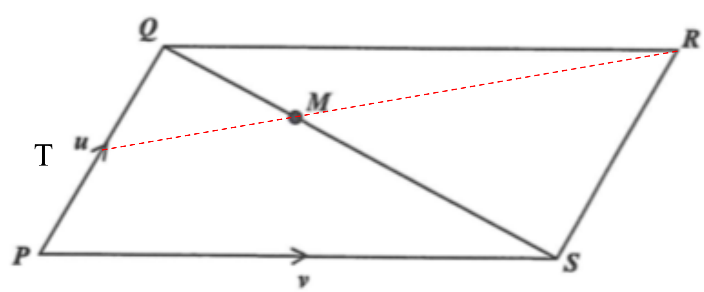
$= \frac{1}{3}(\boldsymbol{u} + 2\boldsymbol{v})$

Q.E.D.

(iii) *T* is the mid-point of *PQ*. Prove that *R*, *M* and *T* are collinear.

SOLUTION:

To prove that *R*, *M* and *T* are collinear, consider the vectors $\overrightarrow{TR}$ and $\overrightarrow{MR}$.



$\overrightarrow{TR} = \overrightarrow{TQ} + \overrightarrow{QR}$

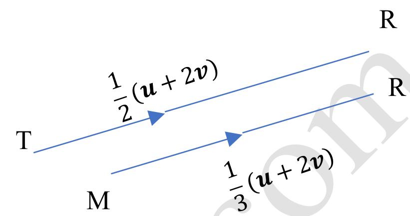
$\overrightarrow{TR} = \frac{1}{2}\boldsymbol{u} + \boldsymbol{v}$

$\overrightarrow{TR} = \frac{1}{2}(\boldsymbol{u} + 2\boldsymbol{v})$

We know from part (ii) that

$$\overrightarrow{MR} = \frac{1}{3}(\boldsymbol{u} + 2\boldsymbol{v})$$

$$\overrightarrow{MR} = \frac{1}{3}(\boldsymbol{u} + 2\boldsymbol{v}) = \frac{2}{3}\left\{\frac{1}{2}(\boldsymbol{u} + \boldsymbol{2v})\right\}$$



$$\overrightarrow{MR} = \frac{2}{3}\overrightarrow{TR}$$

The vector $\overrightarrow{MR}$ is a scalar multiple of $\overrightarrow{TR}$ (The scalar multiple being $\frac{2}{3}$).

Hence, $\overrightarrow{MR}$ is parallel to $\overrightarrow{TR}$.

In both line segments, R is a common point.

Therefore, R, M and T are collinear.

Q.E.D.