FAS-PASS Maths

CSEC MATHEMATICS JANUARY 2019 PAPER 2

SECTION I

1. (a) Evaluate

(i) $3.8\times10^2+1.7\times10^3$, giving your answer in standard form

SOLUTION:
Required to evaluate: $3.8\times10^2+1.7\times10^3$ and write the answer in standard form
Solution:

$$
\begin{aligned}
3.8\times10^2+1.7\times10^3 &= 0.38\times10\times10^2+1.7\times10^3 \\
&= 0.38\times10^3+1.7\times10^3 \\
&= (0.38+1.7)\times10^3 \\
&= 2.08\times10^3
\end{aligned}
$$

expressed in standard form or in scientific notation, as required.

(ii) $\dfrac{\frac{1}{2}\times\frac{3}{5}}{3\frac{1}{2}}$, giving your answer as a fraction in its lowest terms.

SOLUTION:
Required to evaluate:

$$\frac{\frac{1}{2}\times\frac{3}{5}}{3\frac{1}{2}}$$

and give the answer as a fraction in its lowest terms
Solution:
Numerator:

$$\frac{1}{2}\times\frac{3}{5}=\frac{3}{10}$$

Denominator:

$$3\frac{1}{2}=\frac{(3\times2)+1}{2}=\frac{7}{2}$$

So,

$$\frac{\frac{1}{2}\times\frac{3}{5}}{3\frac{1}{2}}=\frac{\frac{3}{10}}{\frac{7}{2}}=\frac{3}{10}\times\frac{2}{7}=\frac{3}{35}$$

as a fraction in its lowest terms.

(b) Express the number 6 as a binary number.

SOLUTION:
Required to express: 6 as a binary number
Solution:
Binary means that the base is 2.

$$\begin{array}{r|l l} 2 & 6 & \\ \hline 2 & 3 & \text{R } 0 \\ \hline 2 & 1 & \text{R } 1 \\ \hline & 0 & \text{R } 1 \end{array}$$

Reading from the bottom upwards, $6_{10} = 110_2$.
Re-checking,

$$\begin{aligned} 110_2 &= (1\times 2^2)+(1\times 2^1)+(0\times 2^0) \\ &= (1\times 4)+(1\times 2)+(0\times 1) \\ &= 4+2+0 \\ &= 6_{10} \end{aligned}$$

(c) John bought a car for \$65 000. If the value of the car depreciates by 8% each year, how much will the car be worth at the end of 2 years?

SOLUTION:
Data: John bought a car for \$65 000 and its value depreciates by 8% each year.
Required to calculate: The value of the car at the end of 2 years
Calculation:
Initial cost of the car is \$65 000.

Depreciation after 1 year is 8% $= \frac{8}{100} \times \$65\,000$
$= \$5\,200$

Value of the car after 1 year $= \$65\,000 - \$5\,200$
$= \$59\,800$

Depreciation after second year $= 8\% \text{ of } \$59\,800$
$= \frac{8}{100} \times \$59\,800$
$= \$4\,784$

Value of the car after the second year $= \$59\,800 - \$4\,784 = \$55\,016$

Alternative Method:

$A = P\left(1 + \frac{R}{100}\right)^n$, where $P =$ the original cost of the car, $R =$ rate of depreciation, $n =$ the number of years, $A =$ the value of the car.

$$A = 65\,000(1 - \tfrac{8}{100})^2$$
$$= 65\,000(0.92)^2$$
$$= \$55\,016$$

(d) The table below shows the results obtained by a student in her CSEC Mathematic examination. The maximum mark for each paper is given in the third column of the table.

Paper	**Percentage Obtained**	**Maximum Mark for Paper**
01	55	30
02	60	50
03	80	20
Total		**100**

Determine, as a percentage, the student's final mark for the Mathematics examination.

SOLUTION:
Data: Table showing the results obtained, as percentages, by a student in each of the three papers in CSEC Mathematics examinations.
Required to determine: The student's final mark as a percentage
Solution:

Actual mark obtained in Paper 01 is $\frac{55}{100} \times 30 = 16.5$

Actual mark obtained in Paper 02 is $\frac{60}{100} \times 50 = 30$

Actual mark obtained in Paper 03 is $\frac{80}{100} \times 20 = 16$

Total mark obtained in Paper 01, Paper 02, and Paper 03 $= 16.5 + 30 + 16$
$= 62.5$

Since the mark is out of 100, the percentage mark is the same as the mark since $\frac{62.5}{100} \times 100 = 62.5\%$

2. (a) (i) Make x the subject of the formula

$$y = \frac{x}{5} + 3p$$

SOLUTION:

Data: $y = \frac{x}{5} + 3p$

Required to make: x the subject

Solution:

$$y = \frac{x}{5} + 3p$$

$$\frac{x}{5} + 3p = y$$

$$\frac{x}{5} = y - 3p$$

$$(\times 5)$$

$$x = 5(y - 3p)$$

(ii) Solve the following equation by factorisation.

$$2x^2 - 9x = 0$$

SOLUTION:

Data: $2x^2 - 9x = 0$

Required to find: x

Solution:

$$2x^2 - 9x = 0$$

$$2x \times x - 9 \times x = 0$$

$$x(2x - 9) = 0$$

So $x = 0$ or $2x - 9 = 0$

$$2x = 9$$

$$x = \frac{9}{2}$$

(b) A farmer wishes to enclose a rectangular plot with a wire fence. The width of the plot is 3 metres less than its length, l.

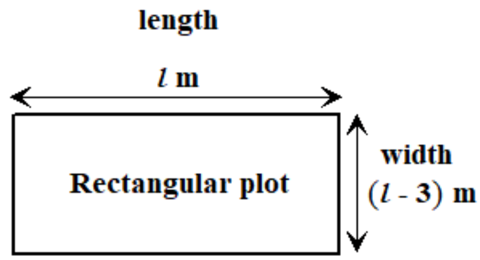
Given that the area enclosed by the fence is 378 square metres, show that $l^2 - 3l - 378 = 0$.

SOLUTION:

Data: The width of a rectangular plot of land is 3 metres less than its length, l. The area of the plot is 378 square metres.

Required to show: $l^2 - 3l - 378 = 0$

Proof:



Since the width is 3 metres less than the length, then the width is $(l-3)$m.

Area $=$ length $\times$ width

$= l \times (l-3)$ m^2

$= (l^2 - 3l)$ m^2

The area is 378 m^2.

So $l^2 - 3l = 378$

and $l^2 - 3l - 378 = 0$

Q.E.D

(c) The force, F, applied to an object is directly proportional to the extension, e, produced by that object.

(i) Represent this information as an equation in terms of F, e and an appropriate constant, k.

SOLUTION:

Data: Force, F, of an object is directly proportional to the extension, e.

Required to write: An equation to represent this information

Solution:

Force is proportional to extension.

F $\propto$ e

$F \propto e$

$F = k \times e$ (where k is the constant of proportionality)

(ii) The incomplete table below shows the corresponding values of F and e.

F	8	25	60	y
e	0.2	x	1.5	3.2

Using the equation obtained in (c)(i), or otherwise, determine the value of x and y.

SOLUTION:
Data: Table showing corresponding values of F and e.
Required to find: The value of x and of y
Solution:

$F = 8$ when $e = 0.2$

$F = ke$

So $8 = k \times 0.2$

$$k = \frac{8}{0.2}$$

$k = 40$

Hence, $F = 40e$

(The same result for k would have been obtained if we had substituted $F = 60$ when $k = 1.5$.)

When $F = 25,\ e = x$

So $25 = 40x$

$$x = \frac{25}{40}$$

$= 0.625$

When $F = y,\ e = 3.2$

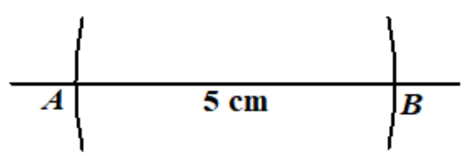
So $y = 40 \times 3.2$

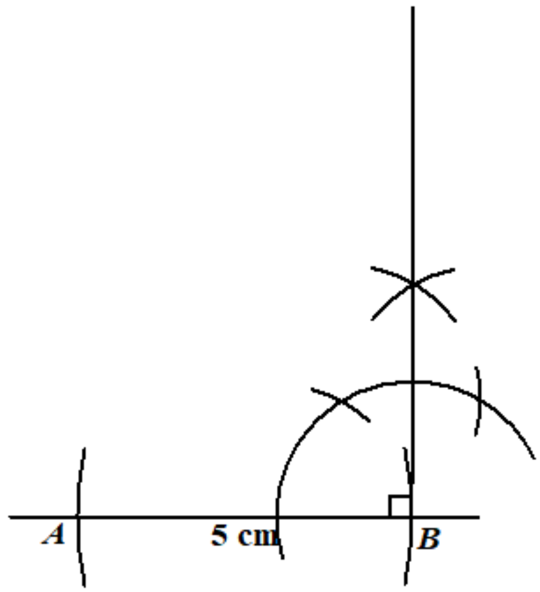
$y = 128$

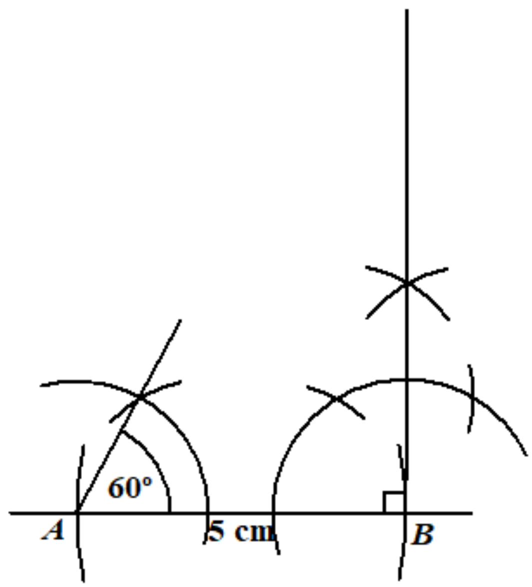
$\therefore x = 0.625$ and $y = 128$

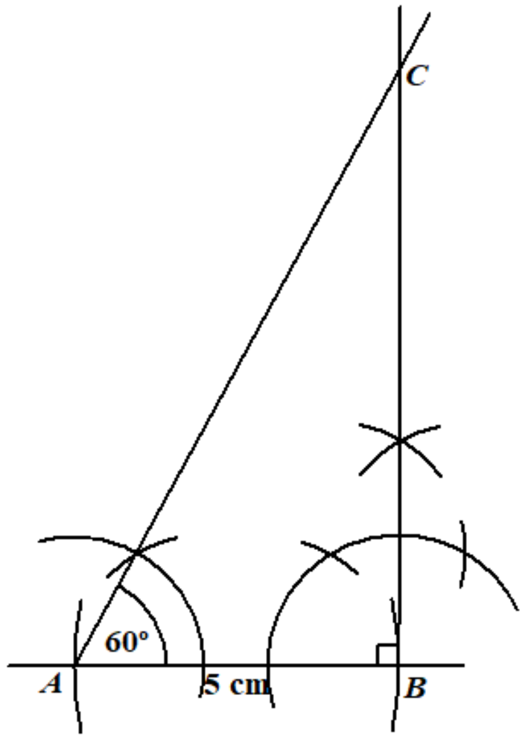
3. (a) Using a ruler, a pencil and a pair of compasses, construct the **right-angled** triangle ABC such that $AB = 5$ cm, $\angle ABC = 90°$ and $\angle BAC = 60°$.

SOLUTION:
Required to construct: The right-angled triangle ABC with $AB = 5$ cm, $\angle ABC = 90°$ and $\angle BAC = 60°$.
Solution:

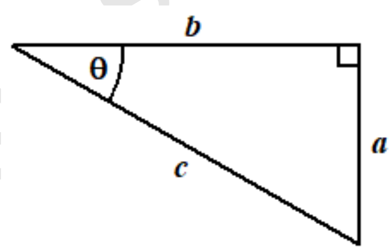



A
5 cm
B


60°
A
5 cm
B



(b) The diagram below shows a right-angled triangle with sides *a* units, *b* units and *c* units.



(i) Using the diagram,

a) Express *c* in terms of *a* and *b*

SOLUTION:
Data: Diagram showing right-angled triangle with sides *a* units, *b* units and *c* units.
Required to express: *c* in terms of *a* and *b*.
Solution:

$c^2 = a^2 + b^2$ (Pythagoras' Theorem)

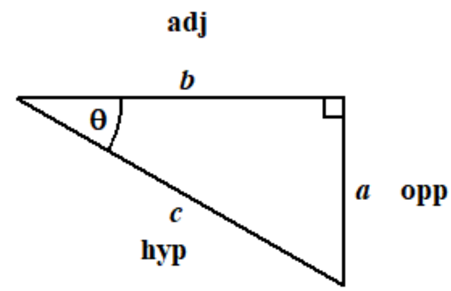
$c = \sqrt{a^2 + b^2}$ (positive root taken)

b) Write, in terms of *a*, *b* and *c*, an expression for $\sin\theta + \cos\theta$.

SOLUTION:

Required to write: $\sin\theta + \cos\theta$, in terms of *a*, *b* and *c*.

Solution:



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{a}{c} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c}$$

$$\sin\theta + \cos\theta = \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

(ii) Using the results from (i) a) and b), show that $(\sin\theta)^2 + (\cos\theta)^2 = 1$.

SOLUTION:

Required to show: $(\sin\theta)^2 + (\cos\theta)^2 = 1$

Proof:

$$\sin^2\theta + \cos^2\theta = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$$

$$= \frac{a^2}{c^2} + \frac{b^2}{c^2}$$

$$= \frac{a^2 + b^2}{c^2} \qquad \text{Re}: c^2 = a^2 + b^2$$

$$= \frac{c^2}{c^2}$$

$$= 1$$

Q.E.D.

4. (a) Given the function $h(x)=\dfrac{2x+3}{5-x}$, determine

(i) the value of x for which the function is undefined

SOLUTION:
Data: $h(x)=\dfrac{2x+3}{5-x}$
Required to find: The value of x for which $h(x)$ is undefined

Solution:
When $x=5$

$$h(5)=\frac{2(5)+3}{5-5}$$
$$=\frac{10+3}{0}$$

Recall that $\dfrac{N}{0}$ is undefined for all values of N and is equal to ∞.
So the function is undefined when $x=5$.

(ii) an expression for $h^{-1}(x)$.

SOLUTION:
Required to find: $h^{-1}(x)$
Solution:
Let $y=h(x)$

$$\therefore y=\frac{2x+3}{5-x}$$

Making x the subject:

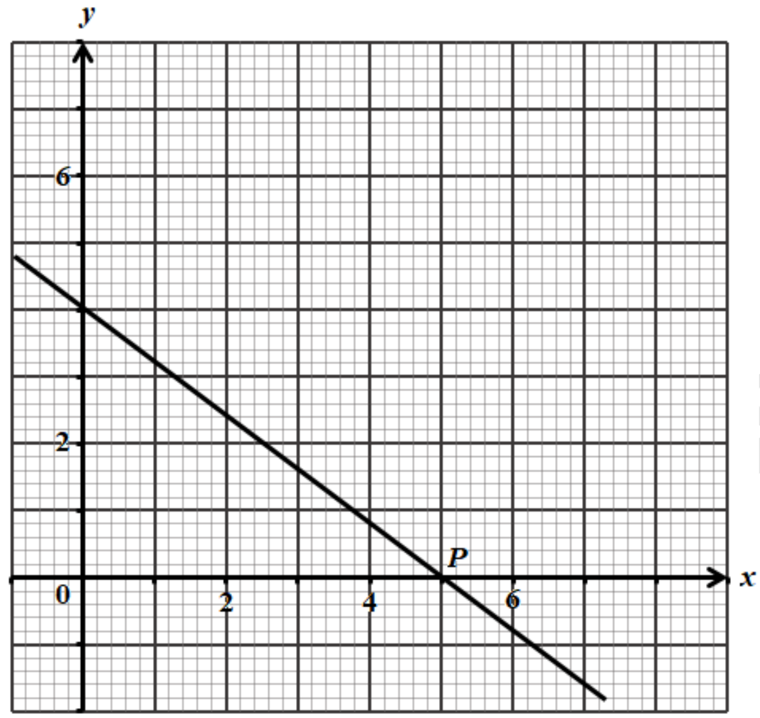
$$y(5-x)=2x+3$$
$$5y-xy=2x+3$$
$$5y-3=2x+xy$$
$$5y-3=x(2+y)$$

So $x=\dfrac{5y-3}{2+y}$

Replace y by x to get:

$$h^{-1}(x)=\frac{5x-3}{2+x},\ x\neq -2$$

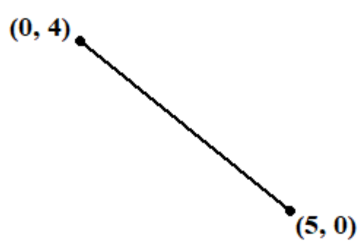
(b) The graph below shows a straight line intersecting the x and y axes.



Using the graph, determine the

(i) gradient of the line

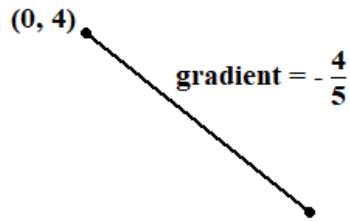
SOLUTION:
Data: The graph of a straight line intersecting the x and y axes.
Required to find: The gradient of the line
Solution:
The graph cuts the y – axis at $(0,\ 4)$ and the x – axis at $(5,\ 0)$.



The gradient of the line is $\dfrac{4-0}{0-5}=-\dfrac{4}{5}$

(ii) Equation of the line

SOLUTION:
Required to find: The equation of the line
Solution:

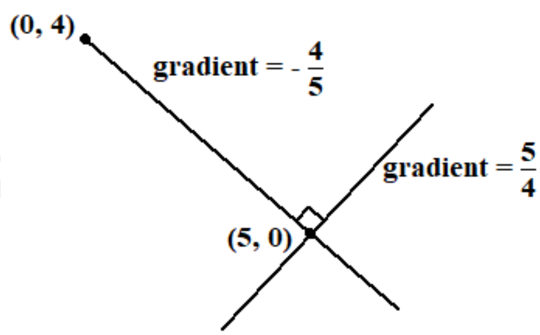


The equation of a straight line is of the form $y = mx + c$, where $m =$ gradient and $c =$ intercept on the $y -$ axis.

Hence, the equation of the line is $y = -\frac{4}{5}x + 4$ or any other equivalent form.

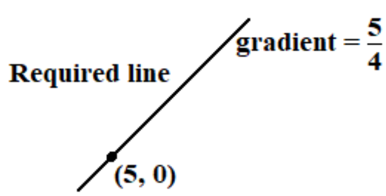
(iii) the equation of the perpendicular line that passes through *P*.

SOLUTION:
Required to find: The equation of a perpendicular line on which *P* lies
Solution:



The gradient of any perpendicular to the given line given is $\frac{-1}{-\frac{4}{5}} = \frac{5}{4}$.

(The product of the gradients of perpendicular lines $= -1$)



The equation of the perpendicular to the given line and which passes through P is

$$\frac{y-0}{x-5}=\frac{5}{4}$$

$$4(y-0)=5(x-5)$$

$$4y=5x-25$$ or any other equivalent form.

5. (a) A survey was conducted among 48 persons to find out what mobile network they used.

The table below shows the results of the survey.

Mobile Network	WireTech	DigiLec	O-Fone	Other
Number of Persons	20	12	10	6

(i) If this information is to be represented on a pie chart, what is the angle for the sector that will represent O-Fone?

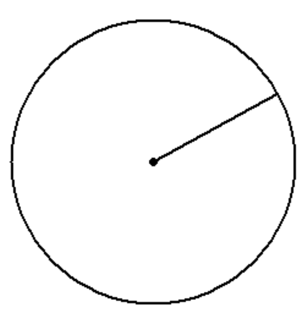
SOLUTION:
Data: Table showing the results after 48 persons were surveyed about the mobile network they used.
Required to find: The angle of the sector representing O-Fone on a pie chart
Solution:
The angle of the sector to represent O-Fone

$$=\frac{\text{Number of persons who use O-Fone}}{\text{Total number of persons in the survey}}\times 360^\circ$$

$$=\frac{10}{48}\times 360^\circ$$

$$=75^\circ$$

(ii) Using the circle below, with radius shown, represent the information in the table above on a clearly labelled pie chart.



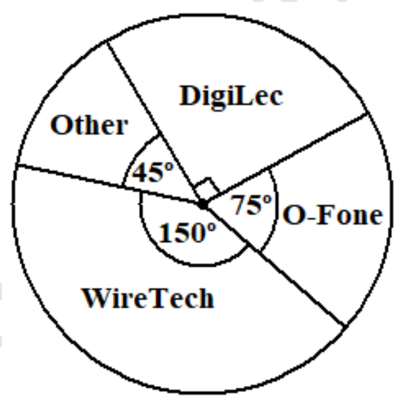
SOLUTION:
Required to represent: The information given in the table as a pie chart
Solution:

Similarly, the angle of the sector to represent WireTech $= \frac{20}{48} \times 360°$
$= 150°$

The angle of the sector to represent DigiLec $= \frac{12}{48} \times 360°$
$= 90°$

The angle of the sector to represent Other $= \frac{6}{48} \times 360°$
$= 45°$

OR The angle of the sector representing Oth $= 360° - (150° + 75° + 90°)$
$= 45°$



(b) The incomplete table below shows the results obtained 200 boys and 250 girls on a Spanish examination.

Grade	Boys	Girls
I		62
II		75
III		90
IV	30	
V	8	
Total	**200**	**250**
Standard deviation	8.2	6.3

(i) A girl is chosen at random. Determine the probability that she achieves a Grade I.

SOLUTION:
Data: Table showing the grades obtained by 200 boys and 250 girls in a Spanish examination.
Required to find: The probability a girl chosen at random achieves a Grade I
Solution:

$$P(\text{girl achieves Grade I}) = \frac{\text{Number of girls who achieved a Grade I}}{\text{Total number of girls}}$$

$$= \frac{62}{250}$$

$$= \frac{31}{125}$$

(ii) What percentage of the boys who took the exam achieved Grades I to III?

SOLUTION:
Required to find: The percentage of boys who achieved Grades I to III
Solution:

The number of boys who achieved Grades I to III $= 200 - (30 + 8)$

$$= 162$$

So, the percentage $= \frac{162}{200} \times 100\%$

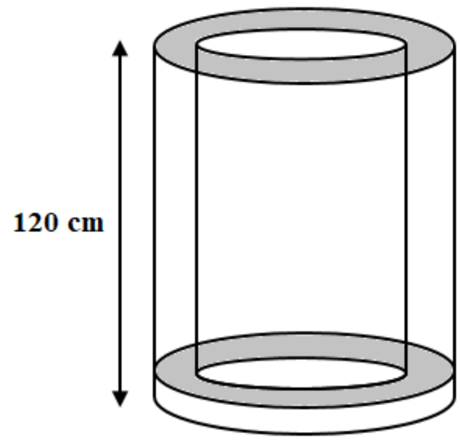
$$= 81\%$$

(iii) Considering the standard deviations in the table, compare the performance of the boys and the girls.

SOLUTION:
Required to compare: The performance of the boys and the girls
Solution:
The standard deviation of the boys' performances is larger than the standard deviation of the girls' performances. Since standard deviation is a measure of the spread of the data it means that the marks of the boys were more 'spread out' or distributed over its range than the marks of the girls. The marks of the girls were more concentrated over a smaller range of values.

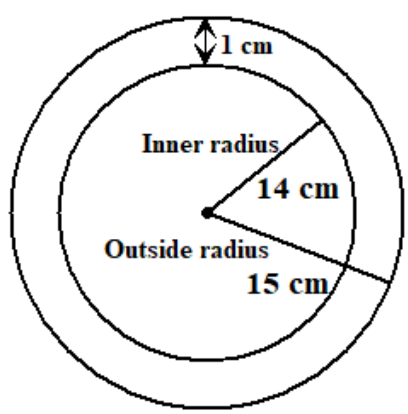
6. The diagram below, **not drawn to scale**, shows an open cylindrical container made of metal with a circular base and **uniform thickness throughout**. The length of the container, from the top to outer bottom, is 120 cm and the inner and outer radii are 14 cm and 15 cm respectively.

Take π to be $\frac{22}{7}$



(a) Draw a cross-sectional view of the container showing the measurements of the inner and the outer radii.

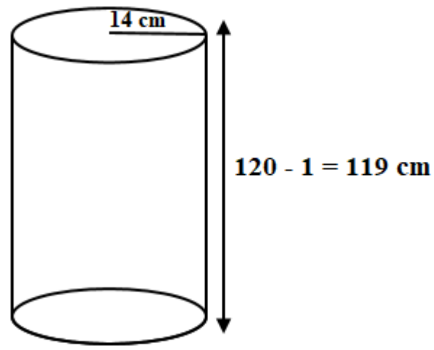
SOLUTION:
Data: Diagram showing an open cylindrical container of uniform thickness with a height of 120 cm and inner and outer radii 14 cm and 15 cm respectively.
Required to draw: The cross-sectional view of the container showing the inner and outer radii
Solution:



(b) Show that the capacity of the container is 73 304 cm^3.

SOLUTION:
Required to show: The capacity of the container is 73 304 cm^3.

Proof:



$$\text{Volume } = \pi r^2 h$$
$$= \pi \times 14 \times 14 \times 119 \text{ cm}^3$$
$$= \frac{22}{7} \times 14 \times 14 \times 119 \text{ cm}^3$$
$$= 73304 \text{ cm}^3$$

Q.E.D

(c) Determine the volume of the material used to make the container.

SOLUTION:
Required to determine: The volume of material used to make the container
Solution:

$$\text{External volume of the cylinder } = \pi \times (15)^2 \times 120 \text{ cm}^3$$
$$= 84857.142 \text{ cm}^3$$

Volume of material used
$= \text{External volume of the container} - \text{Internal volume of the container}$
$= 84857.412 - 73304 \text{ cm}^3$
$= 11553.14\underline{\underline{2}} \text{ cm}^3$
$\approx 11553.14 \text{ cm}^3$ (correct to 2 decimal places)

(d) Given that the density of the material used to make the container is 2.2 g/cm^3, determine the mass, in kg, of the empty container.

$$\left[\text{density} = \frac{\text{mass}}{\text{volume}}\right]$$

SOLUTION:
Data: The density of the material used to make the container is 2.2 g/cm^3.
Required to determine: The mass, in kg, of the empty container
Solution:

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

$$\therefore \text{ Mass} = \text{Density} \times \text{Volume}$$

$$= 2.2 \text{ gcm}^{-3} \times 11553.142 \text{ cm}^3$$

$$= 25\,416.91 \text{ g}$$

$$= \frac{25\,416.91}{1000} \text{ kg}$$

$$= 25.4 \text{ kg}$$

7. A sequence of figures is made from joining polygons with sides of unit length. The first three figures in the sequence are shown below.

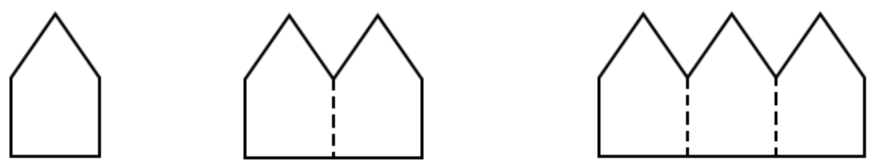


Figure 1 **Figure 2** **Figure 3**

(a) Draw Figure 4 of the sequence.

SOLUTION:
Data: Diagrams showing the first three figures in a sequence of figures made from joining polygons of units length.
Required to draw: The 4^{th} figure of the sequence
Solution:
Figure 4 should look like Figure 1 attached to Figure 3.

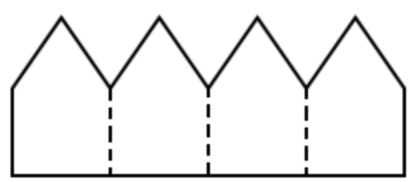


Figure 4

(b) Study the pattern of numbers in each row of the table below. Each row relates to one of the figures in the sequence of figures. Some rows have not been included in the table. Complete the rows numbered (i), (ii) and (iii).

	Figure 1	**Number of Outer Lines of Unit Length**	**Perimeter**
	1	$1+2+2$	5
	2	$2+2+4$	8
	3	$3+2+6$	11
(i)	6		
(ii)			65
(iii)	n		

SOLUTION:
Data: Incomplete table showing the number of outer lines and perimeter of each figure in the sequence.
Required to complete: The table given
Solution:

Figure 1	**Number of Outer Lines of Unit Length (L)**	**Perimeter (P)**
1	$1+2+2$	5
2	$2+2+4$	8
3	$3+2+6$	11

Trying to observe a pattern for L in terms of n.
L is the sum of three numbers. The first is always the same number as that of the figure n. The second is always 2. The third is always twice the number of the figure, which is $2\times n$. So we can figure that

$$L = n+2+2\times n$$
$$= n+2+2n$$

P is the sum of the three numbers in the column for L.

$$P = n+2+2n$$
$$= 3n+2$$

So,

	n	$n+2+2n$	$3n+2$
(i)	$n=6$	$L=6+2+2(6)$ $=6+2+12$	$P=20$

(ii) If $P=65$, then $3n+2=65$

$$3n = 65 - 2$$
$$3n = 63$$
$$n = 21$$
$$L = 21 + 2 + 2(21)$$
$$= 21 + 2 + 42$$

$n = 21$ $\quad L = 21 + 2 + 42 \quad P = 65$

The completed table looks like:

Figure 1	Number of Outer Lines of Unit Length	Perimeter
1	$1+2+2$	5
2	$2+2+4$	8
3	$3+2+6$	11
6	$6+2+12$	20
21	$21+2+42$	65
n	$n+2+2n$	$3n+2$

(c) Show that no figure can have a perimeter of 100 units.

SOLUTION:
Required to show: No figure in the sequence can have a perimeter of 100 units
Proof:
If $P = 100$, then
$$3n + 2 = 100$$
$$3n = 100 - 2$$
$$3n = 98$$
$$n = \frac{90}{3}$$
$$n = 32\frac{2}{3}$$

However, n must be a positive integer and $32\frac{2}{3}$ is not. So, $P \neq 100$ units.

Q.E.D

FAS-PASS Maths

SECTION II

Answer ALL questions.

ALGEBRA, RELATIONS, FUNCTIONS AND GRAPHS

8. (a) (i) Complete the table below for the function $f(x)=3+2x-x^2$.

x	-2	-1	0	1	2	3	4
$f(x)$		0	3	4		0	-5

SOLUTION:
Data: Incomplete table of values for the function $f(x)=3+2x-x^2$ for the domain $-2 \leq x \leq 4$.
Required to complete: The table of values given
Solution:

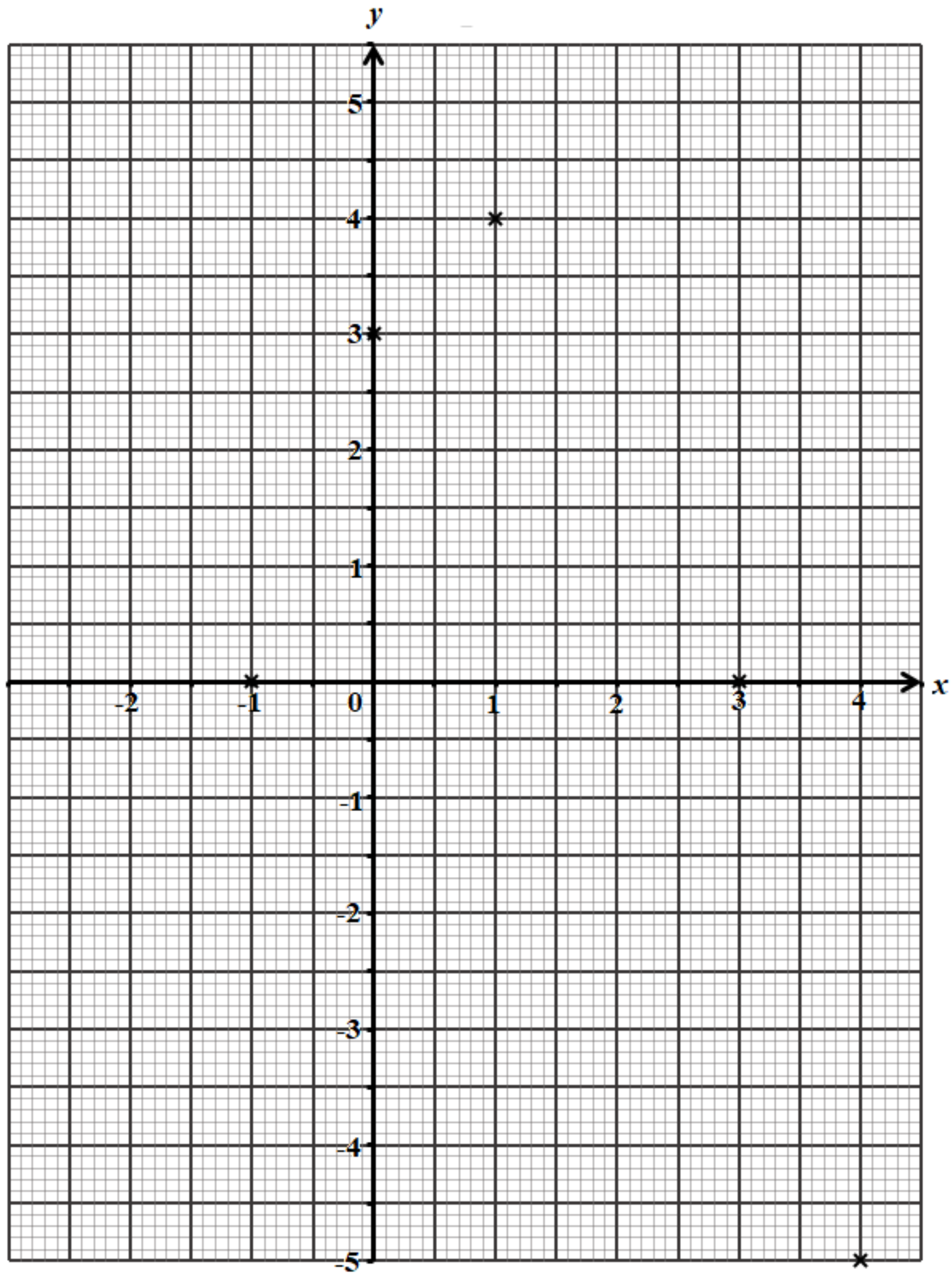
$$f(-2)=3+2(-2)-(-2)^2$$
$$=3-4-4$$
$$=-5$$

$$f(2)=3+2(2)-(2)^2$$
$$=3+4-4$$
$$=3$$

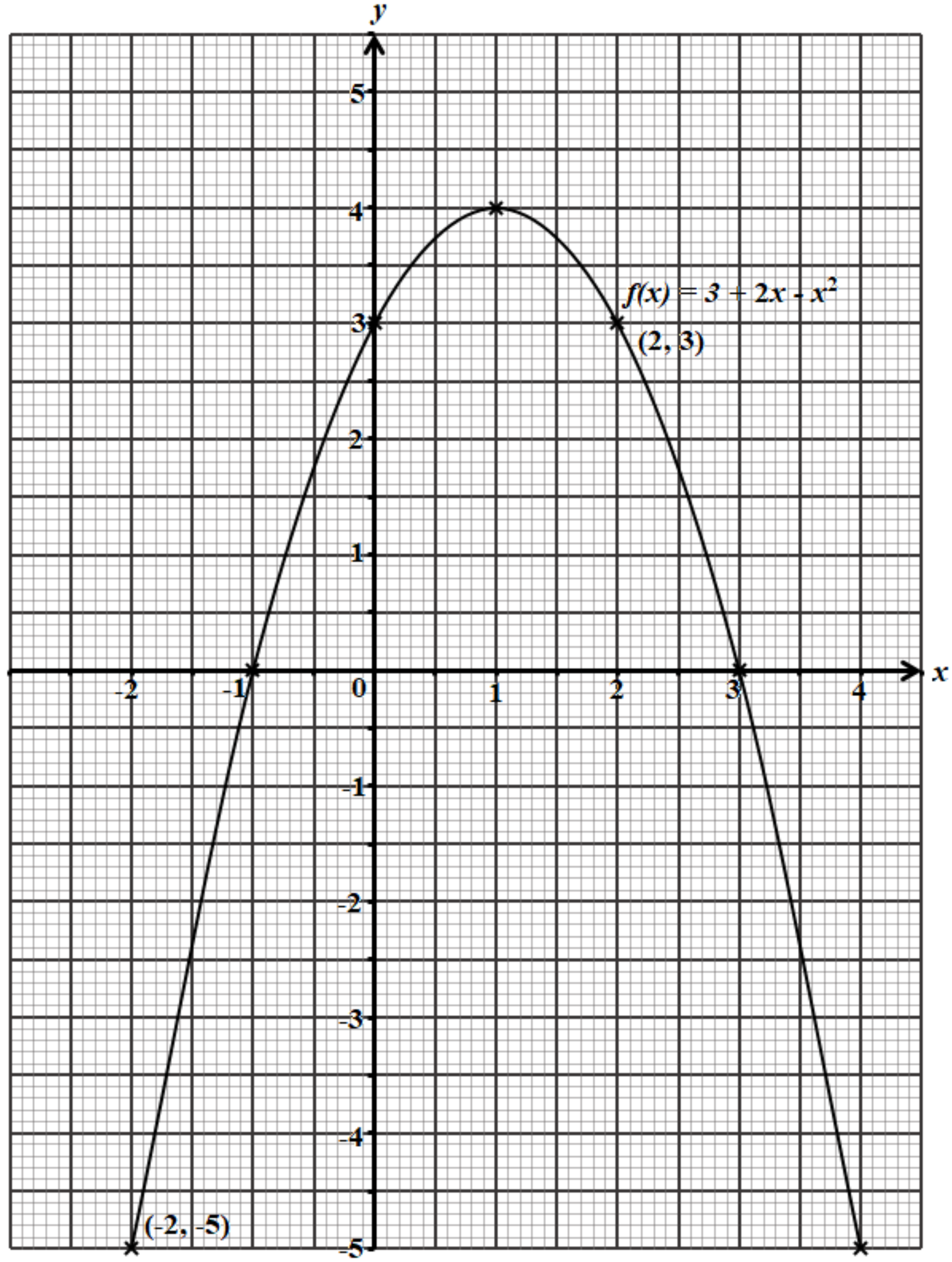
The completed table looks like:

x	-2	-1	0	1	2	3	4
$f(x)$	-5	0	3	4	3	0	-5

(ii) Complete the grid below to show all the points in the table and hence, draw the graph of the function $f(x)=3+2x-x^2$ for $-2 \leq x \leq 4$.



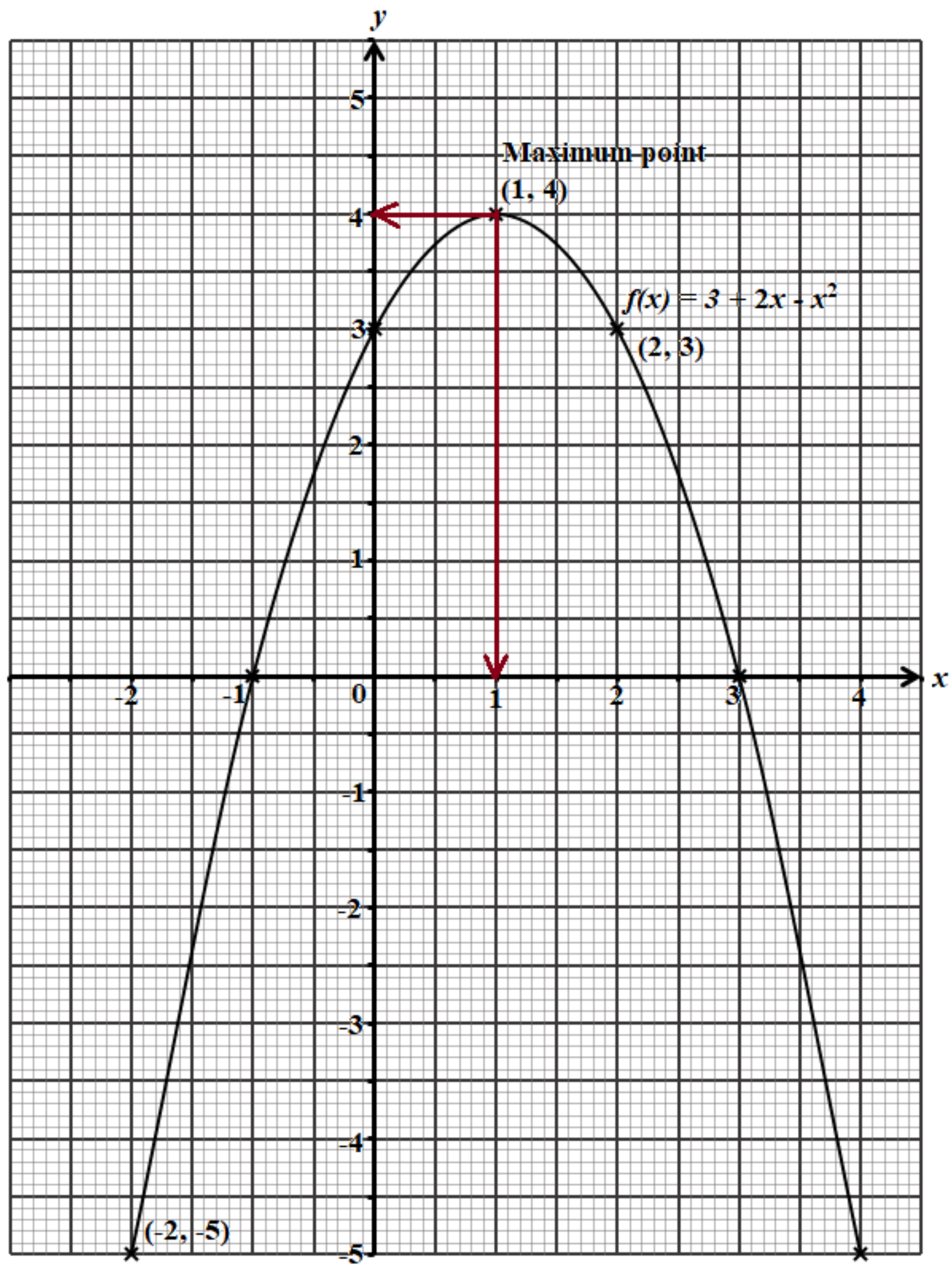
SOLUTION:
Data: Grid showing the points from the table of values plotted.
Required to draw: The graph of $f\left(x\right)=3+2x-x^2$ for $-2\le x\le 4$.
Solution:



(iii) Using the graph above, determine

a) the coordinates of maximum point of $f(x)$

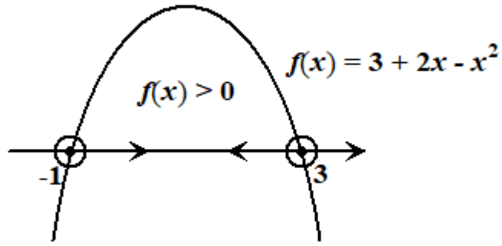
SOLUTION:
Required to determine: The coordinates of the maximum point of $f(x)$.
Solution:



The maximum turning point of $f(x)$ is $(1, 4)$.

b) the range of values of x for which $f(x) > 0$

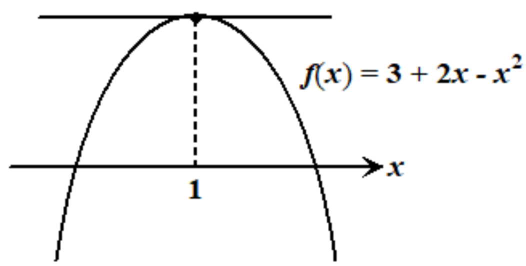
SOLUTION:
Required To Determine: The range of values for x for which $f(x) > 0$.
Solution:



$f(x) > 0$
When $x < 3$ and $x > -1$
$-1 < x < 3$
$\therefore f(x) > 0$ for $\{x : -1 < x < 3\}$

c) the gradient of $f(x)$ at $x = 1$.

SOLUTION:
Required to find: The gradient of $f(x)$ at $x = 1$.
Solution:



The tangent at $x = 1$ is a horizontal line which has a gradient of 0.
Therefore, the gradient of $f(x)$ at $x = 1$ is 0.

(b) Mr. Thomas makes x bottles of juice and y cakes each day. To supply his customers, he makes at least 20 bottles of juice and no more than 15 cakes each day.

(i) Write TWO inequalities to represent this information.

SOLUTION:
Data: Mr. Thomas makes at least 20 bottles of juice and no more than 15 cakes each day, where x represents the number of bottles of juice and y represents the number of cakes made each day.
Required to write: Two inequalities to represent the information given
Solution:
The number of bottles of juice is at least 20.
$x \geq 20$

$x \geq 20$ …❶

The number of cakes is no more than 15.

y $\leq$ 15

$y \leq 15$ …❷

(ii) Each day, Mr. Thomas uses \$163 to make the bottles of juice and the cakes. The cost to make a bottle of juice is \$3.50 while the cost to make a cake is \$5.25.

Write an inequality to represent this information.

SOLUTION:
Data: The cost to make one bottle of juice is \$3.50 and the cost to make one cake is \$5.25. Mr. Thomas spends \$163 each day to make the bottles of juice and the cakes.
Required to write: An inequality to represent this information
Solution:
The question should read that Mr. Thomas uses no more than \$163 to make the bottles of juice and the cakes so that we can derive an inequality.
x bottles of juice at \$3.50 each at y cakes at \$5.25 each cots \$163 or less.
So $3.5 \times x + 5.25 \times y \leq 163$

$$3.5x + 5.25y \leq 163$$

(iii) Show that on any given day, it is NOT possible for Mr. Thomas to make 50 bottles of juice and 12 cakes.

SOLUTION:
Required to show: Mr. Thomas cannot make 50 bottles of juice and 12 cakes on any day.
Proof:
50 bottles of juice at \$3.50 each and 12 cakes at \$5.25 each will cost

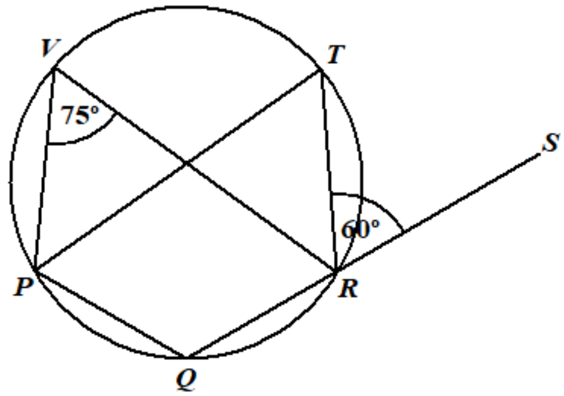
$$50 \times \$3.50 + 12 \times \$5.25 = \$175 + \$63$$
$$= \$238$$

The figure of \$238 exceeds Mr. Thomas' budget of \$163 and so making 50 bottles of juice and 12 cakes is not possible on his budget.

Q.E.D.

GEOMETRY AND TRIGONOMETRY

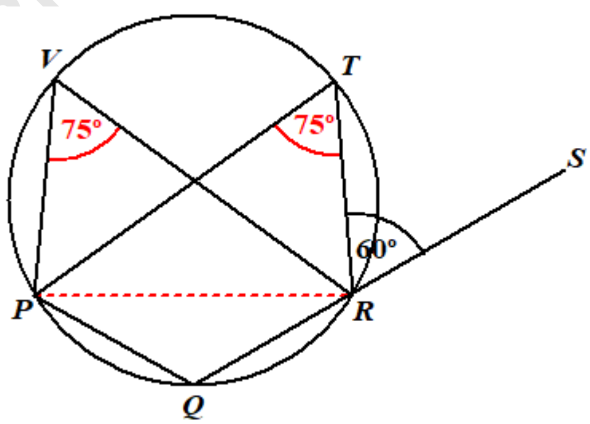
9. (a) The diagram below, **not drawn to scale**, shows a circle. The points *P*, *Q*, *R*, *T* and *V* are on the circumference. *QRS* is a straight line. Angle $PVR = 75°$ and angle $TRS = 60°$.



Determine the value of EACH of the following angles. Show detailed working where necessary and give a reason to support your answer.

(i) Angle *PTR*

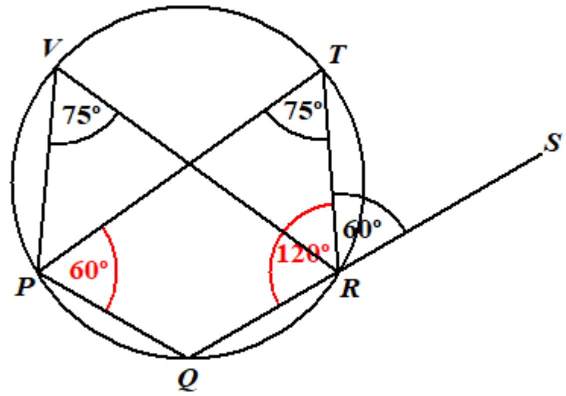
SOLUTION:
Data: Diagram showing the points *P*, *Q*, *R*, *T* and *V* on the circumference. *QRS* is a straight line and angle $PVR = 75°$ and angle $TRS = 60°$.
Required to find: Angle *PTR*
Solution:



Draw the chord *PR*.
$P\hat{T}R = 75°$
(The angles subtended by a chord (*PR*) at the circumference of a circle ($P\hat{V}R$ and $P\hat{T}R$) and standing on the same arc are equal.)

(ii) Angle TPQ

SOLUTION:
Required To Find: Angle TPQ
Solution:



$$T\hat{R}Q = 180° - 60°$$
$$= 120°$$

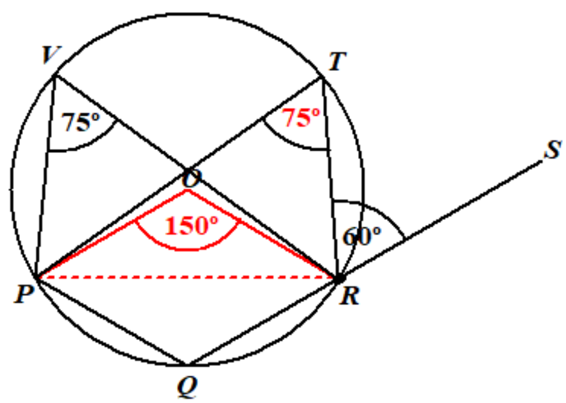
(Angles in a straight line.)

$$T\hat{P}R = 180° - 120°$$
$$= 60°$$

(The opposite angles ($T\hat{P}Q$ and $T\hat{R}Q$) of a cyclic quadrilateral are supplementary.)

(iii) Obtuse angle POR, where O is the center of the circle.

SOLUTION:
Required to find: Obtuse angle POR, where O is the center of the circle
Solution:

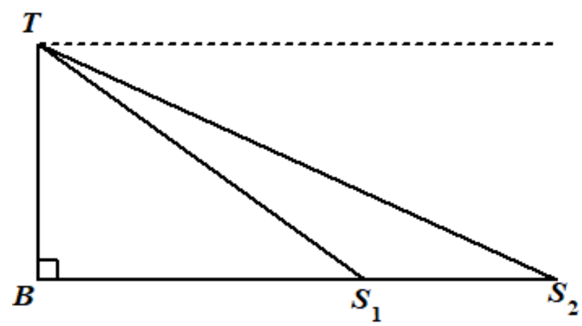


$P\hat{O}R = 2(75°)$
$= 150°$

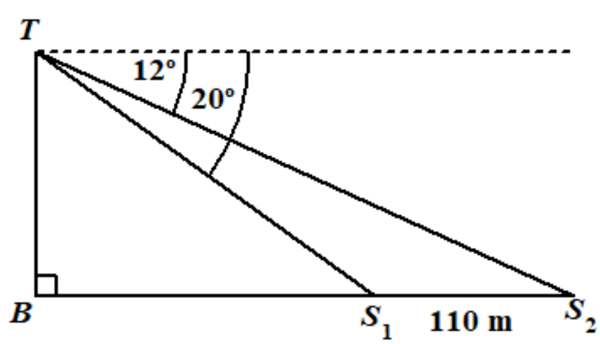
(The angle subtended by a chord (PR) at the center of a circle ($P\hat{O}R$) is twice the angle that the chord subtends at the circumference ($P\hat{T}R$) standing on the same arc.)

(b) The person at the top of a lighthouse, *TB*, sees two ships, S_1 and S_2, approaching the coast as illustrated in the diagram below. The angles of depression are 12° and 20° respectively. The ships are 110 m apart.

(i) Complete the diagram below by inserting the angles of depression and the distance between the ships.



SOLUTION:
Data: Incomplete diagram showing two ships, S_1 and S_2, 110 m apart, with angles of depression 12° and 20°, approaching a lighthouse *TB.*
Required to complete: The diagram with the angles of depression and the distance between the ships
Solution:



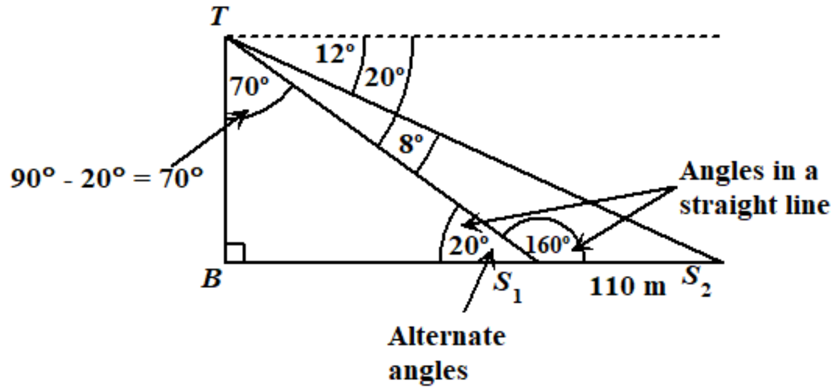
(ii) Determine, to the nearest metre,

a) the distance, TS_2, between the top of the lighthouse and Ship 2

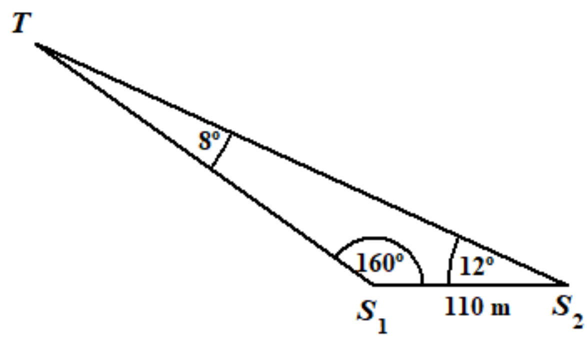
SOLUTION:

Required to determine: The distance TS_2

Solution:



Let us consider ΔTS_1S_2:

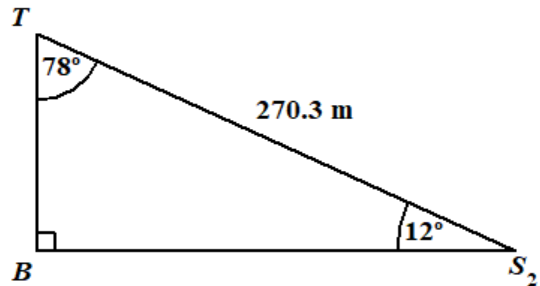


$$B\hat{T}S = 90° - 20°$$
$$= 70°$$
$$T\hat{S_1}B = 20° \qquad \text{(Alternate angles)}$$
$$T\hat{S_1}S_2 = 180° - 20°$$
$$= 160° \qquad \text{(Angles in a straight line)}$$
Angle $S_1\hat{T}S_2$
$$= 12° \qquad \text{(Angles in a triangle total 180°.)}$$

$$\frac{TS_2}{\sin 160°} = \frac{110}{\sin 8°} \qquad \text{(Sine rule)}$$
$$TS_2 = \frac{110 \times \sin 160°}{\sin 8°}$$
$$= 270.\underline{\underline{3}} \text{ m}$$
$$\approx 270 \text{ m to the nearest metre}$$

b) the height of the lighthouse *TB*.

SOLUTION:
Required to determine: *TB*, the height of the lighthouse
Solution:
Consider ΔTBS_2:



$$\sin 12° = \frac{TB}{270.3}$$
$$TB = 270.3 \times \sin 12°$$
$$= 56.1 \text{ m}$$
$$\approx 56 \text{ m correct to the nearest metre}$$

VECTORS AND MATRICES

10. (a) Three matrices are given as follows:

$$P = \begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix}, Q = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } R = \begin{pmatrix} 11 \\ 15 \end{pmatrix}.$$

(i) Using a calculation to support your answer, explain whether matrix *P* is a singular or a non-singular matrix.

SOLUTION:
Data: $P = \begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix}, Q = \begin{pmatrix} a \\ b \end{pmatrix}$ and $R = \begin{pmatrix} 11 \\ 15 \end{pmatrix}$
Required to explain: Whether *P* is singular or non-singular
Solution:

$$P = \begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix}$$

$$\begin{aligned} \det P &= (-1 \times 5) - (2 \times 0) \\ &= -5 - 0 \\ &= -5 \\ &\neq 0 \end{aligned}$$

Hence, P is a non-singular matrix since $\det P$ or $|P| \neq 0$.

(ii) Given $PQ = R$, determine the value of a and of b.

SOLUTION:
Data: $PQ = R$
Required to determine: a and b
Solution:

$$\begin{matrix} P & \times & Q & = & PQ \\ 2\times 2 & & 2\times 1 & = & 2\times 1 \end{matrix}$$

$$\begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ b \end{pmatrix} = \begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix}$$

$$\begin{aligned} e_{11} &= (-1 \times a) + (2 \times b) \\ &= -a + 2b \\ e_{21} &= (0 \times a) + (5 \times b) \\ &= 5b \end{aligned}$$

So $$PQ = \begin{pmatrix} -a + 2b \\ 5b \end{pmatrix}$$

$$\begin{pmatrix} -a + 2b \\ 5b \end{pmatrix} = \begin{pmatrix} 11 \\ 15 \end{pmatrix} = R$$

Equating corresponding entries:

$$5b = 15$$
$$\therefore b = 3$$

$$\begin{aligned} -a + 2b &= 11 \\ -a + 2(3) &= 11 \\ -a &= 11 - 6 \\ a &= -5 \end{aligned}$$

$\therefore a = -5$ and $b = 3$

(iii) State the reason why the matrix product QP is not possible.

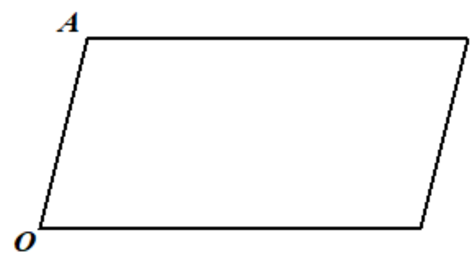
SOLUTION:
Required to state: The reason why the matrix QP cannot be found.
Solution:

$$\begin{array}{ccc} Q & \times & P \\ 2\times 1 & \times & 2\times 2 \\ & \neq & \end{array}$$

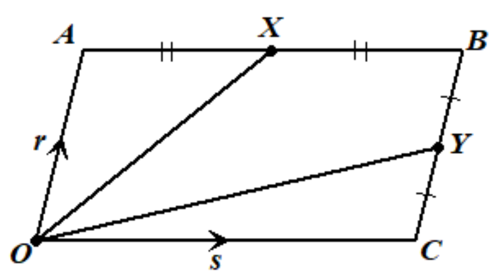
The number of columns of Q (1) is not equal to the number of rows of P (2). So, matrix multiplication QP is not possible.

(b) $OABC$ is a parallelogram. X is the midpoint of AB and Y is the midpoint of BC. $OA = r$ and $OC = s$.

(i) Complete the diagram below to represent ALL the information given above.



SOLUTION:
Data: On a parallelogram $OABC$, X is the midpoint of AB and Y is the midpoint of BC. $OA = r$ and $OC = s$.
Required to complete: The parallelogram given to represent the information given
Solution:



(ii) Given that $OX + OY = k(r+s)$, where k is a constant, using a vector method, find the value of k.

SOLUTION:

Data: $OX + OY = k(r+s)$, where k is a constant

Required to find: k

Solution:

$$AB = OC \quad \text{(Opposite sides of a parallelogram)}$$
$$= s$$

$$\therefore AX = \frac{1}{2}s$$

$$OX = OA + AX$$
$$= r + \frac{1}{2}s$$

$$CB = OA \quad \text{(Opposite sides of a parallelogram)}$$
$$= r$$

$$OY = OC + CY$$
$$= s + \frac{1}{2}r$$

$$OX + OY = r + \frac{1}{2}s + s + \frac{1}{2}r$$
$$= 1\frac{1}{2}r + 1\frac{1}{2}s$$
$$= 1\frac{1}{2}(r+s)$$ and which is of the form $k(r+s)$, where $k = 1\frac{1}{2}$.