CSEC MATHEMATICS MAY-JUNE 2016

SECTION I

1. (a) Using a calculator, or otherwise, calculate

(i) $\dfrac{3\frac{3}{8}-2\frac{1}{4}}{1\frac{1}{2}}$, giving your answer as a fraction in its lowest terms.

SOLUTION:

Required to calculate: $\dfrac{3\frac{3}{8}-2\frac{1}{4}}{1\frac{1}{2}}$

Calculation:

Numerator: $=3\frac{3}{8}-2\frac{1}{4}$

$=1\,\frac{1(3)-2(1)}{8}=\ 1\frac{1}{8}$

$\dfrac{\text{Numerator}}{\text{Denominator}}=\dfrac{1\frac{1}{8}}{1\frac{1}{2}}\ =\dfrac{\frac{9}{8}}{\frac{3}{2}}\ =\frac{9}{8}\times\frac{2}{3}=\frac{18}{24}$

$=\frac{3}{4}$ (as a fraction in its lowest terms)

(ii) $(2.86+0.75)+0.481^2$, giving your answer correct to 2 decimal places.

SOLUTION:

Required to calculate: $(2.86+0.75)+0.481^2$ correct to 2 decimal places

Calculation:

$(2.86+0.75)+0.481^2=(2.86+0.75)+(0.481\times0.481)$

$=3.61+0.231361$ (by the calculator)

$=3.84\underline{\underline{1}}361$

$=3.84$ (correct to 2 decimal places)

(b) Paul bought and sold a computer. He wrote his business activity as follows:

Cost price of computer	$=\$1064$
Marked price of computer	$=\$1399$
Discount on marked price	$=5\%$

(If paid by cash)
Calculate,

(i) The selling price (paid cash)

SOLUTION:

Data: The cost price of a computer is \$1 064. The marked price of the computer is \$1 399 and a discount of 5% is given off the marked price if it is paid for by cash.

Required to calculate: The selling price of the computer
Calculation:
Discount if cash is paid is 5%.

$$\text{Discount} = 5\% \text{ of } \$1399$$
$$= \frac{5}{100} \times \$1399$$
$$= \$69.95$$

$$\text{Selling price} = \$1399 - \$69.95$$
$$= \$1329.05$$

OR
Selling price is (100-5)% of \$1399 = \$1 329.05

(ii) The profit or loss as a percentage of the cost price.

SOLUTION:
Required to calculate: The percentage profit or loss
Calculation:

$$\text{The profit, if cash is paid } = \$1329.05 - \$1064.00$$
$$= \$265.05$$

$$\text{Percentage profit if cash is paid } = \frac{\text{Profit}}{\text{Cost price}} \times 100$$
$$= \frac{\$265.05}{\$1064} \times 100$$
$$= 24.9\underline{\underline{1}}\%$$
$$= 24.9\% \text{ (correct to 1 decimal place)}$$

However, if cash was not paid and some alternative payment was accepted, and there is no discount; this would lead to a different percentage profit.

$\therefore$ Profit $= \$1399 - \$1064 = \$335$

Profit percentage $= \dfrac{\$335}{\$1064} \times 100 = 31.4\underline{\underline{8}}\%$

$= 31.5\%$

(c) Orange juice is sold in cartons on three different sizes.

Carton Size	Selling Price
350 ml	$4.20
450 ml	$5.35
500 ml	$5.80

Which size of orange juice is the most cost-effective buy? Justify your answer.

SOLUTION:

Data: Table showing the prices of three different sizes of cartons of orange juice. The 350 ml volume is $4.20, the 450 ml carton is $5.35 and the 500 ml carton is $5.80.

Required to state: The most cost-effective buy
Solution:
Carton size of 350 ml is sold for $4.20.

Hence, cost per ml $= \dfrac{\$4.20}{350}$

$= 1.2\dot{0}$ cents

Carton size of 450 ml is sold for $5.35.

Hence, cost per ml $= \dfrac{\$5.35}{450}$

$= 1.19$ cents

Carton size of 500 ml is sold for $5.80.

Hence, cost per ml $= \dfrac{\$5.80}{500}$

$= 1.16$ cents

If 'cost-effective' means solely the cheapest buy, then $1.16 < 1.19 < 1.2\dot{0}$ and orange juice in the 500 ml carton is then the most cost-effective.

2. **(a)** Factorise completely:

(i) $4a^2 - 16$

SOLUTION:
Required to factorise: $4a^2 - 16$

Solution:

$$4a^2 - 16 = 4\left(a^2 - 4\right)$$
$$= 4\left\{(a)^2 - (2)^2\right\}$$

The terms within the curly brackets is now expressed as a difference of two squares and can be further factorised.
Hence, $4a^2 - 16 = 4(a-2)(a+2)$

(ii) $3y^2 + 2y - 8$

SOLUTION:
Required to factorise: $3y^2 + 2y - 8$
Solution:

$$3y^2 + 2y - 8 = (3y-4)(y+2)$$

$$\begin{array}{l} 3y^2 + 6y \\ \quad\; -4y - 8 \\ \hline 3y^2 + 2y - 8 \\ \hline \end{array}$$

$$3y^2 + 2y - 8 = (3y-4)(y+2)$$

(b) Solve the simultaneous equations:

$$2x + y = 3$$
$$5x - 2y = 12$$

SOLUTION:

Required to solve: $2x + y = 3$ and $5x - 2y = 12$ simultaneously
Solution:
Method of substitution
$2x + y = 3$ …❶
$5x - 2y = 12$ …❷

From equation ❶:
$y = 3 - 2x \quad \ldots$❸

Substitute equation ❸ into equation ❷:
$$5x - 2(3 - 2x) = 12$$
$$5x - 6 + 4x = 12$$
$$9x = 18$$
$$x = \frac{18}{9}$$
$$x = 2$$

Substitute $x = 2$ into equation ❸:
$$y = 3 - 2(2)$$
$$= 3 - 4$$
$$= -1$$

$\therefore x = 2$ and $y = -1$

Alternative Method:
Method of elimination
$2x + y = 3 \quad \ldots$❶
$5x - 2y = 12 \quad \ldots$❷
Equation ❶ $\times 2$:
$4x + 2y = 6 \quad \ldots$❸

Equation ❷ + Equation ❸:
$$\begin{array}{r} 5x - 2y = 12 \\ +\ 4x + 2y = 6 \\ \hline 9x \quad = 18 \\ \hline \end{array}$$
$$x = \frac{18}{9} = 2$$

Substitute $x = 2$ into equation ❶:
$$2(2) + y = 3$$
$$y = 3 - 4$$
$$= -1$$
$\therefore x = 2$ and $y = -1$

Alternative Method:
Graphical method
Treating the two equations as the equations of straight lines, we draw both lines on the same axes.

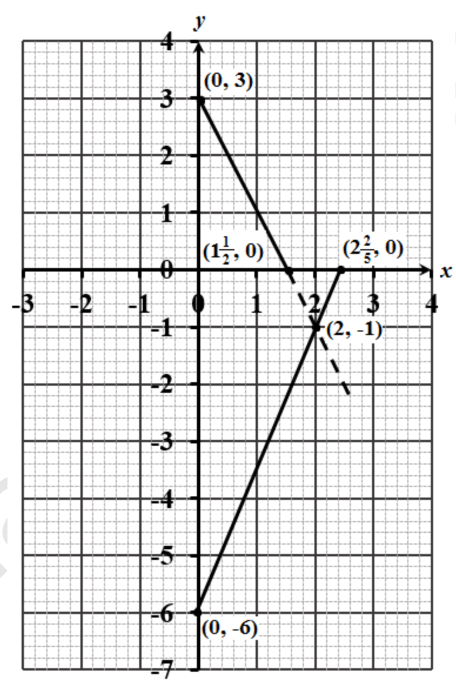
We find the coordinates of two points on both lines and plot them. Then we draw both lines extending either or both, if necessary, till they meet.

$2x + y = 3$

x	y
0	3
$1\frac{1}{2}$	0

$5x - 2y = 12$

x	y
0	-6
$2\frac{2}{5}$	0



The point of intersection is $(2, -1)$.
$\therefore x = 2$ and $y = -1$

Alternative Method:
The matrix method
$2x + y = 3 \quad \ldots$❶
$5x - 2y = 12 \quad \ldots$❷

$$\begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \end{pmatrix}$$ …matrix equation

Let $A=\begin{pmatrix}2 & 1\\ 5 & -2\end{pmatrix}$

$$|A|=(2\times -2)-(1\times 5)$$
$$=-9$$

$$A^{-1}=\frac{1}{-9}\begin{pmatrix}-2 & -(1)\\ -(5) & 2\end{pmatrix}$$
$$=\begin{pmatrix}\frac{2}{9} & \frac{1}{9}\\ \frac{5}{9} & -\frac{2}{9}\end{pmatrix}$$

Matrix equation $\times A^{-1}$:

$$\begin{pmatrix}\frac{2}{9} & \frac{1}{9}\\ \frac{5}{9} & -\frac{2}{9}\end{pmatrix}\begin{pmatrix}2 & 1\\ 5 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\frac{2}{9} & \frac{1}{9}\\ \frac{5}{9} & -\frac{2}{9}\end{pmatrix}\begin{pmatrix}3\\ 12\end{pmatrix}$$

$$I\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\frac{2}{9} & \frac{1}{9}\\ \frac{5}{9} & -\frac{2}{9}\end{pmatrix}\begin{pmatrix}3\\ 12\end{pmatrix}$$

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\left(\frac{2}{9}\times 3\right)+\left(\frac{1}{9}\times 12\right)\\ \left(\frac{5}{9}\times 3\right)+\left(-\frac{2}{9}\times 12\right)\end{pmatrix}$$

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}2\\ -1\end{pmatrix}$$

Equating corresponding entries:
$\therefore x=2$ and $y=-1$

(d) The table below shows corresponding values of the variables x and y, where y varies directly as x.

$\boldsymbol{x}$	6	10	t
$\boldsymbol{y}$	3	u	9

Calculate the value of t and of u.

SOLUTION:

Data: Two variables are such that y varies directly as x. A table of corresponding values of x and y

Required to calculate: The value of t and of u.

Calculation:

$$y \underbrace{\text{varies directly as}}_{\propto} x$$

$$\therefore y \propto x$$

Hence,

$y = kx$ (where k is the constant of proportion)

$x = 6$ when $y = 3$ (data)

$$\therefore 3 = k \times 6$$

$$k = \frac{1}{2}$$

So, $y = \frac{1}{2}x$

When $x = 10$

$$y = \frac{1}{2}(10)$$

$$u = 5$$

When $y = 9$

$$9 = \frac{1}{2}x$$

$$x = 18$$

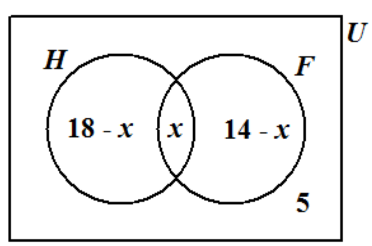
$$\therefore t = 18$$

3. (a) The Venn diagram below shows the number of students who study History and French in a class of 30 students.

$U = \{\text{students in the class}\}$

$H = \{\text{students who study History}\}$

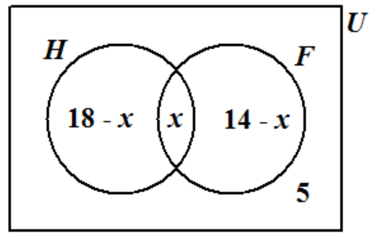
$F = \{\text{students who study French}\}$



(i) Write an expression, in x, in its simplest form, for the TOTAL number of students in the class.

SOLUTION:

Data: A Venn diagram showing the number of students who study History and French in a class of 30 students.



Required to write: An expression for the total number of students, in terms of x

Solution:

The total number of the students in the class $= (18 - x) + x + (14 - x) + 5$

$= 37 - x$ (in its simplest form)

(ii) State whether the following relationships are true or false.

- $H \cup F = U$
- $H \cap F' = \phi$

SOLUTION:

Required to state: Whether the relationships $H \cup F = U$ and $H \cap F' = \phi$ are true or false.

Solution:

- $(H \cup F)' = 5 \neq 0$
  $\therefore H \cup F \neq U$
  Hence, the statement given is false.

- $H \cap F'$ is equal to the set of elements common to H and F', which is the same as the elements belonging to H only. From the diagram, H only, is not empty.
  Recall: $37 - x = 30$
  $x = 7$
  $H \cap F' = 18 - 7$
  $= 11$
  $H \cap F' \neq 0$
  $H \cap F' \neq \phi$
  Hence, the statement given is false.

(iii) Determine the number of students who study BOTH History and French.

SOLUTION:
Required to determine: The number of students who study both History and French
Solution:
There are a total of 30 students in the class. (data)

$$\therefore 37 - x = 30$$
$$x = 37 - 30$$
$$x = 7$$

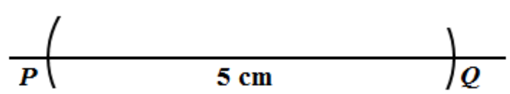
The number of students who study both History and French is expressed as $n(H \cap F) = x = 7$.

(b) (i) Using **a ruler, a pencil and a pair of compasses**, construct triangle *PQR* with $PQ = 5\text{cm}$, $\angle PQR = 60°$ and $\angle QPR = 90°$.

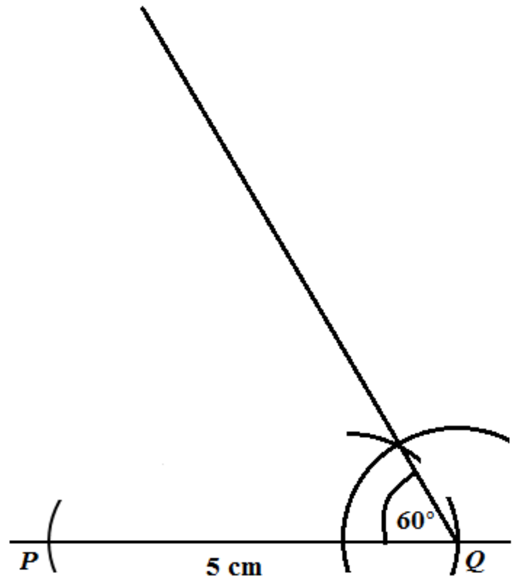
SOLUTION:

Data: Triangle *PQR* has $PQ = 5\text{cm}$, $\angle PQR = 60°$ and $\angle QPR = 90°$.
Required to construct: Triangle *PQR*, using a ruler, a pencil and a pair of compasses
Solution:

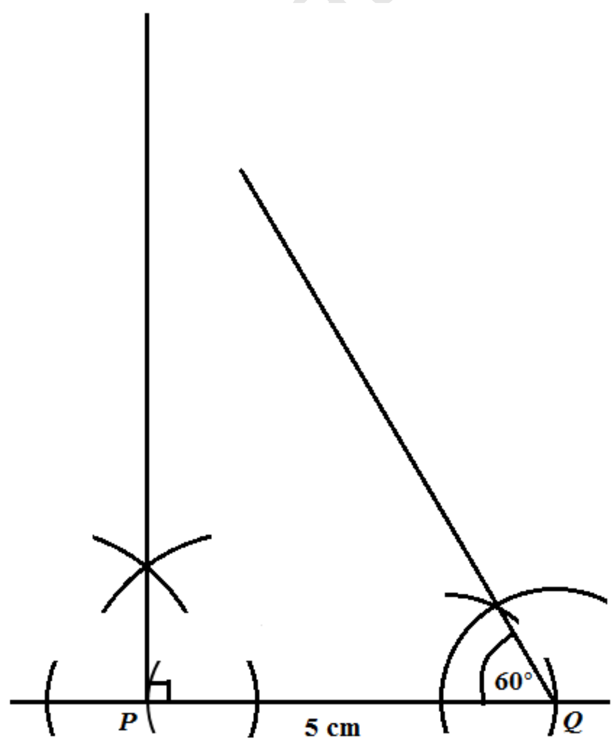
Step 1: Construct *PQ* = 5 cm. We draw a straight line longer that 5 cm and cut off *PQ* = 5 cm



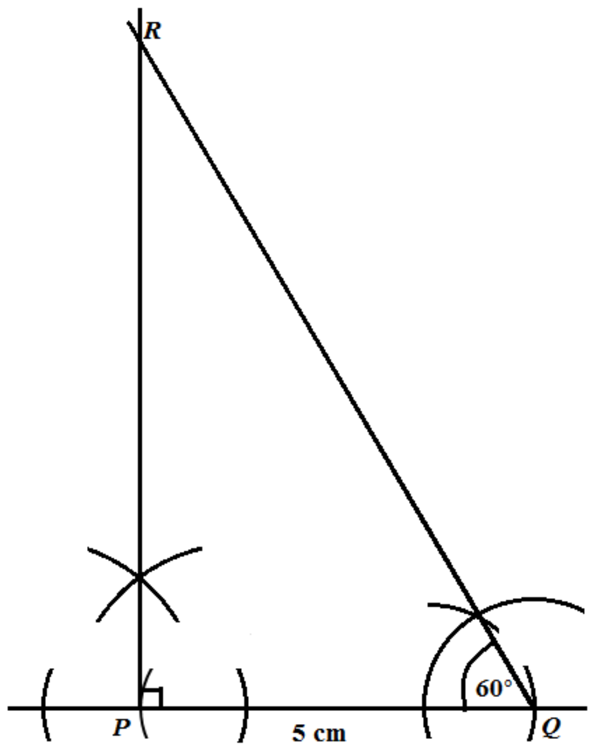
Step 2: Construct an angle of 60 degrees at *Q*.



Step 3: Construct an angle of 90 degrees at *P*



Step 4: By extending the arms of the triangle at *P* and at *Q*, we locate *R*, the third point of the triangle.



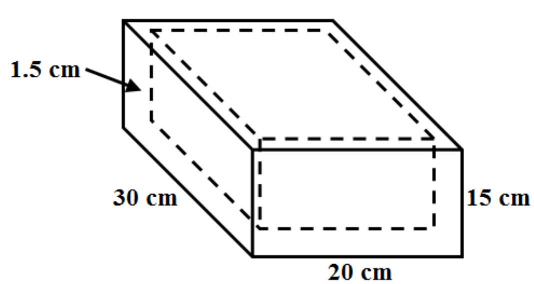
(ii) Measure and state

- the length of PR
- the measure of $\angle PRQ$

SOLUTION:
Required to measure: And state the length of PR and $\angle PRQ$.
Solution:

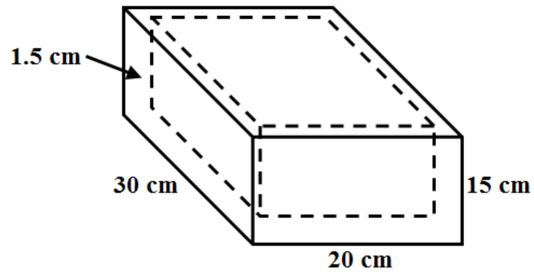
- The length of $PR = 8.7$ cm (by measurement)
- The measure of $\angle PRQ = 30°$ (by measurement)

4. The diagram below, **not drawn to scale**, shows a silver box with no lid. The sides and bottom of the box are 1.5 cm thick.



(a) Calculate the volume of the box using the external dimensions.

SOLUTION:
Data: Diagram showing a silver box with no lid and of dimensions 30 cm by 20 cm by 15 cm and thickness 1.5 cm.



Required to calculate: The volume of the box using the external dimensions
Calculation:
External volume of the box $= (30 \times 20 \times 15)\ \text{cm}^3$
$= 9000\ \text{cm}^3$

(b) Complete EACH of the following statements.

(i) The internal length of the box is $30\ \text{cm} - 2 \times 1.5\ \text{cm} =$

Solution:
The internal length of the box is $30\ \text{cm} - 2 \times 1.5\ \text{cm} = 27\text{cm}$

(ii) The internal width of the box is ……

Solution:
The internal width of the box is $20 - 2(1.5)\ \text{cm} = 17\ \text{cm}$.

(iii) The internal depth of the box is ……

Solution:
The internal depth of the box is $15 - 1(1.5)\ \text{cm} = 13.5\ \text{cm}$

(c) Calculate the internal volume of the box.

SOLUTION:
Required to calculate: The interval volume of the box.
Calculation:
The internal volume of the box $= (27 \times 17 \times 13.5)\ \text{cm}^3$
$= 6196.5\ \text{cm}^3$

(d) The box is made of silver which has a mass of 10.5 g for each cm^3. Calculate the mass of the silver box, giving your answer in kg.

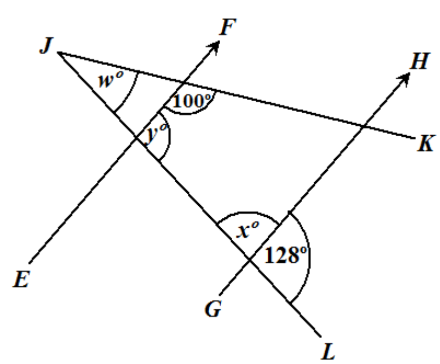
SOLUTION:

Data: The box is made of silver which has a mass of 10.5 g for each cm^3.

Required to calculate: The mass of the silver box, in kg.
Calculation:
The volume of silver $=$ External volume of the box – Internal volume of the box
$$= 9000 - 6196.5$$
$$= 2803.5 \text{ cm}^3$$

Mass of silver is 10.5 g cm^{-3}.
$$\therefore \text{Mass of box} = 2803.5 \times 10.5 \text{ g}$$
$$= 29436.75 \text{ g}$$
$$= \frac{29436.75}{1000} \text{ kg}$$
$$= 29.43\underline{\underline{6}}5 \text{ kg}$$
$$\approx 29.44 \text{ kg (correct to 2 decimal places)}$$

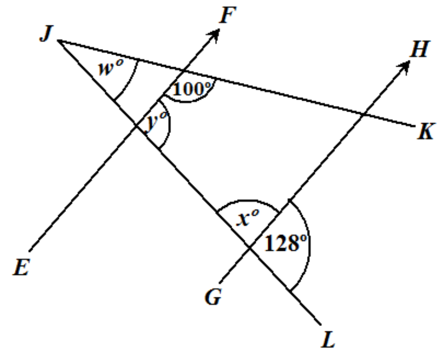
5. (a) The diagram below, **not drawn to scale**, shows two straight lines, *JK* and *JL*, intersecting a pair of parallel lines, *EF* and *GH*.



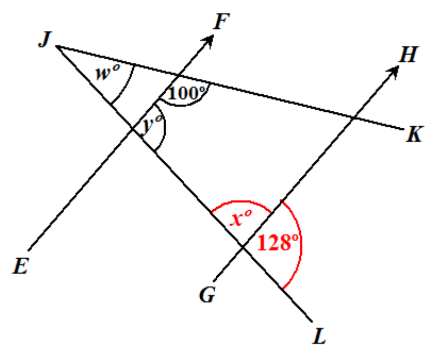
Determine, giving reasons for EACH of your answers, the value of:

(i) x

SOLUTION:
Data: Diagram showing two straight lines, *JK* and *JL*, intersecting a pair of parallel lines, *EF* and *GH*.



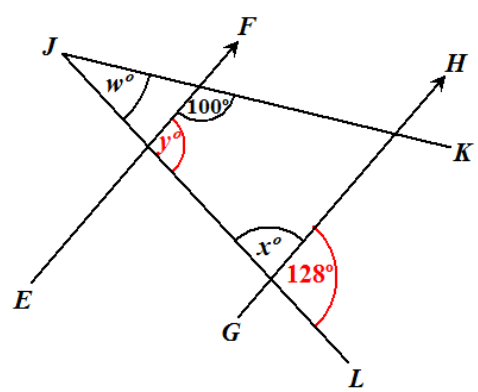
Required to determine: The value of *x*
Solution:



$x = 180 - 128$
$= 52$ (*x* is a numerical value and does not carry units)
(Angle in a straight line or a straight angle = 180^0)

(ii) *y*

SOLUTION:
Required to determine: The value of *y*
Solution:

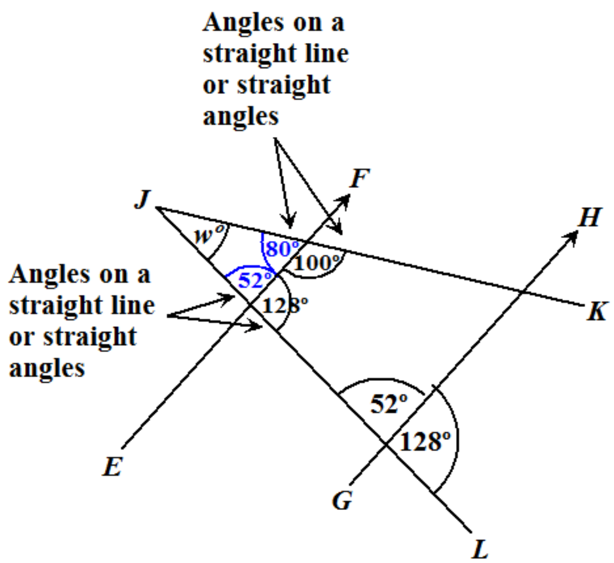


$y = 128$ (*y* is a numerical value and does not carry units)

(Corresponding angle to the angle marked 128°)

(iii) w

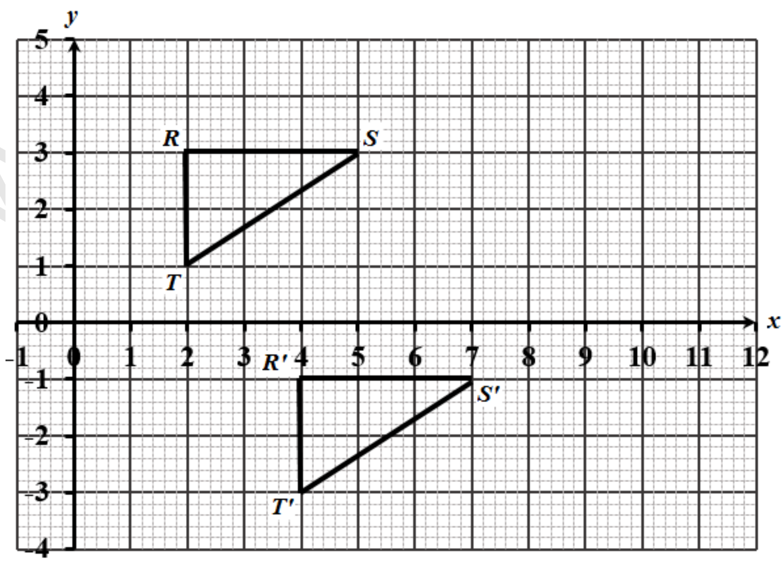
SOLUTION:
Required To Determine: The value of w.
Solution:



$$w° + 52° + 80° = 180° \text{ (Sum of angles in a triangle } = 180°\text{)}$$
$$w = 180 - (80 + 52)$$
$$= 48$$

(w is a numerical value and does not carry units)

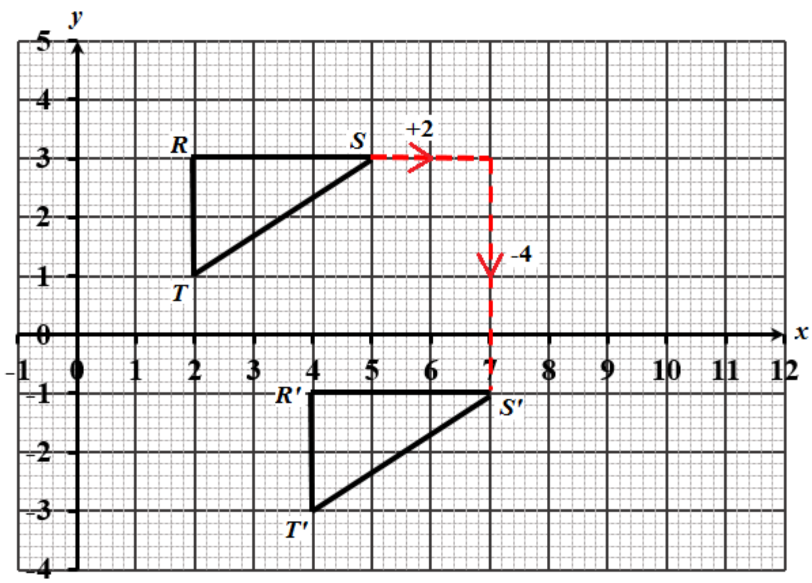
(b) The diagram below shows a triangle RST and its image $R'S'T'$ after a transformation.



(i) Describe FULLY the transformation which maps RST onto $R'S'T'$.

SOLUTION:
Data: Diagram showing a triangle *RST* and its image $R'S'T'$ after a transformation.

Required to describe: The transformation which maps *RST* onto $R'S'T'$ fully
Solution:



The image $R'S'T'$ is congruent to object *RST*. There is no flip (lateral inversion), re-orientation or change in size (with respect to the object). The transformation is deduced to be a translation.
Considering one set of object-image point, say, $S \to S'$.
The shift is 2 units horizontally to the right and 4 units vertically downwards. Hence, the transition, *T*, can be defined as $T = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$.

(ii) Triangle *RST* is reflected in the line, $x = 6$. On the graph above, draw triangle, $R''S''T''$, the image of ΔRST, after the reflection.

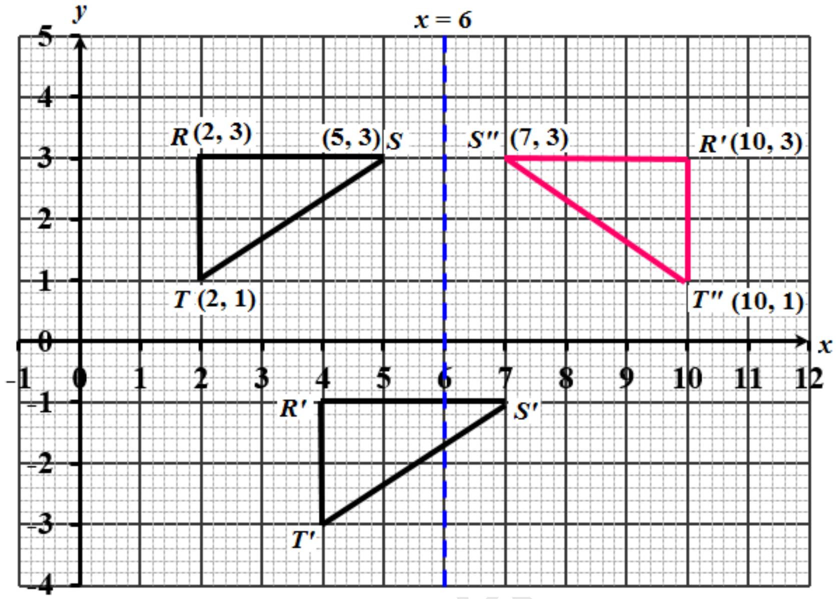
Write down the coordinates of R''.

SOLUTION:

Data: Triangle *RST* is reflected in the line $x = 6$ to give triangle $R''S''T''$.
Required to write: The coordinates of R''.

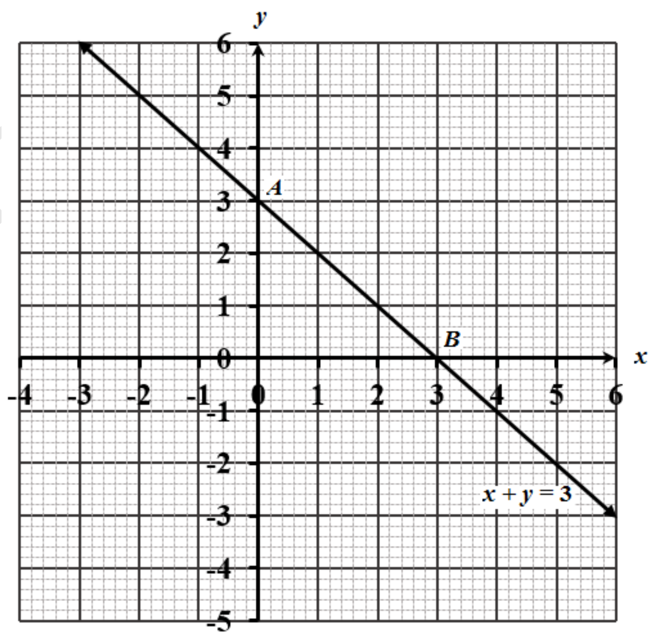
Solution:
The line $x = 6$ is shown dotted and in blue

The image of each vertex of the object is measured the same perpendicular distance on the opposite side of the reflection plane. The three are joined to obtain the image.



$R'' = (10, 3)$

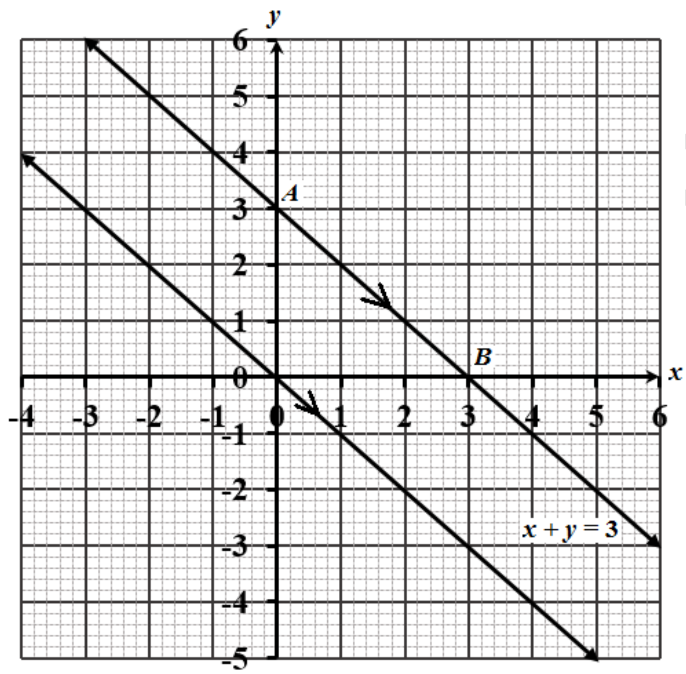
6. (a) The diagram below shows the graph of the straight line, $x + y = 3$.



Determine the equation of the line which is

(i) Parallel to the line $x + y = 3$ and passes through the origin.

SOLUTION:
Data: Diagram showing the graph of the straight line, $x+y=3$.
Required to determine: The equation of the line which is parallel to $x+y=3$ and passes through the origin
Solution:



If $x+y=3$, then $y=-x+3$ which is of the form $y=mx+c$, where $m=-1$ is the gradient of the line.
The line through O and parallel to $x+y=3$ will also have gradient of -1 (Parallel lines have the same gradient).

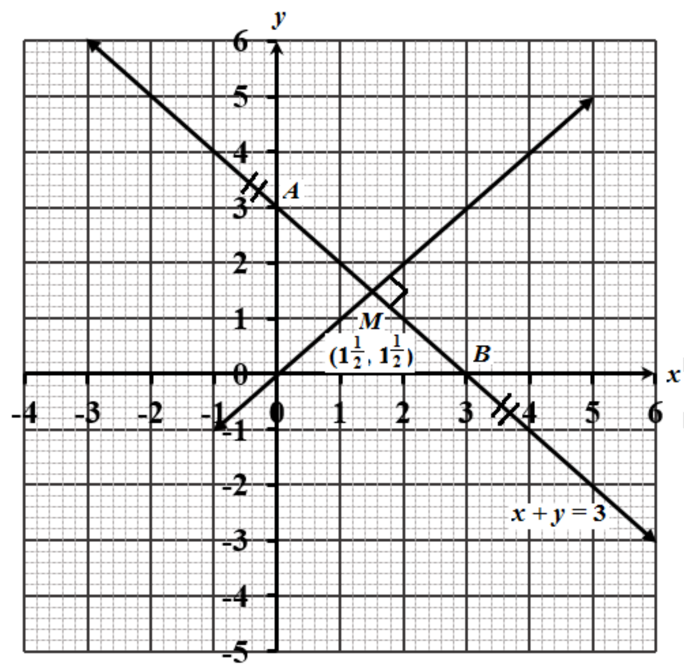
Equation is $\frac{y-0}{x-0}=-1$

$$y-0=-1(x-0)$$

$$y=-x \text{ or } x+y=0$$

(ii) Perpendicular to the line $x+y=3$ and passes through the midpoint of AB.

SOLUTION:
Required to determine: The equation of the line perpendicular to the line $x+y=3$ and passes through the midpoint of AB
Solution:



Let the midpoint of *AB* be *M*.

$$M = \left(\frac{3+0}{2}, \frac{0+3}{2}\right)$$
$$= \left(1\frac{1}{2}, 1\frac{1}{2}\right)$$

Any line perpendicular to *AB* has a gradient of $\dfrac{-1}{\text{gradient of } AB} = \dfrac{-1}{-1} = 1$

(The product of the gradients of perpendicular lines $= -1$).
The equation of the perpendicular to *AB* and passing through the midpoint (that is, the perpendicular bisector of *AB*) is

$$\frac{y - 1\frac{1}{2}}{x - 1\frac{1}{2}} = 1$$

$$y - 1\frac{1}{2} = x - 1\frac{1}{2}$$
$$y = x$$

(b) The function $y = x^2 - 2x - 3$ is defined in the domain, $-2 \le x \le 4$. The table below shows the corresponding values of *y* for five selected values of *x*.

x	−2	−1	0	1	2	3	4
y	5	0	−3		−3		5

(i) Complete the table by calculating and inserting the missing values of y.

SOLUTION:
Data: Table showing five corresponding values of x and y for the function $y = x^2 - 2x - 3$ for the domain, $-2 \le x \le 4$.
Required to complete: The table by inserting the missing values of y
Solution:

When $x = 1$

$y = (1)^2 - 2(1) - 3$

$y = -4$

When $x = 3$

$y = (3)^2 - 2(3) - 3$

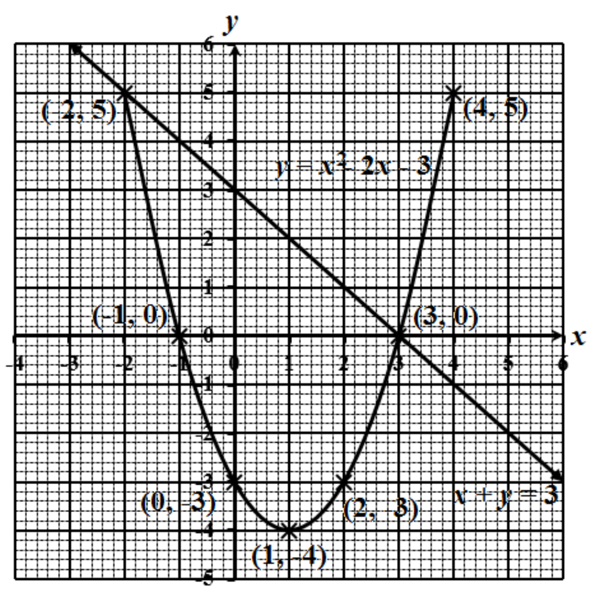
$y = 0$

The completed table will now look like:

x	-2	-1	0	1	2	3	4
y	5	0	-3	-4	-3	0	5

(ii) On the same axes used in Part (a), draw the graph of, $y = x^2 - 2x - 3$.

SOLUTION:
Required to draw: The graph of $y = x^2 - 2x - 3$ on the same axes in part (a)
Solution:



(iii) Using information from your graph, complete EACH of the following statements.

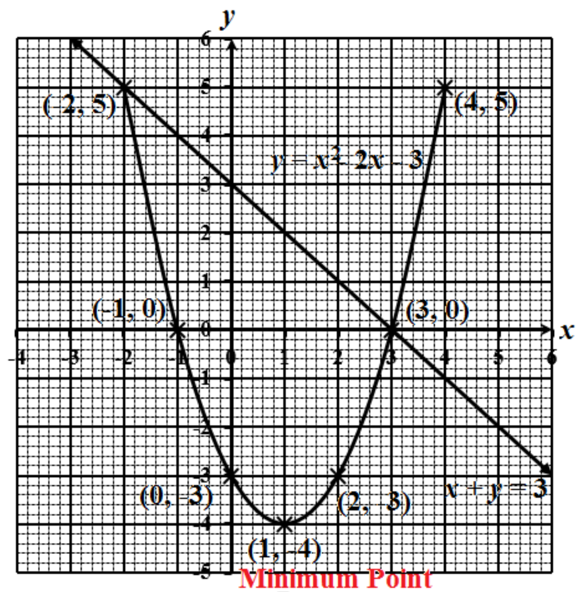
- The minimum value of $y = x^2 - 2x - 3$ occurs when $x = \ldots\ldots$

- The values of x for which $x^2 - 2x - 3 = -x + 3$ are $x = \ldots\ldots$ and $x = \ldots\ldots$

SOLUTION:

Required to complete: The statements given using information from the graph

Solution:



- The minimum point on the graph of $y = x^2 - 2x - 3$ occurs at $x = \dfrac{-(-2)}{2(1)}$. This is the equation of the axis of symmetry and hence is the x coordinate of the minimum point.
  $= 1$
  When $x = 1,\ y = -4$
  $\therefore$ The minimum point has coordinates $(1, -4)$

  The minimum value of $y = x^2 - 2x - 3$ occurs when $x = 1$.

- When $x^2 - 2x - 3 = -x + 3$, the curve $y = x^2 - 2x - 3$ and the straight line $y = -x + 3$ meet.
  This occurs at $(-2, 5)$ and at $(3, 0)$.

  Hence, the values of x for which $x^2 - 2x - 3 = -x + 3$ are $x = -2$ and $x = 3$.

7. Twenty bags of sugar were weighed. The weights, to the nearest kg, are as follows:

3	38	17	33	28
12	43	38	31	30
11	8	23	18	26
50	22	35	39	5

(a) Complete the frequency table for the data shown above.

Weight (kg)	Tally	Number of Bags
1 – 10		
11 – 20		
21 – 30		
31 – 40		
41 – 50		

SOLUTION:

Data: The weights of 20 bags of bags to the nearest kg.
Required to complete: The frequency table for the data shown
Solution:
The measure of weight is a continuous variable.
The modified frequency table looks like:

Weight (kg) **Lower Class Limit – Upper Class Limit**	**Class Boundaries** **Upper Class Boundary – Lower Class Boundary**	**Tally**	**Number of Bags**
1 – 10	$0.5 \le x < 10.5$	|||	3
11 – 20	$10.5 \le x < 20.5$	||||	4
21 – 30	$20.5 \le x < 30.5$	~~||||~~	5
31 – 40	$30.5 \le x < 40.5$	~~||||~~ |	6
41 – 50	$40.5 \le x < 50.5$	||	2
		Total	20

(b) For the class interval 21 – 30, state:

(i) The upper class boundary

SOLUTION:
Required to state: The upper class boundary for the class interval 21 – 30.
Solution:
The upper class boundary of the class interval 21 -30 is 30.5.

(ii) The class width

SOLUTION:
Required to state: The class width for the class interval 21 – 30
Solution:
The class width of the class interval 21 – 30

$= \text{Upper class boundary} - \text{Lower class boundary}$

$= 30.5 - 20.5$

$= 10$

(iii) The class midpoint

SOLUTION:
Required to state: The class midpoint for the class interval 21 – 30
Solution:

$$\text{The class midpoint } = \frac{\text{Upper class boundary} + \text{Lower class boundary}}{2}$$

OR

$$= \frac{\text{Upper class limit} + \text{Lower class limit}}{2}$$

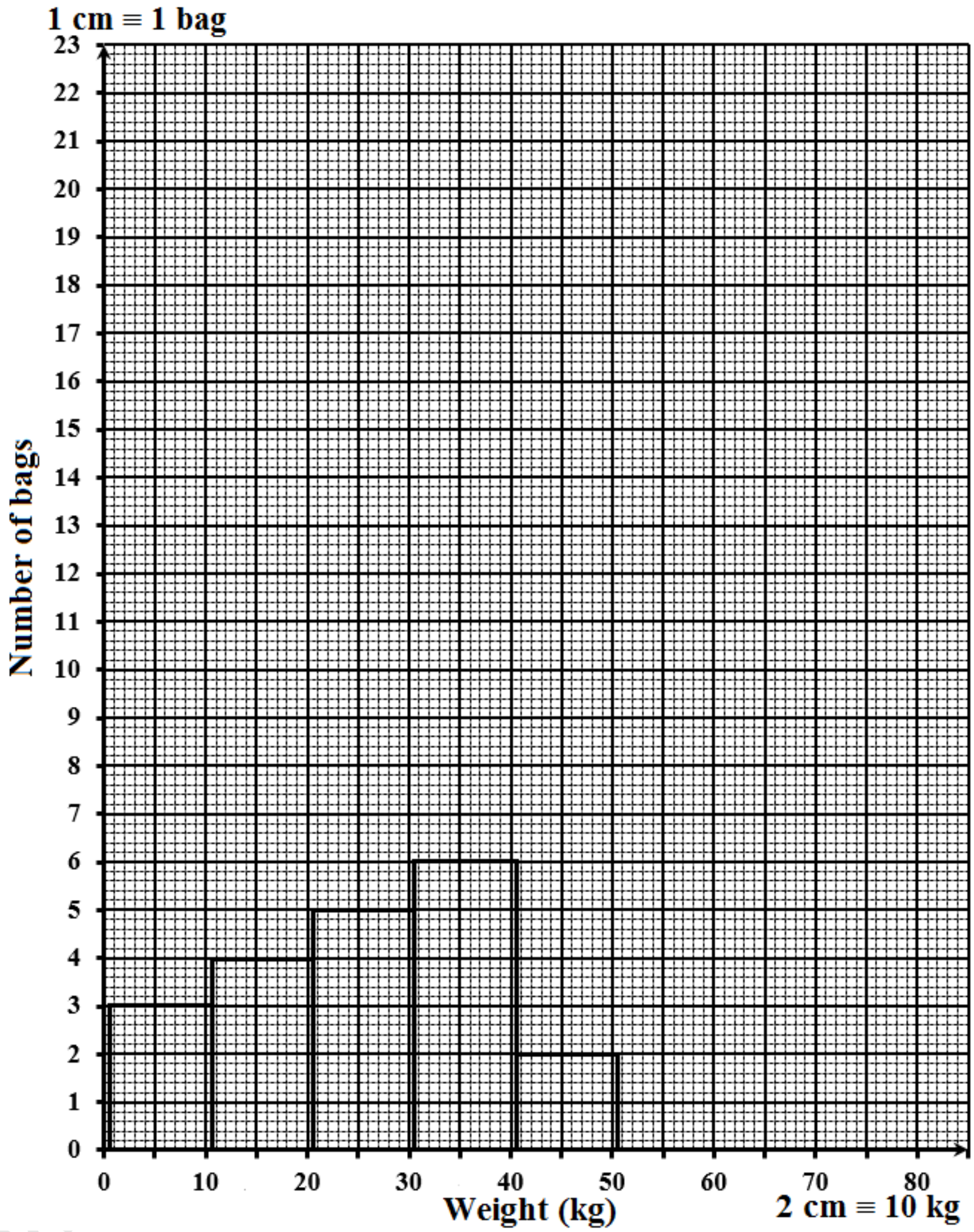
$$= \frac{20.5 + 30.5}{2} \text{ or } \frac{21 + 30}{2}$$

$$= 25.5$$

(c) On the grid, using a scale of **2 cm to represent 10 kg on the x – axis and 1 cm to represent 1 bag on the y – axis**, draw a histogram to represent the data contained in your frequency table above.

SOLUTION:
Required to draw: A histogram to represent the data given in the table

Solution:


1 cm ≡ 1 bag
Number of bags
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
0
10
20
30
40
50
60
70
80
Weight (kg)
2 cm ≡ 10 kg

8. The diagram below shows the first three figures in a sequence. Each figure is made up of knots and strings. Each knot connects exactly 3 strings.

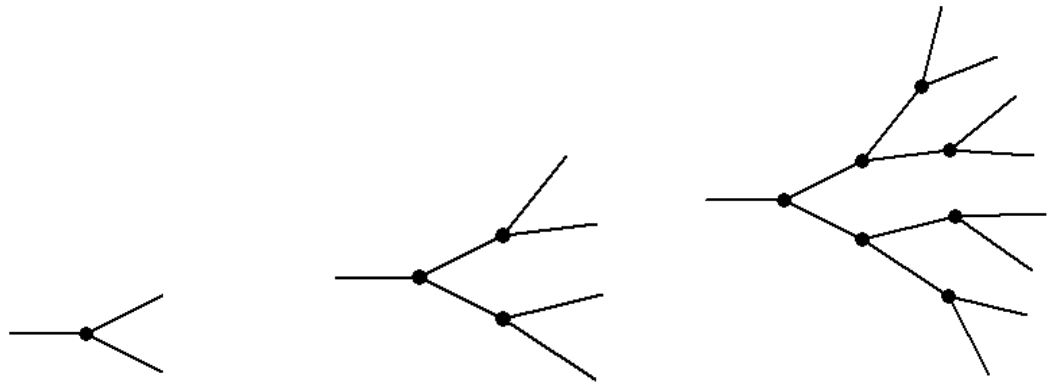
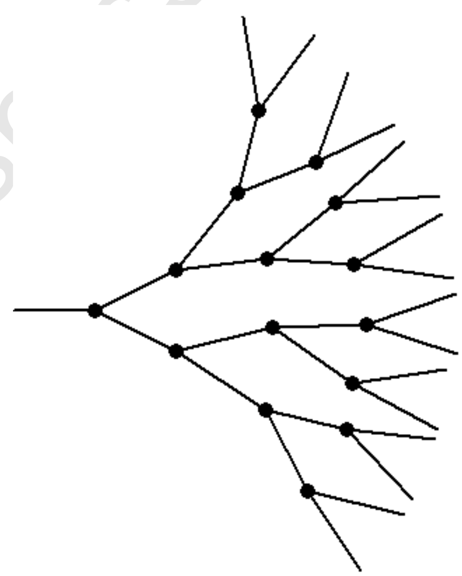


Figure 1 Figure 2 Figure 3

(a) Draw Figure 4 of the sequence.

SOLUTION:

Data: Diagram showing the first three figures in a sequence. Each figure is made up of knots and strings. Each knot connects exactly 3 strings.
Required to draw: Figure 4 of the sequence.
Solution:



(b) The table below shows the number of knots and strings in each figure used to draw Figures 1, 2 and 3. Complete the table by inserting the missing values in the rows numbered (i), (ii) and (iii).

	Figure Number (N)	Number of Knots (K)	Number of Strings (S)
	1	1	3
	2	3	7
	3	7	15
(i)	4	…………	……………
(ii)	…………	…………..	255
(iii)	10	…………..	…………….

SOLUTION:

Data: Table showing the number of knots and strings in each figure used to draw Figures 1, 2 and 3.
Required to complete: The table by inserting the missing values in the rows numbered (i), (ii) and (iii)
Solution:
For $S = 3$ Notice 3 = 2 (***K***=1) + 1
For $S = 7$ Notice 7 = 2 (***K***=3) + 1
For $S = 15$ Notice 15 = 2 (***K***=7) + 1

$$\therefore S = 2(K) + 1$$
$$S = 2K + 1$$

When $S = 255$
$$255 = 2K + 1$$
$$254 = 2K$$
$$K = 127$$

For $N = 1$ $K = 2^1 - 1 = 1$
For $N = 2$ $K = 2^2 - 1 = 3$
For $N = 3$ $K = 2^3 - 1 = 7$

$$\therefore K = 2^N - 1$$

The completed table looks like:

	Figure Number (N)	Number of Knots (K)	Number of Strings (S)
	1	1	3
	2	3	7
	3	7	15
(i)	4	15	31
(ii)	7	127	255
(iii)	10	1 023	2 047

Alternative Method:

When $N = 4$

$K = 2^N - 1$

$K = 2^4 - 1$

$= 16 - 1$

$= 15$

When $K = 15$

$S = 2(K) + 1$

$S = 2(15) + 1$

$S = 31$

When $S = 255$

$2K + 1 = 255$

$2K = 254$

$K = 127$

When $N = 10$

$K = 2^{10} - 1$

$= 1024 - 1$

$= 1023$

When $K = 1023$

$N = 2(1023) + 1$

$= 2047$

The completed table looks like:

	Figure Number (N)	Number of Knots (K)	Number of Strings (S)
	1	1	3
	2	3	7
	3	7	15
(i)	4	15	31
(ii)	7	127	255
(iii)	10	1 023	2 047

ALTERNATIVE METHOD:

Figure Number (N)	Number of Knots (K)	Number of Strings (S)
1	1	3
2	3	7
3	7	15

By observation:

K	S
1	$3=2(1)+1$
3	$7=2(3)+1$
7	$15=2(7)+1$

$\therefore S=2K+1$

The value of S in row 1 = The value of K in row 2
The value of S in row 2 = The value of K in row 3
The value of S in row 3 = The value of K in row 4
And so on.

Hence, we may choose to complete the table by first computing the S value, say

$S = 15$ when $n = 3$ and use this S value for the next K value when $n = 4$. This process is repeated until all the unknown values are found.

Figure (N)	**Number of Knots (K)**	**Number of Strings (S)** $S = 2K + 1$
1	1	3
2	3	7
3	7	15
4	15	$2(15)+1=31$
5	31	$2(31)+1=63$
6	63	$2(63)+1=127$
7	127	$2(127)+1=255$
8	255	$2(255)+1=511$
9	511	$2(511)+1=1023$
10	1 023	$2(1023)+1=2047$

Hence, the completed table looks like:

	Figure Number (N)	**Number of Knots (K)**	**Number of Strings (S)**
	1	1	3
	2	3	7
	3	7	15
(i)	4	15	31
(ii)	7	127	255
(iii)	10	1 023	2 047

SECTION II

ALGEBRA AND RELATIONS, FUNCTIONS AND GRAPHS

9. (a) The functions $f(x)$ and $g(x)$ are defined as $f(x)=2x-7$ and $g(x)=x^2+1$, respectively

(i) Write an expression, in terms of *x*, for EACH of the following:

- $f^{-1}(x)$

SOLUTION:
Data: $f(x)=2x-7$ and $g(x)=x^2+1$
Required to write: An expression for $f^{-1}(x)$, in terms of *x*.
Solution:
$f(x)=2x-7$
Let $y=2x-7$
$y+7=2x$
$\frac{y+7}{2}=x$
Replace *y* by *x*:
$\therefore f^{-1}(x)=\frac{x+7}{2}$

- $g^{-1}(x)$

SOLUTION:
Required to write: An expression for $g^{-1}(x)$, in terms of *x*.
Solution:
$g(x)=x^2+1$
Let $y=x^2+1$
$y-1=x^2$
$\sqrt{y-1}=x$
Replace *y* by *x*:
$\therefore g^{-1}(x)=\sqrt{x-1}$

- $fg(x)$

SOLUTION:
Required to write: An expression for $fg(x)$, in terms of x.
Solution:

$$\begin{aligned} fg(x) &= 2(g(x)) - 7 \\ &= 2(x^2 + 1) - 7 \\ &= 2x^2 + 2 - 7 \\ &= 2x^2 - 5 \end{aligned}$$

- $(fg)^{-1}x$

SOLUTION:
Required to write: An expression for $(fg)^{-1}x$, in terms of x.
Solution:
Let $y = 2x^2 - 5$

$$y + 5 = 2x^2$$

$$\frac{y+5}{2} = x^2$$

$$x = \sqrt{\frac{y+5}{2}}$$

Replace y by x:

$$(fg)^{-1}x = \sqrt{\frac{x+5}{2}}$$

(ii) Show that, $(fg)^{-1}(5) = g^{-1}f^{-1}(5)$.

SOLUTION:
Required to prove: $(fg)^{-1}(5) = g^{-1}f^{-1}(5)$
Proof:

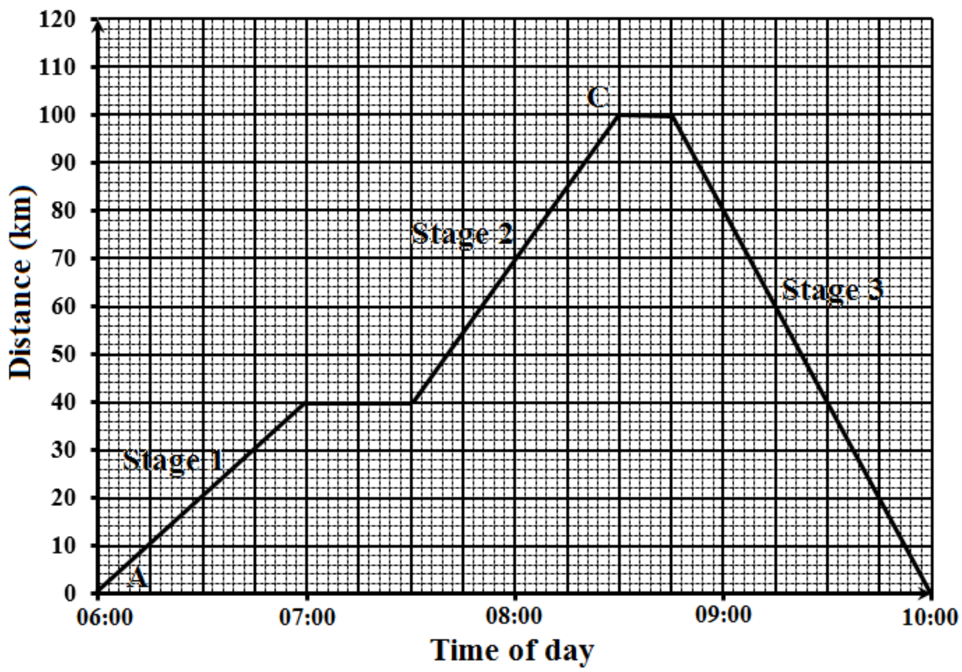
$$\begin{aligned} (fg)^{-1}(5) &= \sqrt{\frac{5+5}{2}} \\ &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} f^{-1}(5) &= \frac{5+7}{2} \\ &= 6 \end{aligned}$$

$$g^{-1}f^{-1}(5) = g^{-1}(6)$$
$$= \sqrt{6-1}$$
$$= \sqrt{5}$$
$$\therefore (fg)^{-1}(5) = g^{-1}f^{-1}(5)$$

Q.E.D.

(b) The distance-time graph below shows the three stage journey of a car travelling from Town *A* to Town *C* and back to Town *A*.

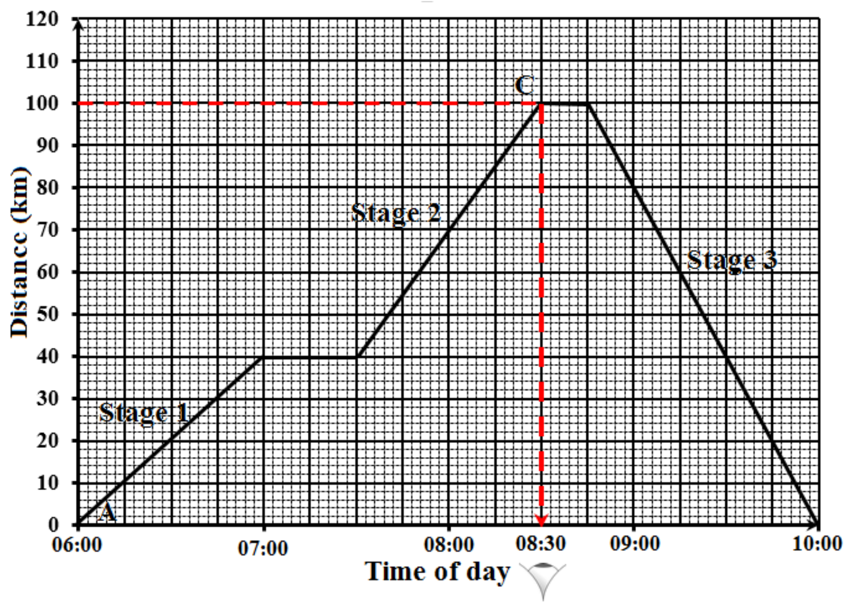


(i) State the time of day at which the car arrived at Town *C*.

SOLUTION:

Data: A distance-time graph showing the three stage journey of a car travelling from Town *A* to Town *C* and back to Town *A*.
Required to state: The time of day at which the car arrived at Town *C*

Solution:



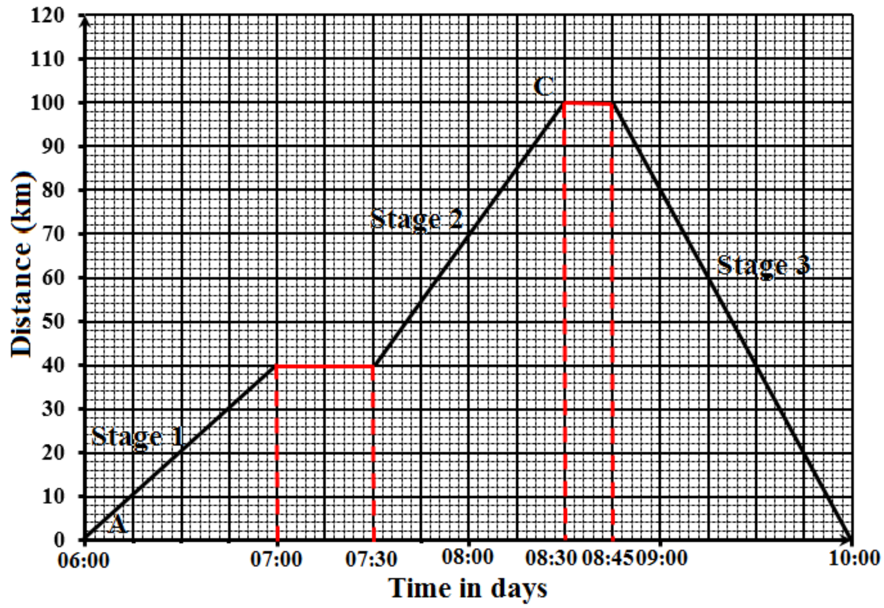
The car arrived at Town C at 08:30 (obtained by a read off as shown in red).

(ii) Calculate the TOTAL time, in minutes, for which the car stopped during the journey.

SOLUTION:

Required to calculate: The total time for which the car stopped during the journey

Calculation:
The stoppage period of the car is shown by the horizontal branches.



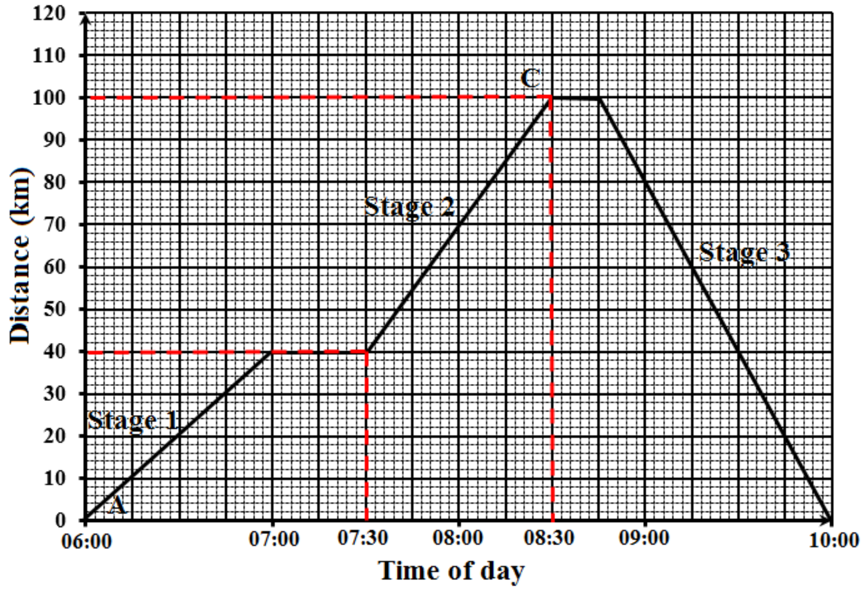
Hence, the total stoppage time of the car occurs from 07:00 to 07:30 (30 mins) and 08:30 to 08:45 (15 mins)

$$= (30 + 15) \text{ minutes}$$
$$= 45 \text{ minutes}$$
$$= \frac{3}{4} \text{ hour}$$

(iii) Determine the constant speed of the car during Stage 2 of the journey.

SOLUTION:
Required to determine: The constant speed of the car during Stage 2 of the journey

Solution:



Since the 'branch' is straight, the speed is constant.

$$\text{The constant speed} = \frac{\text{Distance covered}}{\text{Time taken}}$$

$$= \frac{(100-40)\text{ km}}{8:30-7:30}$$

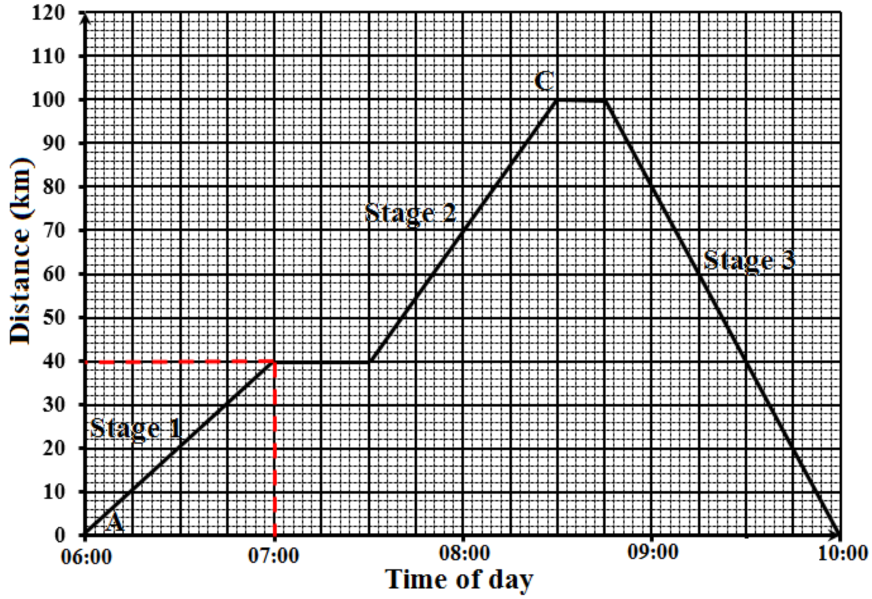
$$= \frac{60\text{ km}}{1\text{ hour}}$$

$$= 60\,\text{kmh}^{-1}$$

(iv) Calculate the average speed of the car for the time during which it was moving.

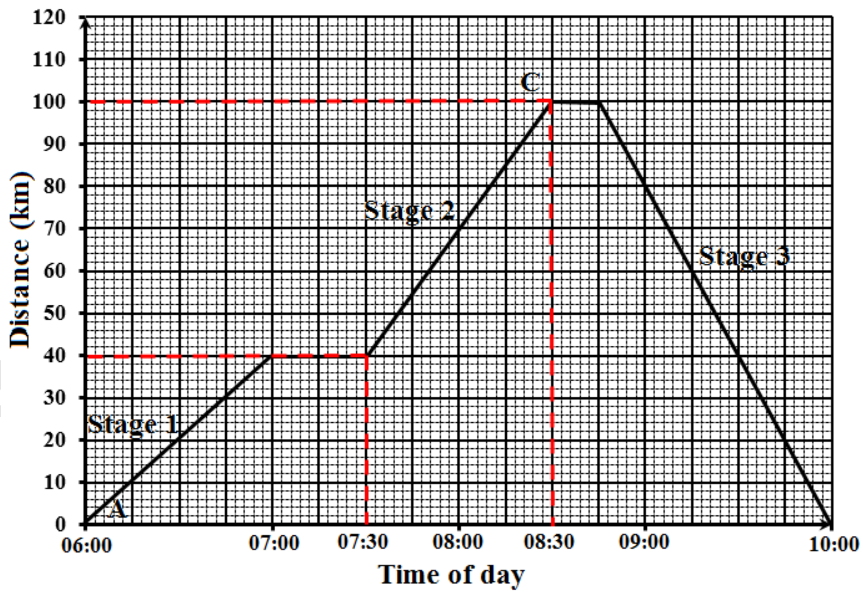
SOLUTION:

Required to calculate: The average speed of the car for the time during which it was moving
Calculation:
Stage 1



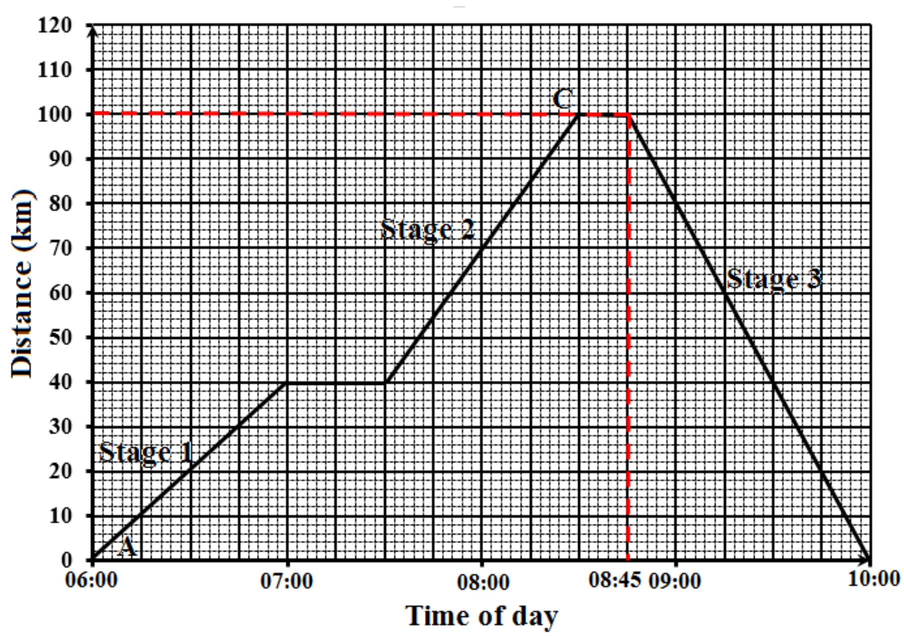
$(40-0)$ km $= 40$ km from 06:00 to 07:00.
40 km in 1 hour

Stage 2



$(100-40)$ km $= 60$ km from 07:30 to 08:30
60 km in 1 hour

Stage 3



$(100-0)$ km $=100$ km from 08:45 to 10:00

100 km in $1\frac{1}{4}$ hours

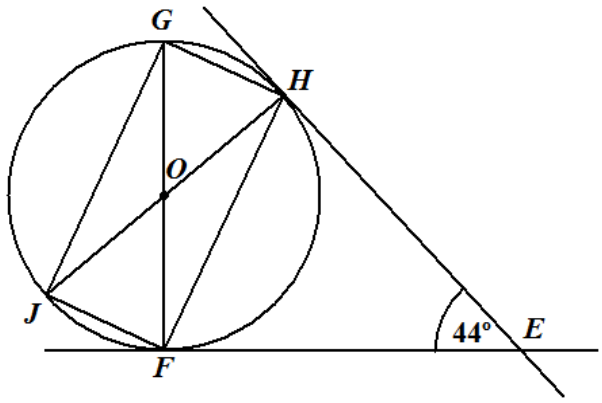
$$\text{Total distance covered} = 40+60+100 = 200 \text{ km}$$

$$\text{Total time taken} = 1+1+1\frac{1}{4} = 3\frac{1}{4} \text{ hours}$$

$$\text{Average speed} = \frac{\text{Total distance covered}}{\text{Total time taken}} = \frac{200 \text{ km}}{3\frac{1}{4} \text{ hour}} = \frac{800}{12} \text{ kmh}^{-1} \approx 61.54 \text{ kmh}^{-1}$$

MEASUREMENT, GEOMETRY AND TRIGONOMETRY

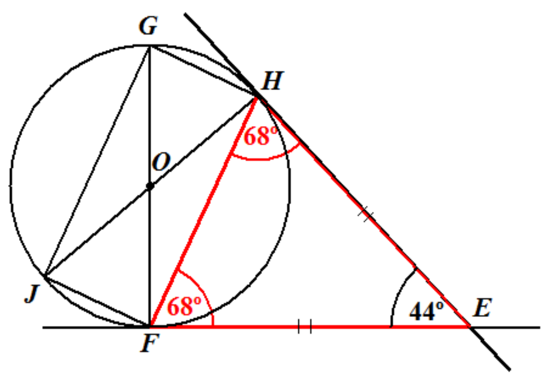
10. (a) The diagram below, **not drawn to scale**, shows a circle, center *O*. *EH* and *EF* are tangents to the circle. *FOG* and *JOH* are straight lines. The measure of $\angle FEH = 44°$.



Calculate, **giving reasons for your answer**, the measure of:

(i) $\angle EHF$

SOLUTION:
Data: Diagram showing a circle, center *O*. *EH* and *EF* are tangents to the circle. *FOG* and *JOH* are straight lines. The measure of $\angle FEH = 44°$.
Required to calculate: $\angle EHF$
Calculation:



The two tangents that can be drawn to a circle from a point outside the circle are equal in length.
Hence, $EH = EF$ and triangle EHF is isosceles.
$\therefore E\hat{H}F = E\hat{F}H$ (the base angles of an isosceles triangle are equal)
Each would therefore be

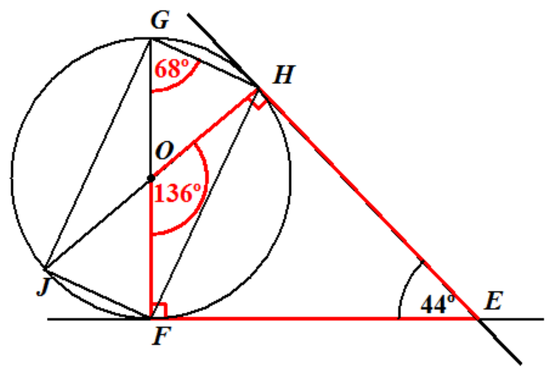
$$= \frac{180° - 44°}{2} \quad \text{(Sum of angles in a triangle = 180}^0\text{)}$$

$$= \frac{136°}{2}$$

$$= 68°$$

(ii) $\angle FGH$

SOLUTION:
Required to calculate: $\angle FGH$
Calculation:



$O\hat{H}E$ and $O\hat{F}E = 90°$
(The angle made by a tangent to a circle and a radius, at the point of contact, is a right angle).

$$F\hat{O}H = 360° - (90° + 90° + 44°)$$
$$= 136°$$

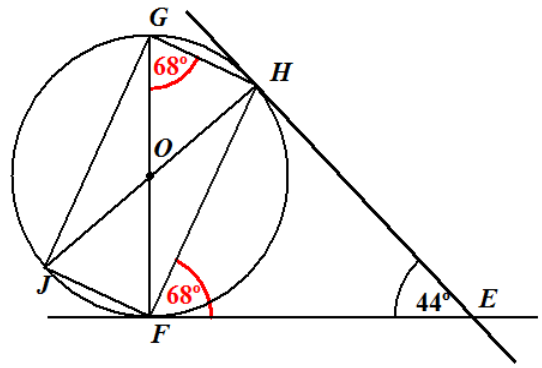
(The sum of the angles of a quadrilateral = 360^0)

$$F\hat{G}H = \frac{1}{2}(136°)$$
$$= 68°$$

(The angle subtended by a chord at the center of a circle, in this case $F\hat{O}H$, is twice the angle that the chord subtends at the circumference, standing on the same arc).

$\therefore F\hat{G}H = 68°$

Alternative Method:



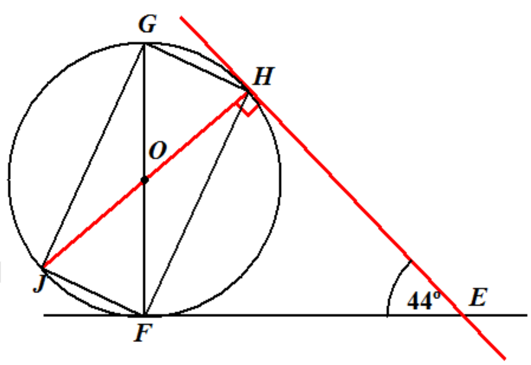
$F\hat{G}H = H\hat{F}E$

$= 68°$

(The angle made by the tangent to a circle and a chord, at the point of contact is equal to the angle in the alternate segment).

(iii) $\angle JHE$

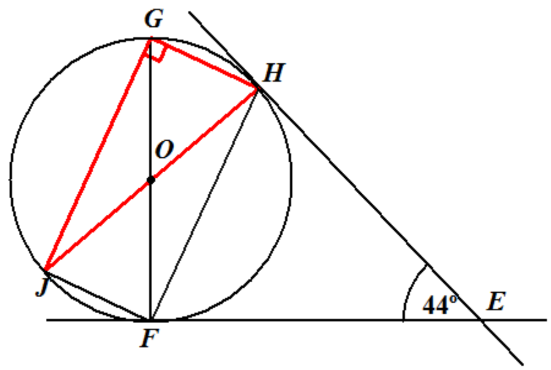
SOLUTION:
Required to calculate: $\angle JHE$
Calculation:



$O\hat{H}E = 90°$
(The angle made by a tangent to a circle and a radius, at the point of contact, is a right angle).

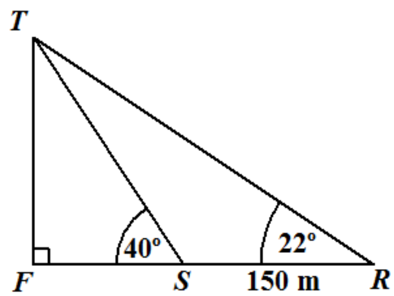
(iv) $\angle JGH$

SOLUTION:
Required to calculate: $\angle JGH$
Calculation:



$J\hat{G}H = 90°$
(Angle in a semi-circle is a right angle.)

(b) The diagram below**, not drawn to scale**, shows two ships, R and S at anchor on a lake of calm water. FT is a vertical tower. FSR is a straight line and $RS = 150$ m. The angles of elevation of T, the top of a tower, from R and S, are 22° and 40° respectively. F is the foot of the tower.



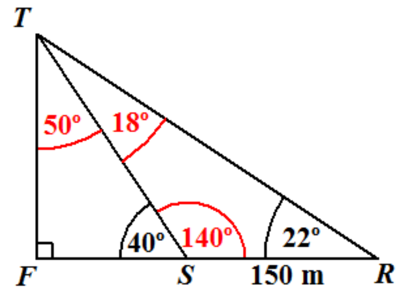
Calculate, giving your answer to 1 decimal place where appropriate

(i) the measure of $\angle RTS$

SOLUTION:

Data: Diagram showing two ships, R and S at anchor on a lake of calm water. FT is a vertical tower. FSR is a straight line and $RS = 150$ m. The angles of elevation of T, the top of a tower, from R and S, are 22° and 40° respectively. F is the foot of the tower.

Required to calculate: $\angle RTS$
Calculation:



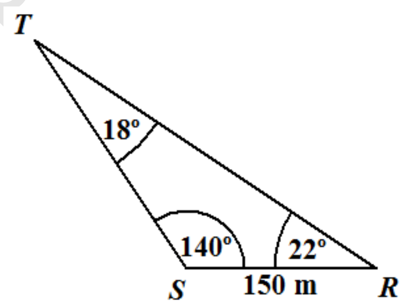
$$F\hat{T}S = 180° - (90° + 40°)$$
$$= 50°$$
(Sum of the angles in a triangle is equal to 180°)

$$T\hat{S}R = 180° - 40°$$ (Angle in a straight line = 180^0)
$$= 140°$$

$$R\hat{T}S = 180° - (140° + 22°)$$
$$= 18°$$ (Sum of the angles in a triangle is equal to 180°)

(ii) The length of *ST*

SOLUTION:
Required to calculate: *ST*
Calculation:
Applying the sine rule to triangle *TSR*.

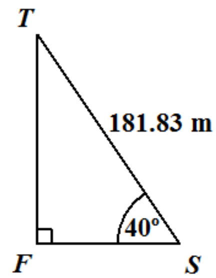


Using the sine rule

$$\frac{ST}{\sin 22°} = \frac{150}{\sin 18°}$$
$$\therefore ST = \frac{150 \times \sin 22°}{\sin 18°}$$
$$ST = 181.8\underline{\underline{3}}$$
$$= 181.8 \text{ m (correct to 1 decimal place)}$$

(iii) The height of the tower, *FT*.

SOLUTION:
Required To calculate: *FT*
Calculation:



$$\frac{TF}{181.83} = \sin 40°$$

$$\therefore TF = 181.83 \times \sin 40°$$

$$= 116.87 \text{ m}$$

$$= 116.9 \text{ m (correct to 1 decimal place)}$$

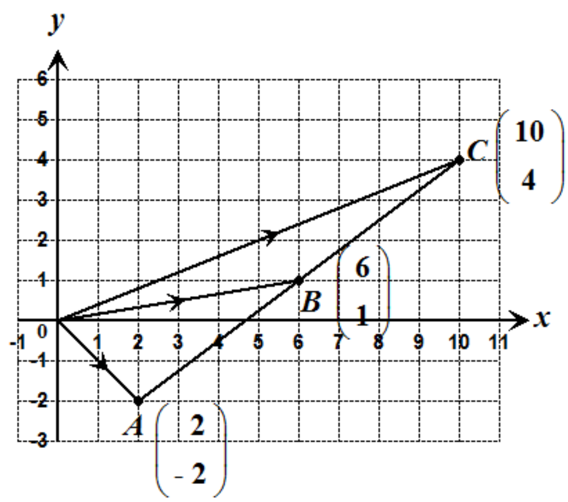
VECTORS AND MATRICES

11. (a) The position vectors of points A, B and C, relative to the origin O, are $OA = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $OB = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$ and $OC = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$ respectively.

(i) Express in the form $\begin{pmatrix} x \\ y \end{pmatrix}$ the vectors

- $\overrightarrow{AB}$

SOLUTION:
Data: The position vectors of points A, B and C, relative to the origin O, are $OA = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $OB = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$ and $OC = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$ respectively.



Required to express: $\overrightarrow{AB}$ in the form $\begin{pmatrix} x \\ y \end{pmatrix}$.

Solution:

$$AB = AO + OB$$
$$= -\begin{pmatrix} 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ 3 \end{pmatrix} \text{ is of the form } \begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } x = 4 \text{ and } y = 3.$$

- $\overrightarrow{AC}$

SOLUTION:

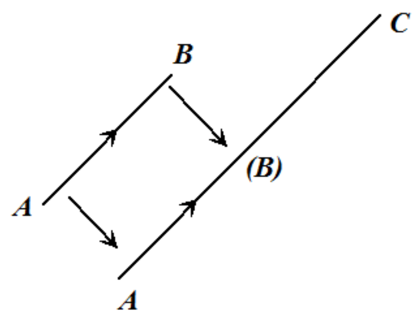
Required to express: $\overrightarrow{AC}$ in terms of $\begin{pmatrix} x \\ y \end{pmatrix}$.

Solution:

$$AC = AO + OC$$
$$= -\begin{pmatrix} 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} 8 \\ 6 \end{pmatrix} \text{ is of the form } \begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } x = 8 \text{ and } y = 6$$

(ii) Hence, determine whether A, B and C are collinear, **giving the reasons for your answer**.

SOLUTION:
Required to determine: Whether A, B and C are collinear
Solution:



$$AB = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$
$$AC = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$$
$$= 2\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$\overrightarrow{AC}$ is a scalar multiple (= 2) of $\overrightarrow{AB}$. Hence, AC is parallel to AB.
A is a common point. Hence, A and B both lie on AC and A, B and C are collinear.

(b) Determine the value of x for which the matrix $\begin{pmatrix} 3 & x \\ 2 & 4 \end{pmatrix}$ is singular.

SOLUTION:

Required to determine: The value of x for which $\begin{pmatrix} 3 & x \\ 2 & 4 \end{pmatrix}$ is a singular matrix

Solution:

Let $A = \begin{pmatrix} 3 & x \\ 2 & 4 \end{pmatrix}$

$$|A| = (3\times4) - (x\times2)$$
$$= 12 - 2x$$

If A is a singular matrix, then $|A|$ or $\det A = 0$.

$\therefore 12 - 2x = 0$

$x = 6$

(c) N and P are 2×2 matrices such that $N = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix}$.

(i) Determine NP.

SOLUTION:

Data: N and P are 2×2 matrices such that $N = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix}$.

Required to determine: NP

Solution:

$$NP = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$$
$$2\times2 \quad \times \quad 2\times2 \quad = \quad 2\times2$$

$$e_{11} = (4\times1) + (1\times2) = 4 + 2 = 6$$
$$e_{12} = (4\times5) + (1\times1) = 20 + 1 = 21$$
$$e_{21} = (3\times1) + (2\times2) = 3 + 4 = 7$$
$$e_{22} = (3\times5) + (2\times1) = 15 + 2 = 17$$

$$\therefore NP = \begin{pmatrix} 6 & 21 \\ 7 & 17 \end{pmatrix}$$

(ii) Given that $PN = \begin{pmatrix} 19 & 11 \\ 11 & 4 \end{pmatrix}$, determine whether matrix multiplication is commutative.

SOLUTION:

Data: $PN = \begin{pmatrix} 19 & 11 \\ 11 & 4 \end{pmatrix}$

Required to determine: Whether matrix multiplication is commutative

Solution:

If $PN = \begin{pmatrix} 19 & 11 \\ 11 & 4 \end{pmatrix}$ and for the previous part, it was found that

$NP = \begin{pmatrix} 6 & 21 \\ 7 & 17 \end{pmatrix}$, then $NP \neq PN$ and so, matrix multiplication is not commutative (except in the case of the inverse, and where $AA^{-1} = A^{-1}A$).

(iii) Determine N^{-1}, the inverse of *N*.

SOLUTION:

Required to determine: N^{-1}

Solution:

$$N = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\begin{aligned} \det N &= (4 \times 2) - (1 \times 3) \\ &= 8 - 3 \\ &= 5 \end{aligned}$$

$$\begin{aligned} \therefore N^{-1} &= \frac{1}{5}\begin{pmatrix} 2 & -(1) \\ -(3) & 4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix} \end{aligned}$$

(iv) Hence, calculate the values of *x* and *y* for which $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

SOLUTION:

Data: $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Required to calculate: *x* and *y*

Calculation:

$$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Multiply by N^{-1}

$$\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$N^{-1}N\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \left(\frac{2}{5}\times 1\right) + \left(-\frac{1}{5}\times 2\right) \\ \left(-\frac{3}{5}\times 1\right) + \left(\frac{4}{5}\times 2\right) \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Equate corresponding entries.

$\therefore x = 0$ and $y = 1$