FAS-PASS Maths

CSEC MATHEMATICS JANUARY 2015

Section I

1. (a) Using a calculator, or otherwise, calculate the EXACT value of

$$(12.8)^2 - (30 \div 0.375).$$

SOLUTION:

Required to calculate: The exact value of $(12.8)^2 - (30 \div 0.375)$

Calculation:

We first work out each of the calculations that are within the brackets, using either basic arithmetic or the calculator. Then, we simplify to obtain the final answer.

$$\begin{aligned}(12.8)^2 - (30 \div 0.375) &= (12.8)(12.8) - (30 \div 0.375) \\ &= 163.84 - 80 \quad \text{(by the calculator)} \\ &= 83.84 \quad \text{(in exact form)}\end{aligned}$$

(b) Mark spends $\frac{3}{8}$ of his monthly income on housing. Of the REMAINDER, he spends $\frac{1}{3}$ on food and saves what is left.

(i) Calculate the fraction of his monthly income spent on food.

SOLUTION:

Data: Mark spends $\frac{3}{8}$ of his income on housing and $\frac{1}{3}$ of the remainder on food. Mark saves the rest of his income.

Required to calculate: The fraction spent on food

Calculation:

Let us first consider Mark's entire income as the whole.

When $\frac{3}{8}$ is spent on housing, the fraction that is the remainder is

$$1 - \frac{3}{8} = \frac{8}{8} - \frac{3}{8} = \frac{5}{8}$$

Recall, according to the data that $\frac{1}{3}$ of this remainder is spent on food.

Hence, the fraction of Mark's income which is spent on food $= \frac{1}{3} \times \frac{5}{8}$

$$= \frac{5}{24}$$

(ii) Calculate the fraction of his monthly income that he saved.

SOLUTION

Required to calculate: The fraction of Mark's income that is saved
Calculation:

The fraction of Mark's income spent on housing $= \frac{3}{8}$ (data)

The fraction of Mark's income spent on food $= \frac{5}{24}$ from part(i)

Hence, the fraction of Mark's income that is spent on both housing and food will be

$$= \frac{3}{8} + \frac{5}{24}$$
$$= \frac{3 \times 3}{8 \times 3} + \frac{5}{24}$$
$$= \frac{9}{24} + \frac{5}{24}$$
$$= \frac{9+5}{24}$$
$$= \frac{14}{24}$$

The question says that the rest of Mark's income is saved.
Hence, the fraction of Mark's income that is saved will be

$$1 - \frac{14}{24} = \frac{24}{24} - \frac{14}{24}$$
$$= \frac{10}{24}$$
$$= \frac{5}{12}$$

(c) (i) At Bank A, US $1.00 = BD $1.96. Calculate the value of US $700 in BD$.
US$ means United States dollars and BD$ means Barbados dollars.

SOLUTION

Data: At Bank A, US $1.00 ≡ BD $1.96
Required to calculate: The value of US $700 in BD$.
Calculation:

$$\text{US } \$1.00 \equiv \text{BD } \$1.96$$
$$\text{Hence, US } \$700 \equiv \text{BD } \$1.96 \times 700$$
$$= \text{BD } \$1\ 372$$

(ii) At Bank B, the value of US $700 is BD $1 386. Calculate the value of US $1.00 in BD$ at this bank.

SOLUTION

Data: At Bank B, US $\$700 \equiv$ BD $1 386.
Required to calculate: The value of US $1.00 in BD$ at Bank B.
Calculation:

$$\text{US } \$700 \equiv \text{BD } \$1\ 386$$
$$\text{Hence, US } \$1.00 \equiv \text{BD } \frac{\$1\ 386}{700}$$
$$= \text{BD } \$1.98$$

2. (a) Simplify

$$p^3q^2 \times pq^5$$

SOLUTION
Required to simplify: $p^3q^2 \times pq^5$
Simplification:
Let us group the common terms together, for convenience, and then apply the sum law of indices to the terms in p and then the terms in q.

$$p^3q^2 \times pq^5 = p^3 \times p \times q^2 \times q^5$$
$$= p^{3+1} \times q^{2+5}$$
$$= p^4 \times q^7$$
$$= p^4q^7$$

(b) Express as a single fraction in its simplest form

$$\frac{a}{5} + \frac{3a}{2}$$

SOLUTION
Required to express: $\frac{a}{5} + \frac{3a}{2}$ as a single fraction in its simplest form.
Solution:
The LCM of 5 and 2 is 10

So,

$$\frac{a}{5}+\frac{3a}{2}$$

$$\frac{2(a)+5(3a)}{10}=\frac{2a+15a}{10}$$

$$=\frac{17a}{10}$$ (as a single fraction in its simplest form)

(c) Factorise completely:

(i) x^2-5x+4

Required to factorise: x^2-5x+4
Solution:

$$x^2-5x+4=x^2-x-4x+4$$

$$=x(x-1)-4(x-1)$$

$$=(x-1)(x-4)$$

OR
We could have found the two numbers whose sum is -5 and product is 4. These are -1 and -4. Hence, $(x-1)(x-4)$

(ii) m^2-4n^2

Required to factorise completely: m^2-4n^2
Solution:

$$m^2-4n^2=(m)^2-(2n)^2$$

This is now expressed as the difference of two squares, and which is of a standard form.
And so,

$$m^2-4n^2=(m-2n)(m+2n)$$

(d) (i) Solve for x

$$2x-7\le 3$$

SOLUTION

Data: $2x-7\le 3$
Required to solve: For x.

Solution:

$$2x - 7 \le 3$$
$$2x \le 3 + 7$$
$$2x \le 10$$
$$\div 2$$
$$x \le \frac{10}{2}$$
$$x \le 5$$

Hence, $x \le 5$. This is better expressed as $\{x : x \le 5\}$.
We use set builder notation as we cannot write out all the solutions.
This may also be illustrated on the number line as:

(ii) If x is a positive integer, list the possible values of x.

SOLUTION

Data: x is a positive integer.
Required to list: The possible values of x
Solution:
The solution is $x \le 5$.
Hence, x will be all the positive integers that are less than or equal to 5.
$x = 5, 4, 3, 2$ or 1. (Notice, 0 is not included as 0 is not positive)

(e) Find the value of $2\pi\sqrt{\frac{l}{g}}$
where $\pi = 3.14$, $l = 0.625$ and $g = 10$.

SOLUTION

Data: $T = 2\pi\sqrt{\frac{l}{g}}$, $\pi = 3.14$, $l = 0.625$ and $g = 10$.
Required to calculate: T

Calculation:
We substitute the given values in the expression

$$T = 2\pi\sqrt{\frac{l}{g}}$$
$$= 2(3.14)\sqrt{\frac{0.625}{10}}$$
$$= 2(3.14)\sqrt{0.0625} \text{ (using the calculator)}$$
$$= 6.28\sqrt{0.0625}$$
$$= 6.28 \times 0.25$$
$$= 1.57 \text{ (exact)}$$

3. (a) In a survey of 30 families, the findings were that:

15 families owned dogs
12 families owned cats
x families owned BOTH dogs and cats
8 families owned NEITHER dogs NOR cats

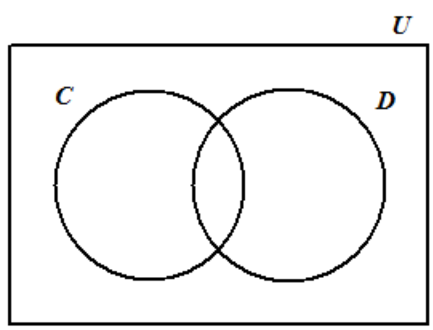
(i) Given that:

$U = \{\text{Families in the survey}\}$

$C = \{\text{Families who owned cats}\}$

$D = \{\text{Families who owned dogs}\}$

Use the given information to complete the Venn diagram below.



SOLUTION

Data: From the 30 families surveyed, 15 owned dogs, 12 owned cats, x owned both dogs and cats and 8 did not own either a cat or a dog.

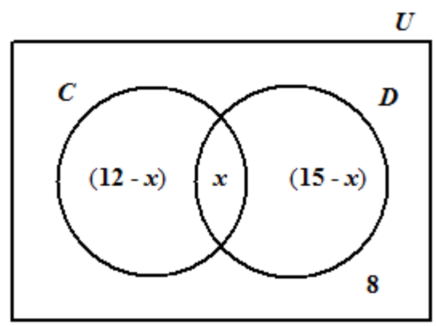
(A family cannot OWN neither a cat nor a dog-This is better expressed as 8 families do not own either a cat or a dog as is stated above)

$U = \{\text{Families in the survey}\}$

$C = \{\text{Families who owned cats}\}$

$D = \{\text{Families who owned dogs}\}$

Required to complete: The Venn diagram that is given
Solution:



(ii) Write an expression, in *x*, which represents the TOTAL number of families in the survey.

SOLUTION

Required to write: An expression, in *x*, which represents the total number of families in the survey
Solution:
Total number of families in the survey
= Sum of the all the families in the all the four subsets of the Universal set.

$= (12 - x) + x + (15 - x) + 8$

$= 12 + 15 + 8 - x + x - x$

$= 35 - x$

(iii) Write an equation which may be used to solve for *x*.

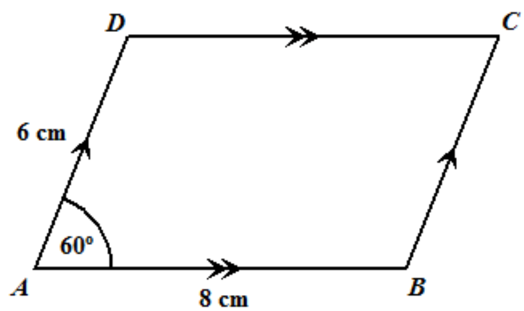
SOLUTION

Required to write: An equation which may be used to solve for *x*
Solution:
The total number of families in all the subsets of the universal set, will be equal to the number of families surveyed in the data.

Hence, $35 - x = 30$

(The question did not ask for the solution of x but if the value of x was required, then $x = 5$)

(b) The diagram below, **not drawn to scale**, shows a parallelogram *ABCD*.



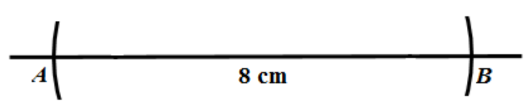
Using a ruler, a pencil and a pair of compasses only, construct parallelogram *ABCD* with $AB = 8$ cm, $AD = 6$ cm and $\angle DAB = 60°$.

Marks will be awarded for construction lines clearly shown.

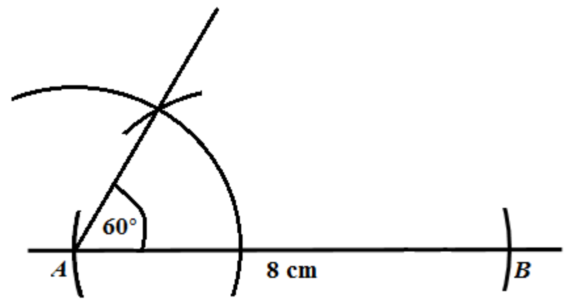
Required to construct: The parallelogram *ABCD* with $AB = 8$ cm, $AD = 6$ cm and $\angle DAB = 60°$.

SOLUTION
Construction: Shown in steps

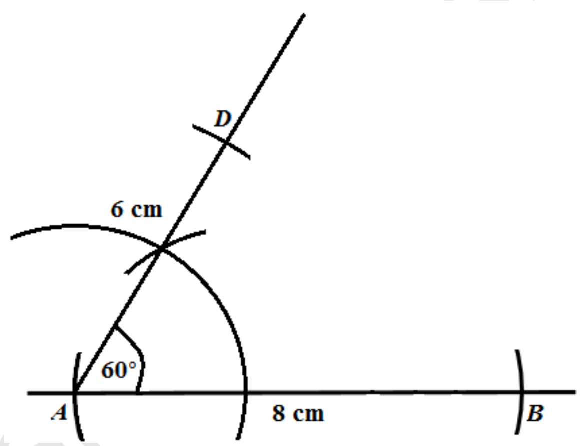
Step (1)
First, we draw a straight line which is longer than 8 cm and with the pair of compasses, we draw two arcs to mark off *A* and *B* so that *AB* is 8 cm long. The arcs are to be clearly shown.



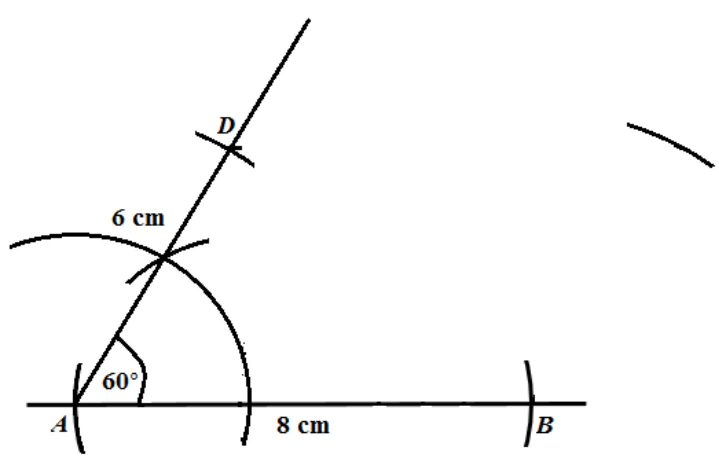
Step (2)
At *A*, we construct an angle of 60^0. This is illustrated in the diagram below.



Step (3)
With the pair of compasses, we cut off *D*, so that *AD* is 6 cm. This is shown below. Notice the arc that cuts off ***D*** is clearly shown.



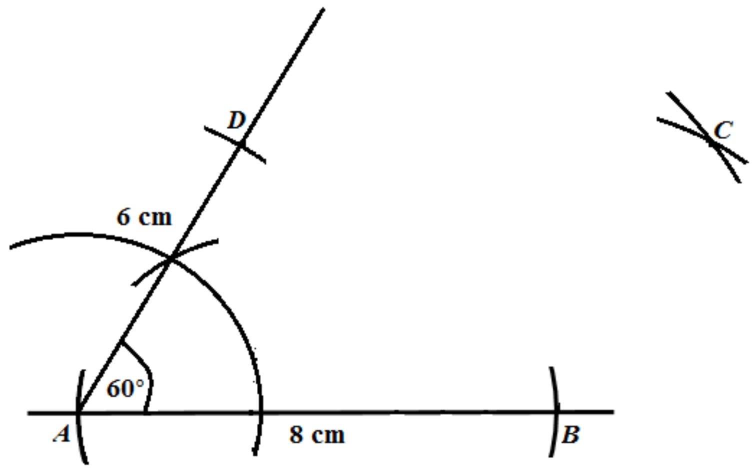
Step (4)
With center *B*, an arc of radius of 6 cm is drawn above *B*



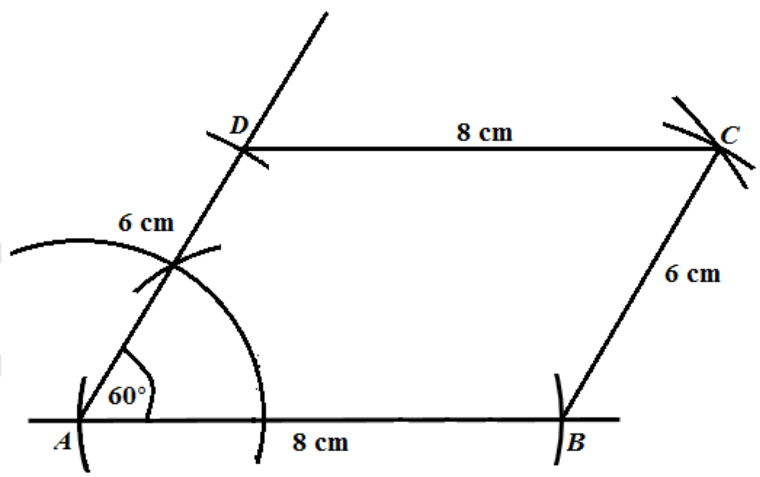
Step (5)

The opposite sides of a parallelogram are equal in length.
So, with center *D*, an arc of radius 8 cm is drawn to the right of *D*.

The arcs drawn from *B* and from *D* intersect at C.

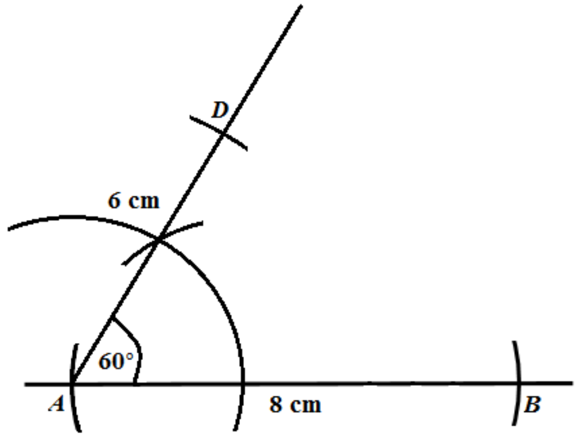


Step (6)
The parallelogram *ABCD* is now completed.



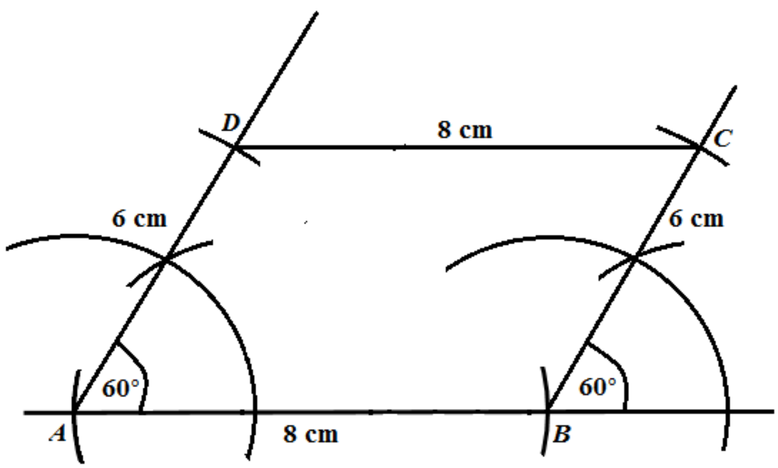
ALTERNATIVE
The construction could also have been done as:

Steps (1-3) are the same as above.



(4) At B we construct an angle of 60^0
(5) We cut off point C so that BC is 6 cm
(6) Join D to C to complete the parallelogram $ABCD$**.**

BC would be parallel to ***AD*** and equal in length. Remember, the opposite sides of a quadrilateral being both parallel and equal, gives a parallelogram.



4. An electrician charges a fixed fee for a house visit plus an additional charge based on the length of time spent on the job.

   The total charges, y, are calculated using the equation $y = 40x + 75$, where x represents the time in hours spent on the job.

   (a) Complete the table of values for the equation, $y = 40x + 75$.

x **(time in hours)**	0	1	2	3	4	5	6
y **(total charges in $)**	75	115		195		275	315

SOLUTION:

Data: Electrician charges a fixed fee for a home visit plus charges based on time spent on the job. The fixed fee means that this fee is paid even when no work has been done, i.e. just for the visit of the electrician's appearance at the site.
Total charges = y
Number of hours spent on the job = x
$y = 40x + 75$
Required to complete: The table given
Solution:
$y = 40x + 75$
When $x = 2$, we substitute to get
$y = 40(2) + 75$
$= 155$

When $x = 4$ we substitute to get

$y = 40(4) + 75$
$= 235$

The completed table will now look like:

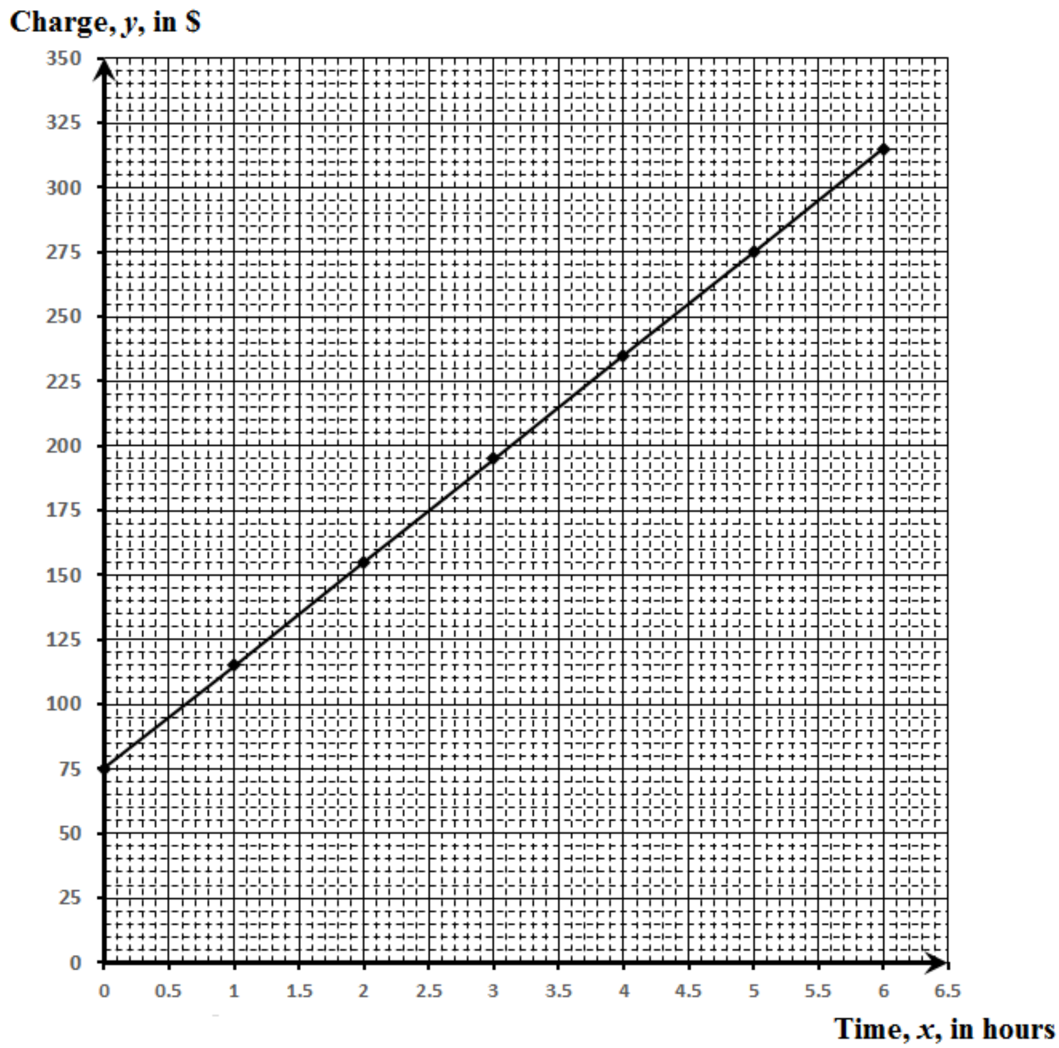
***x* (time in hours)**	0	1	2	3	4	5	6
***y* (total charges in $)**	75	115	155	195	235	275	315

(b) On the grid given, using a scale of **2 cm to represent 1 hour on the *x* – axis and 2 cm to represent 50 dollars on the *y* – axis**, plot the 7 pairs of values shown in your completed table. Draw a straight line through all the plotted points.

Required to plot: The values from the table, given the scale to use on both axes

SOLUTION: On carefully labelled axes we obtain

2 cm ≡ \$50

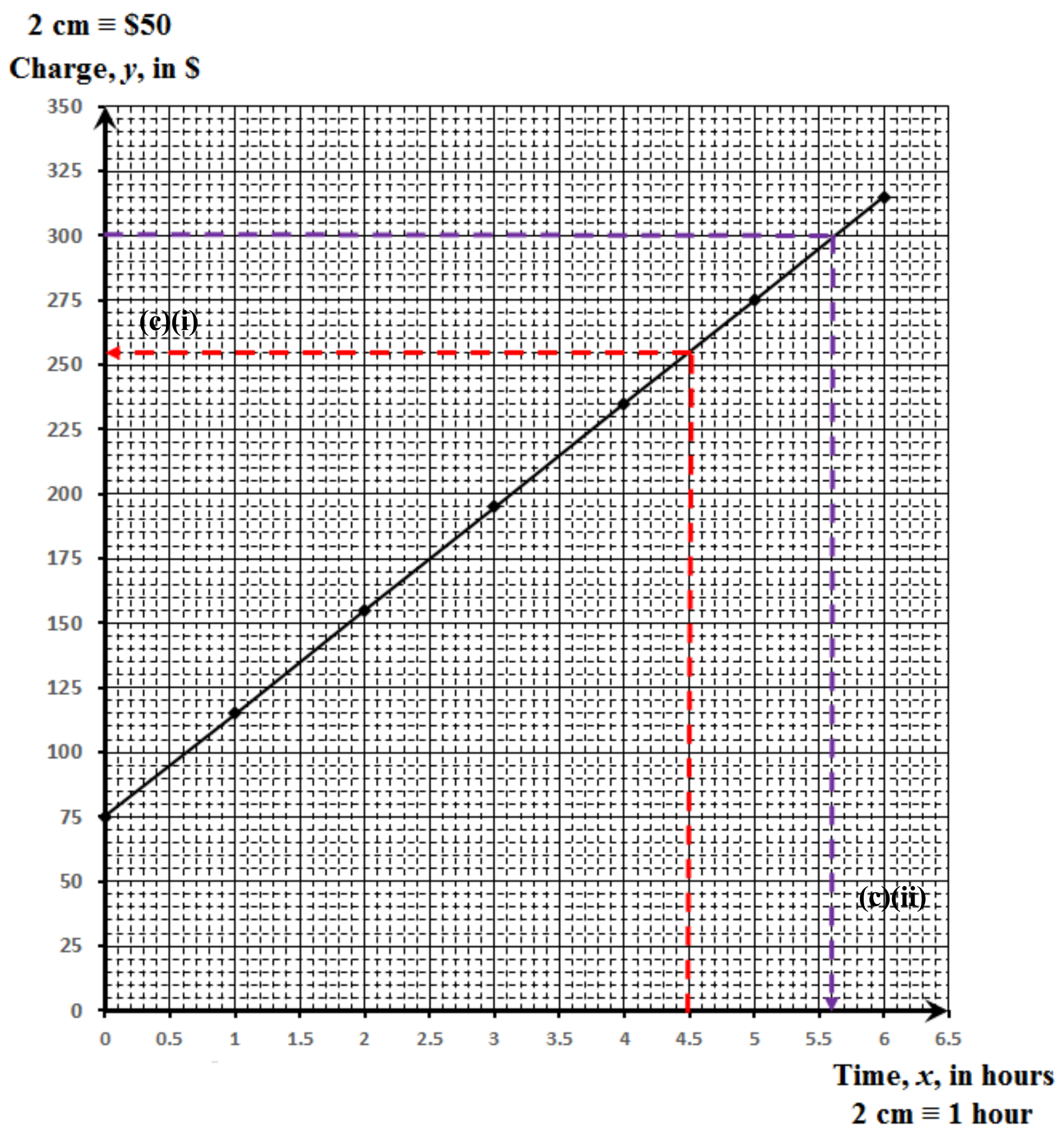


2 cm ≡ 1 hour

(c) Using your graph, determine

(i) the total charges when the job took 4.5 hours.

SOLUTION



Required to determine: The charge when the time $x = 4.5$ hours.

Solution:

From the graph, we draw a vertical at $x = 4.5$ to meet the straight line. At the point of meeting, a horizontal is drawn to meet the vertical axis, for the read off.

When $x = 4.5$, $y = 255$

$\therefore$ The total charges after 4.5 hours is $255.

(ii) the time, in hours, spent on a job if the total charges are $300.

Solution

Required to determine: The time, x, spent on the job if the charge is $300.

Solution:

From the graph, we draw a horizontal at $y = 300$ to meet the straight line.

At the point of meeting, a vertical is drawn to meet the horizontal axis, for the read off.

When $y = 300$, $x = 5.6$

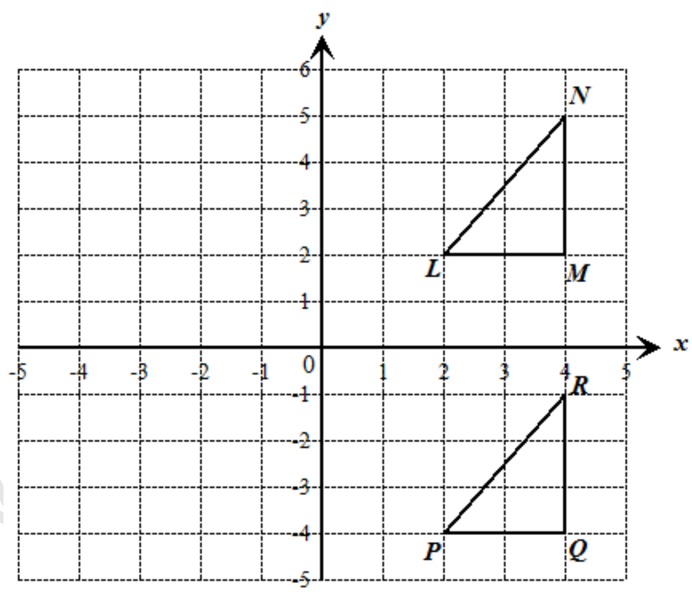
$\therefore$ A total of \$300 occurred when the job lasted 5.6 hours.

(iii) The fixed charge for a visit.

SOLUTION

Required to determine: The fixed charge for a visit
Solution:
The fixed fee will be obtained when the time, $x = 0$
When $x = 0$, $y = 75$.
$\therefore$ The fixed fee for the home visit is therefore \$75.

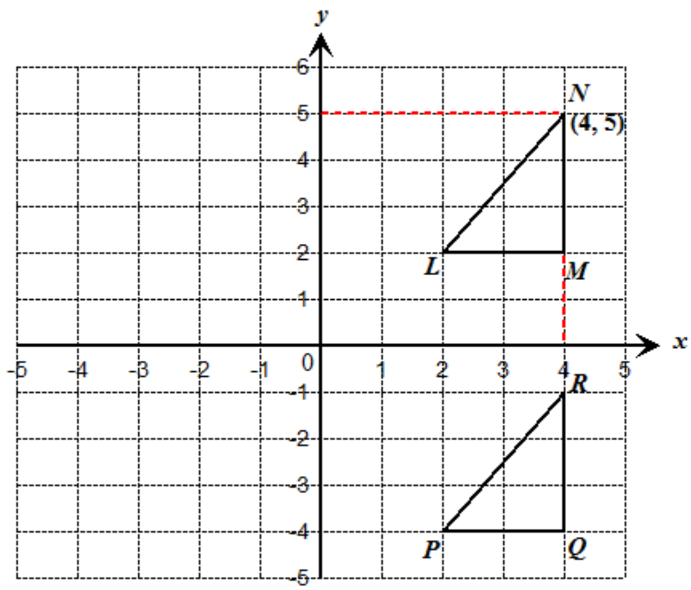
5. The diagram below shows ΔLMN and its image ΔPQR after a transformation.



(i) Write down the coordinates of *N*.

SOLUTION

Data: Diagram showing triangles *PQR* and *LMN*.
Required to state: The coordinates of *N*
Solution:

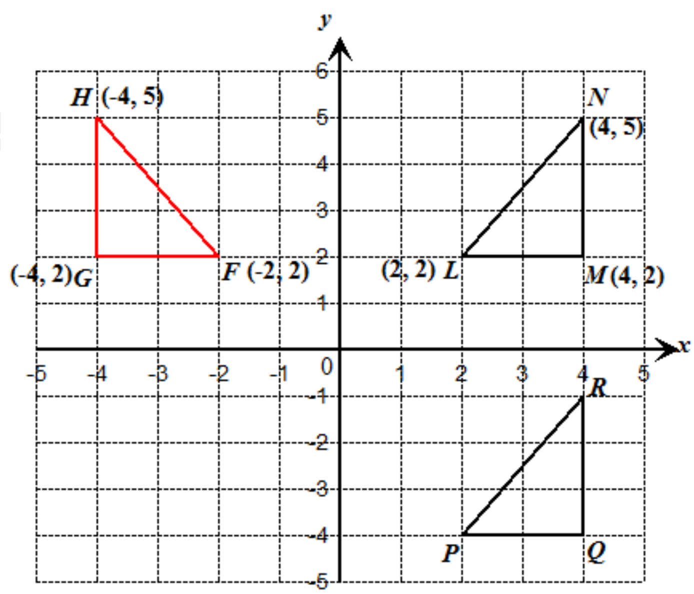


$\therefore$ The coordinates of N is $(4, 5)$ obtained by a read off.

(ii) On the grid above, draw ΔFGH, the reflection of ΔLMN in the y – axis.

SOLUTION

Required to draw: ΔFGH, the reflection of ΔLMN in the y – axis.
Solution:
The image is on the same perpendicular distance as the object and on the opposite side of the reflection plane. We reflect each of the three vertices in turn.

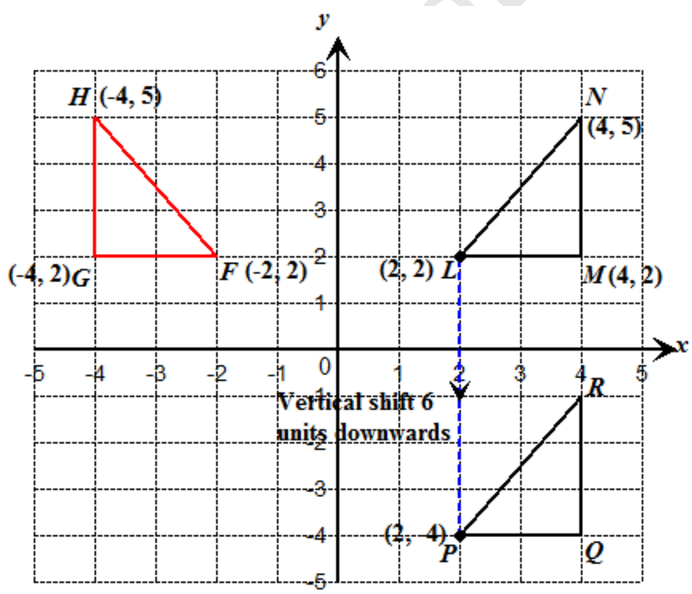


(iii) Using vector notation, describe the transformation which maps ΔLMN onto ΔPQR.

SOLUTION
Data: ΔLMN is mapped onto ΔPQR.
Required to describe: The transformation
Solution:
The object ΔLMN and the image ΔPQR are congruent and there is no re-orientation of the image observed with respect to the object.
Hence, the transformation is deduced as a translation.
Let us consider any one of the object points, say L, and its corresponding image point P, to obtain the translation vector. This procedure could have been carried out with any of the other two object-image points.

Now, L is mapped onto P by a vertical shift of 6 units downwards.
The translation, T can be represented by the vector,

$$T = \begin{pmatrix} 0 \\ -6 \end{pmatrix}$$



Hence, $L \xrightarrow{\begin{pmatrix} 0 \\ -6 \end{pmatrix}} P$ and the transformation that maps ΔLMN onto ΔPQR is a translation described by $T = \begin{pmatrix} 0 \\ -6 \end{pmatrix}$.

(iv) Complete the following statement:

ΔPQR is mapped onto ΔFGH by a combination of two transformations. First, ΔPQR is mapped onto ΔLMN by a ………………………, parallel to the

…………………………; then ΔLMN is mapped onto ΔFGH by a ………………………….. in the ……………………………. .

SOLUTION:
Required to complete: The statement given

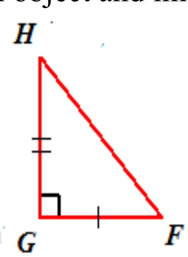
Solution:
ΔPQR is mapped onto ΔFGH by a combination of two transformations. First, ΔPQR is mapped onto ΔLMN by a translation of +6 units, parallel to the $y-$axis; then ΔLMN is mapped onto ΔFGH by a reflection in the $y-$axis.

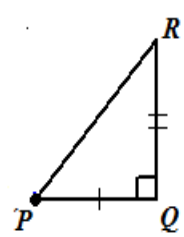
(v) ΔPQR and ΔFGH are congruent.
State TWO reasons why they are congruent.

SOLUTION:

Data: ΔPQR and ΔFGH are congruent.
Required to state: Two reasons why they are congruent
Solution:

Looking at both object and image and comparing, we obtain



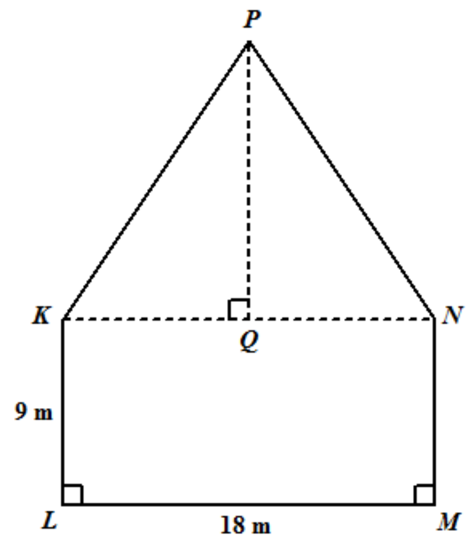


In triangles PQR and FGH:

i. $\hat{Q} = \hat{G}$ (given as 90°)
ii. $PQ = GF$ (given as 2 units)
iii. $QR = GH$ (given as 3 units)

$\therefore \Delta PQR \equiv \Delta FGH$ (Reason for congruency-two sides and the included angle)

6. (a) The diagram below is the scale drawing of the side view of a building. Q is the mid-point of KN and $\angle KLM = \angle LMN = 90°$.



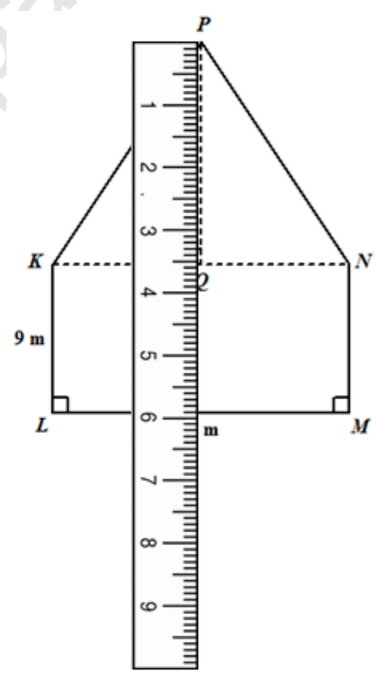
(i) Measure and state the length of PQ on the drawing.

$PQ =$ ……………………..

SOLUTION

Required to measure: And state the length of PQ

Solution:

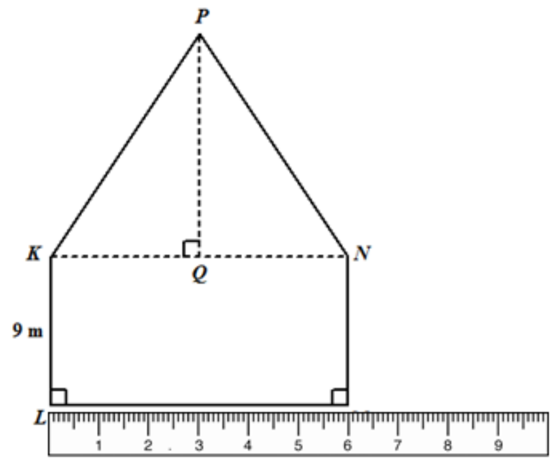


PQ is found to be 3.5 cm by using the ruler.

(ii) Determine the scale of the drawing.

SOLUTION

Required to determine: The scale of the drawing
Solution:



The measure of $LM = 6$ cm (on the diagram) found by using the ruler.
The actual length of $LM = 18$ m

$\therefore 6 \text{ cm} \equiv 18 \text{ m}$
$= 18 \times 100 \text{ cm}$

Hence, the scale on the drawing is 6:1 800
The ratio is reduced its simplest form to 1:300.

(iii) Calculate the actual area of the face *LMNPK* on the building.

SOLUTION

Required to calculate: The actual area of the face *LMNPK* on the building
Calculation:

The compound shape of the face *LMNPK* can be divided into two simple shapes, a rectangle and a triangle.
The area of the rectangle $LMNK = 18 \times 9 \text{ m}^2$
$= 162 \text{ m}^2$

On the diagram, the height of the triangle is 3.5 cm.
Since the scale is 1:300, the actual height of the triangle is 3.5 × 300 cm.

Converting to metres for consistency of units

$=\dfrac{3.5\times 300}{100}$

$=10.5$ metres

The actual area of the triangle $KNP=\dfrac{18\times 10.5}{2}\ \text{m}^2=94.5\ \text{m}^2$

Hence, the area of the entire face of the building

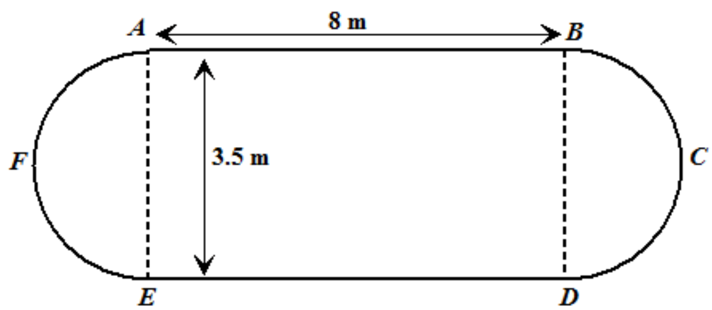
= The area of rectangle *LMNK* + The area of triangle *KNP*

$=(162+94.5)\ \text{m}^2$

$=256.5\ \text{m}^2$

(b) The diagram below, not drawn to scale, shows the plan of a swimming pool in the shape of a rectangle and two semicircles. The rectangle has dimensions 8 metres by 3.5 metres.

[Use $\pi=\dfrac{22}{7}$]

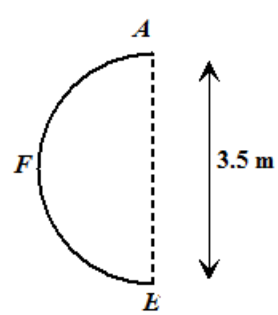


(i) State the length of the diameter of the semi-circle, *AFE*.

SOLUTION

Required to state: The length of the diameter of the semi-circle, *AFE*

Solution:



The length of the diameter of the semicircle, $AFE = 3.5$ m, according to the data.

(ii) Calculate the perimeter of the swimming pool.

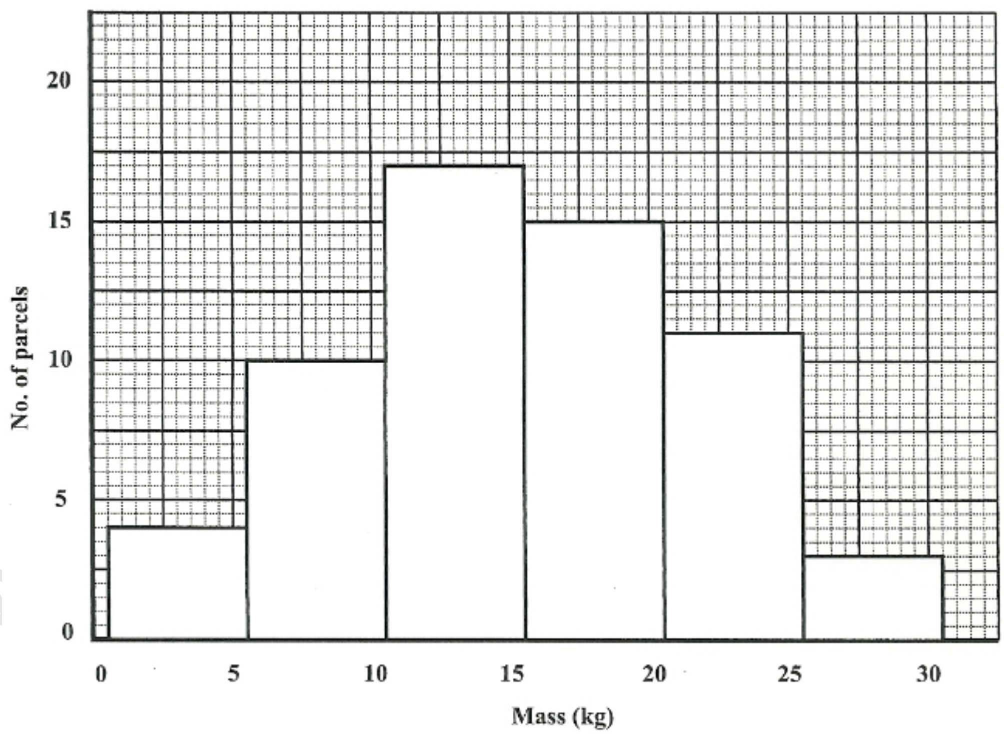
SOLUTION

Required to calculate: The perimeter of the swimming pool
Calculation:
The perimeter of the swimming pool, starting and ending at the point A
= The length of straight side AB + the length of the semi-circular arc BCD
+ the length of straight side DE + the length of semi-circular arc EFA

$$= \left\{8 + \frac{1}{2}\left(3.5 \times \frac{22}{7}\right) + 8 + \frac{1}{2}\left(3.5 \times \frac{22}{7}\right)\right\} \text{ m}$$

$$= 27 \text{ m}$$

7. The masses of 60 parcels collected at a post office were grouped and recorded as shown in the histogram below.



(a) (i) We are to copy and complete the table below to show the information given in the histogram.

Mass (kg)	No. of Parcels	Cumulative Frequency
1 – 5	4	4
6 – 10	10	14
11 – 15	17	31
16 – 20		46
21 – 25	11	
26 – 30		60

SOLUTION:

Data: Table and a histogram showing the masses of 60 parcels, in kg, obtained at the post office.

(i) **Required to copy:** And complete the table using the information given in the histogram.

(ii) **Required to copy**: And complete the column headed 'Cumulative Frequency'

From the definition of cumulative frequency we can calculate to fill the missing blocks

Mass (kg)	No. of Parcels	Cumulative Frequency
1 – 5	4	4
6 – 10	10	14
11 – 15	17	31
16 – 20	46−31= 15	46
21 – 25	11	46+11 = 57
26 – 30	60−57=3	60

(b) On the grid provided, using a scale of **2 cm to represent 5 kg on the *x* – axis** and **2 cm to represent 10 parcels on the *y* – axis**, draw the cumulative frequency curve for the data.

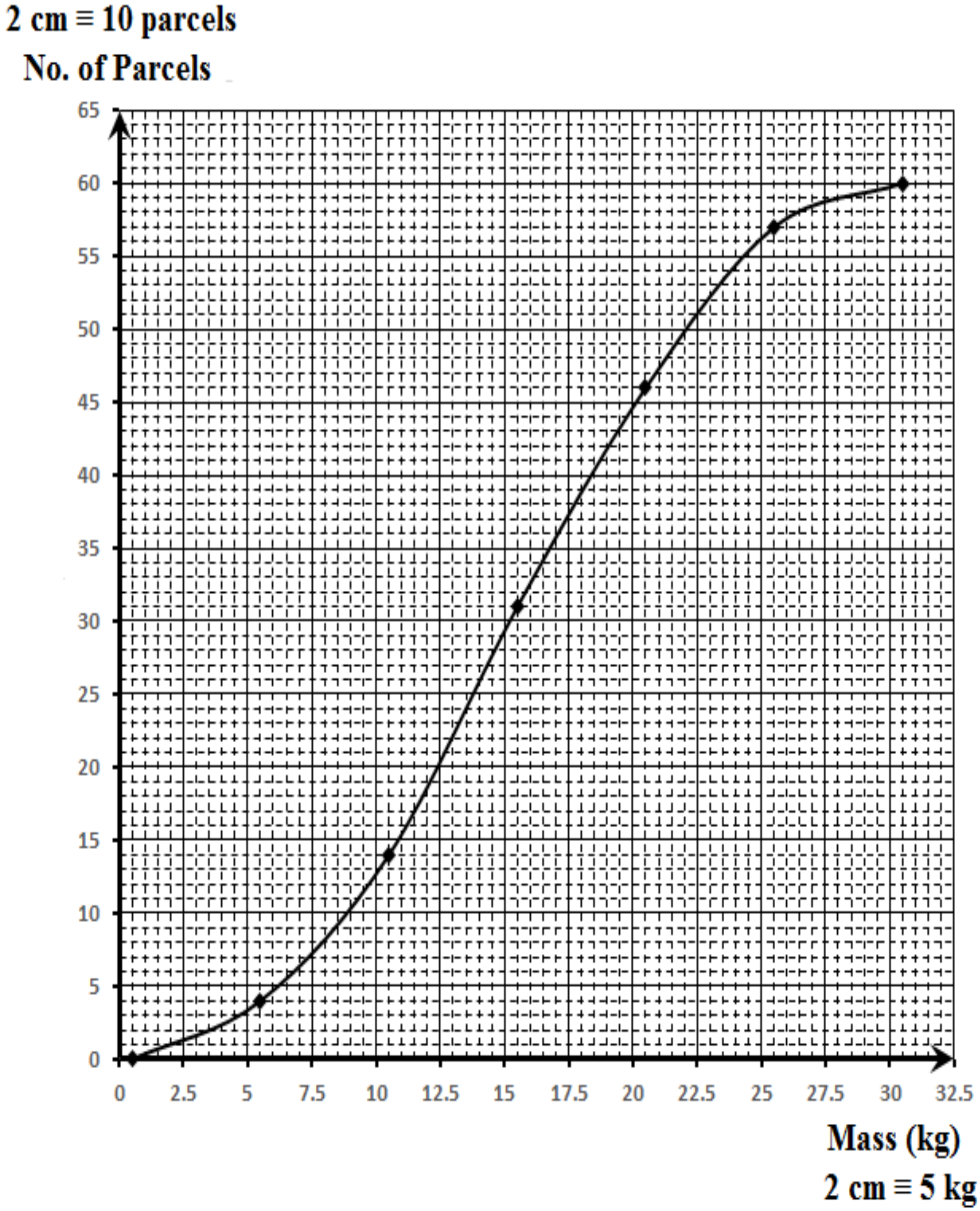
Required To Draw: The cumulative frequency curve to represent the data given in the table

SOLUTION

Mass is a continuous variable. Recreating the table of values to read:

L.C.L. U.C.L. (lower and upper class boundary)	**L.C.B. U.C.B** (lower and upper class limit)	**No. of Parcels, Frequency (*f*)**	**Cumulative Frequency (C.F.)**	**Points to be Plotted (U.C.B, C.F.)**
				$(0.5, 0)$
1 – 5	$0.5 \le m \le 5.5$	4	4	$(5.5, 4)$
6 – 10	$5.5 \le m \le 10.5$	10	14	$(10.5, 14)$
11 – 15	$10.5 \le m \le 15.5$	17	31	$(15.5, 31)$
16 – 20	$15.5 \le m \le 20.5$	15	46	$(20.5, 46)$
21 – 25	$20.5 \le m \le 25.5$	11	57	$(25.5, 57)$
26 – 30	$25.5 \le m \le 30.5$	3	60	$(30.5, 60)$

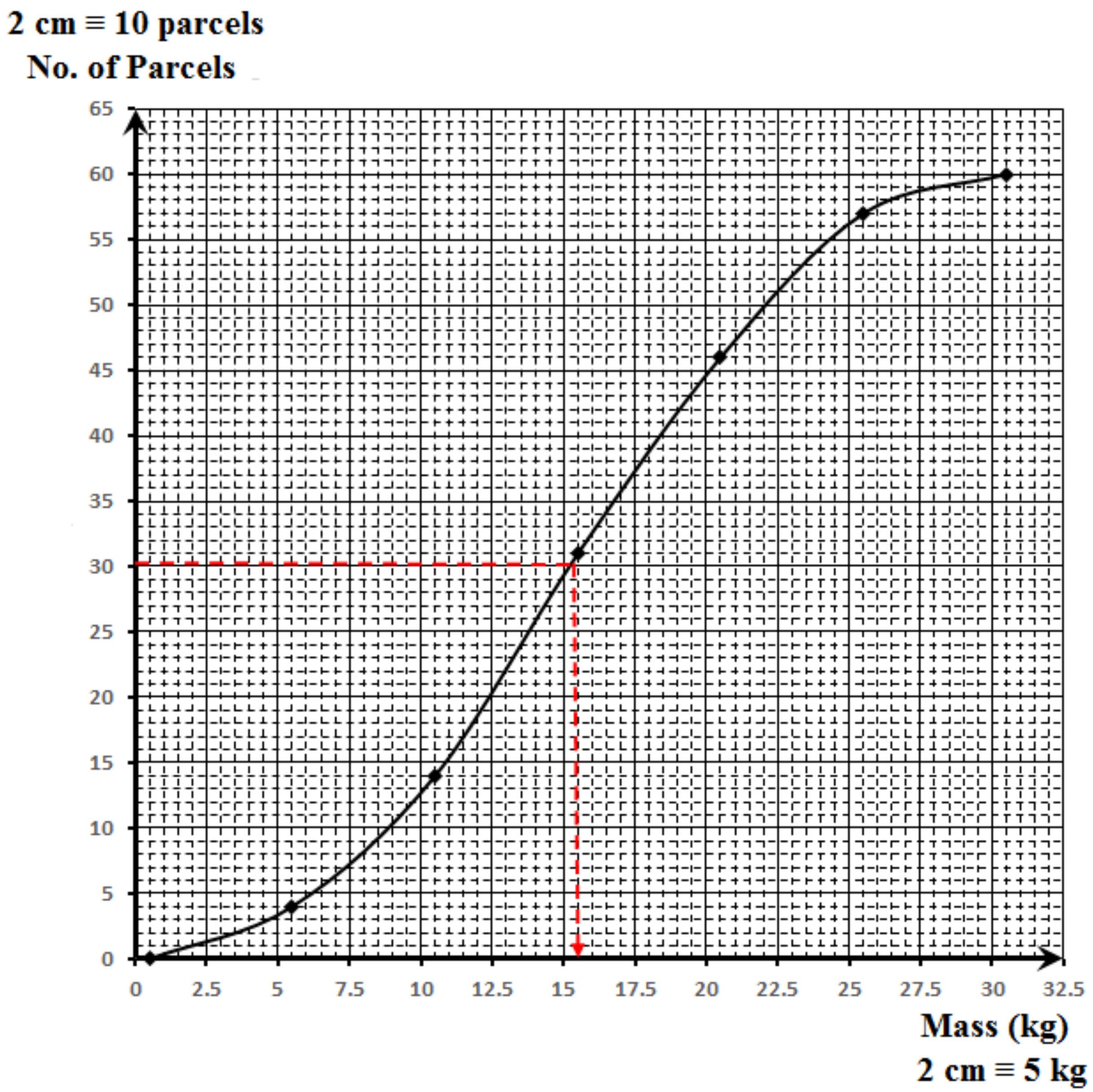
A cumulative frequency curve is expected to start from the horizontal axis and so we 'check backwards' to obtain $(0.5, 0)$ as the starting point of the curve.


2 cm ≡ 10 parcels
No. of Parcels
0
5
10
15
20
25
30
35
40
45
50
55
60
65
0
2.5
5
7.5
10
12.5
15
17.5
20
22.5
25
27.5
30
32.5
Mass (kg)
2 cm ≡ 5 kg

(c) Use the graph drawn at (b) to estimate the median mass of the parcels.
Draw lines on your graph to show how this estimate was obtained.

Required to estimate: The median mass of the parcels using the cumulative frequency curve

SOLUTION

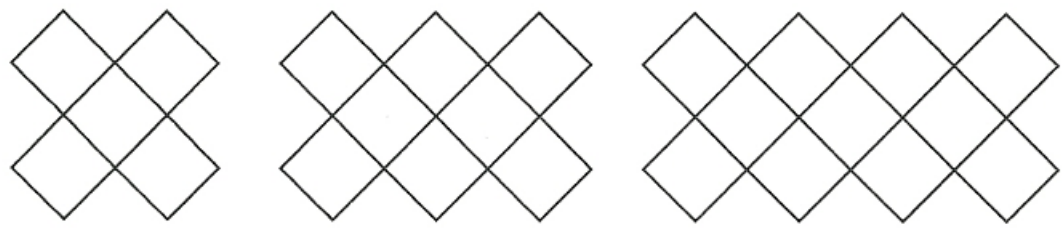


The median lies at ½ of 60 = 30.
The horizontal at 30 is drawn to meet the CF curve. At this point, the vertical is drawn to meet the horizontal axis at 15.5 as indicated.

The median mass of parcels = 15.5 kg

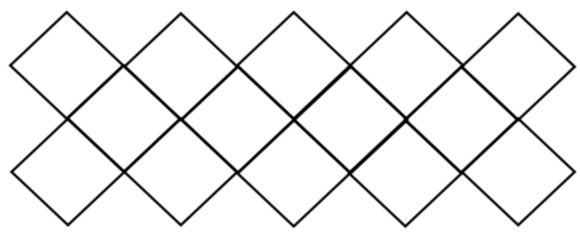
8. The diagram below shows the first three figures in a sequence of figures.

Figure 1 Figure 2 Figure 3



(a) Draw the fourth figure in the sequence.
SOLUTION:

Data: Diagram showing the first three figures in a sequence of figures.
Required to draw: Based on the first three diagrams, the fourth figure in the sequence is drawn to show as:



(b) The table shows the number of squares in each figure. Study the pattern in the table and complete the table by inserting the missing values in the rows numbered (i), (ii), (iii) and (iv).

	Figure (n)	**No. of squares**
	1	5
	2	8
	3	11
(i)	4	………..
(ii)	10	………..
(iii)	………..	50
(iv)	n	…………

SOLUTION

Required do complete: The table by inserting the missing values.
Solution:

Figure (n)	No. of squares (S)
1	$5 = 3(1) + 2$
2	$8 = 3(2) + 2$
3	$11 = 3(3) + 2$

In each case, we notice that the number of squares, *S*, is two (2) added to three times the number (n) of the figure. We conclude that

$S = (3 \times n) + 2$

$S = 3n + 2$

So, we have now created a formula for the number of squares, *S*, in terms of the number of figures, *n*. Now, we can easily answer (i) to (iv) by a simple substitution in each case to fill the incomplete block.

(i) When $n = 4$

$S = 3(4) + 2$
$= 12 + 2$
$= 14$

(ii) When $n = 10$

$S = 3(10) + 2$
$= 30 + 2$
$= 32$

(iii) When $S = 50$

$50 = 3n + 2$
$3n = 48$
$n = 16$

Alternatively, the pattern can be derived as follows:

n	Rule
1	5
2	5+3
3	5+3(2)
4	5+3(3)
n	5+3(n−1)

Note that this rule can be expressed as

S = 5+3(n−1) = 5+ 3n – 3

$= 3n - 2$

(iv) And $S = 3n + 2$ (found from before)

The completed table would now look like:

	Figure (n)	**No. of squares**
	1	5
	2	8
	3	11
(i)	4	14
(ii)	10	32
(iii)	16	50
(iv)	n	$3n+2$

Section II

ALGEBRA AND RELATIONS, FUNCTIONS AND GRAPHS

9. (a) The functions $f(x)$ and $g(x)$ are defined as:

$$f(x)=\frac{5x-4}{3} \qquad g(x)=x^2-1$$

(i) Evaluate $f(7)$.

SOLUTION

Data: $f(x)=\frac{5x-4}{3}$ and $g(x)=x^2-1$

Required to evaluate: $f(7)$

Solution:

We substitute $x = 7$ in the expression for $f(x)$ to obtain

$$f(7)=\frac{5(7)-4}{3}$$
$$=\frac{35-4}{3}$$
$$=\frac{31}{3} \text{ or } 10\frac{1}{3}$$

(ii) Write an expression, in terms of x, for $f^{-1}(x)$.

SOLUTION:

Required to find: $f^{-1}(x)$

Solution:

Let $y=\frac{5x-4}{3}$

Making x the subject of the equation:

$$3y=5x-4$$
$$3y+4=5x$$
$$5x=3y+4$$
$$x=\frac{3y+4}{5}$$

Now, we replace y by x to get the inverse function, $f^{-1}(x)=\frac{3x+4}{5}$

(iii) Write an expression, in terms of x, for $fg(x)$.

SOLUTION:

Required to find: $fg(x)$

Solution:

We replace x in the function f, by the function $g(x)$ to obtain $fg(x)$.
This gives

$$fg(x) = \frac{5(x^2 - 1) - 4}{3}$$

$$= \frac{5x^2 - 5 - 4}{3}$$

$$= \frac{5x^2 - 9}{3} \text{ or } \frac{5x^2}{3} - 3$$

when simplified.

(b) (i) Express the quadratic function $f(x) = 3x^2 + 6x - 2$, in the form $a(x+h)^2 + k$, where a, h and k are constants.

SOLUTION:

Required to express: $f(x) = 3x^2 + 6x - 2$ in the form $a(x+h)^2 + k$, where a, h and k are constants.

Solution:

Looking at the terms in x and in x^2

$$f(x) = 3\underbrace{x^2 + 6}x - 2$$

$$= 3(x^2 + 2x) - 2$$

To introduce the square we find half the coefficient of x and which in this case is $\frac{1}{2}(2) = 1$.

So,

$$f(x) = 3(x+1)^2 + *$$

$\downarrow$ (* is an unknown to be calculated)

$$3(x+1)(x+1) = 3x^2 + 6x + 3$$

Hence, $3 + * = -2$ (which is the constant in the original equation)

$* = -5$

So $3x^2+6x-2=3(x+1)^2-5$ and is of the form $a(x+h)^2+k$, where $a=3$, $h=1$ and $k=-5$.

Alternative Method:

In this method, we expand the desired form of $a(x+h)^2+k$ and equate coefficients with the original equation $f(x)=3x^2+6x-2$

$$\begin{aligned} f(x) &= 3x^2+6x-2 \\ &\equiv a(x+h)^2+k \\ &= a(x+h)(x+h)+k \\ &= a\left(x^2+2hx+h^2\right)+k \\ &= ax^2+2ahx+ah^2+k \end{aligned}$$

Equating coefficients in x^2:
This gives $a=3$
Equating coefficients in x:

$$\begin{aligned} 2ah &= 6 \\ 2(3)h &= 6 \\ h &= 1 \end{aligned}$$

Equating the constant term:

$$\begin{aligned} ah^2+k &= -2 \\ 3(1)^2+k &= -2 \\ k &= -2-3 \\ &= -5 \end{aligned}$$

$$\therefore 3x^2+6x-2=3(x+1)^2-5$$

(ii) Hence, or otherwise, state the **minimum** value of $f(x)=3x^2+6x-2$.

SOLUTION:

Required to state: The minimum value of $f(x)=3x^2+6x-2$.

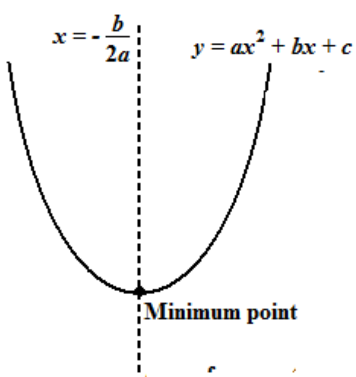
Solution: We begin by recalling the fact that any quantity that is being squared must be greater than or equal to zero, regardless of the variables involved.

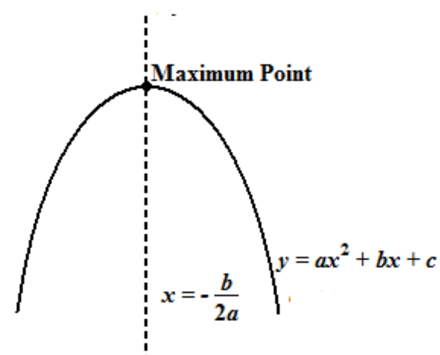
$$f(x) = 3x^2 + 6x - 2$$
$$= 3(x+1)^2 - 5$$
$$(x+1)^2 \geq 0, \forall x$$

Hence, the minimum value of $f(x) = 3(0) - 5$ occurring when $(x+1)^2 = 0$
So, the minimum value of $f(x) = -5$

Alternative Method:
The minimum or maximum value of $f(x) = ax^2 + bx + c$ occurs at $x = \frac{-b}{2a}$.

This is because the vertical line with equation $x = \frac{-b}{2a}$ is the axis of symmetry of a quadratic graph and this vertical passes through the maximum or the minimum point, thereby giving $x = \frac{-b}{2a}$ as the x-coordinate of the minimum or the maximum point.





$$f(x) = 3x^2 + 6x - 2$$

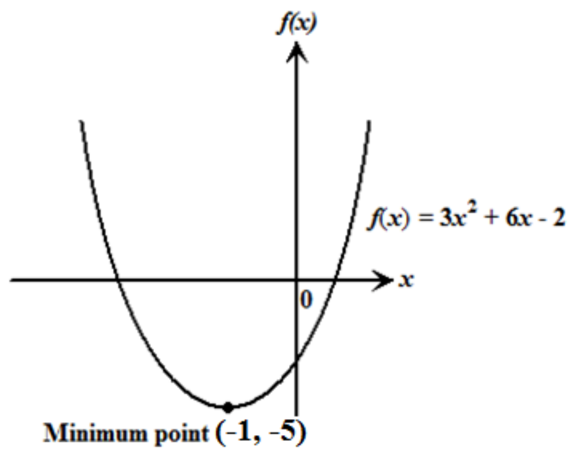
So, the minimum value occurs at $x = \frac{-(6)}{2(3)} = -1$

When $x = -1$
$$f(-1) = 3(-1)^2 + 6(-1) - 2$$
$$= 3 - 6 - 2$$
$$= -5$$

Alternative Method:

x	*f(x)*
Choose values of *x*	Determine the corresponding values of *f*.

Using these values, we sketch the graph of $f(x) = 3x^2 + 6x - 2$ to obtain



$\therefore$ The minimum value of $f(x) = -5$ at $x = -1$ both obtained by a read-off from the graph drawn.

Alternative Method:
Some advanced students of mathematics may find it convenient to use differential calculus. No method was specified.

$$f(x) = 3x^2 + 6x - 2$$
$$f'(x) = 3(2x) + 6$$
$$= 6x + 6$$

At a stationary point. $f'(x) = 0$.

Let $f'(x) = 0$

$$6x + 6 = 0$$
$$x = -1$$

$$f(-1) = 3(-1)^2 + 6(-1) - 2$$
$$= 3 - 6 - 2$$
$$= -5$$

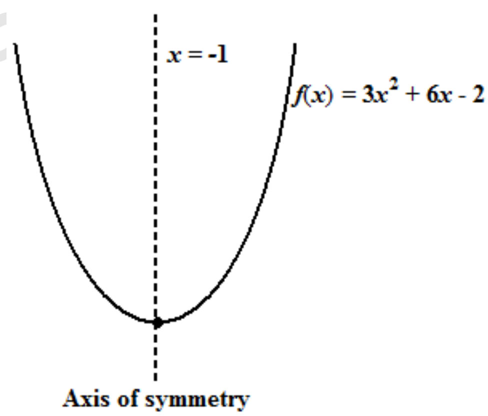
$f''(x) = 6 \quad (> 0) \Rightarrow$ Minimum value
$\therefore$ The minimum value of $f(x) = -5$ at $x = -1$.

(iii) State the equation of the axis of symmetry of the function, $f(x) = 3x^2 + 6x - 2$.

SOLUTION

Required to state: The equation of the axis of symmetry of $f(x) = 3x^2 + 6x - 2$.
Solution:
Recall, when $f(x) = ax^2 + bx + c$, the axis of symmetry occurs at $x = \frac{-b}{2a}$.

Hence, the equation of the axis of symmetry of $f(x) = 3x^2 + 6x - 2$ occurs at $x = \frac{-(6)}{2(3)}$
$$x = -1$$



(iv) Sketch the graph of $y = 3x^2 + 6x - 2$, showing on your sketch

a) the intercept on the y – axis.

b) the coordinates of the minimum point.

SOLUTION

Required to sketch: The graph of $y = 3x^2 + 6x - 2$.
Solution:
In the expression, $ax^2 + bx + c$, $b^2 > 4ac$

In the equation of $y = 3x^2 + 6x - 2$, b = 6, a = 3 and c = -2
$(6)^2 > 4(3)(-2)$

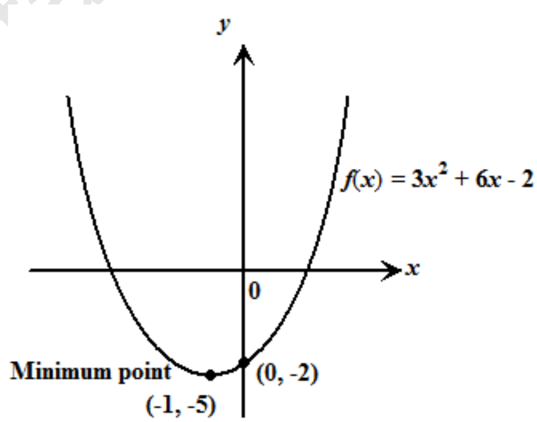
We get $36 > -24$.
Therefore we shall have two real and distinct solutions.
$\therefore$ The quadratic graph cuts the $x-$axis at two distinct points.

The coefficient of $x^2 > 0 \Rightarrow$ the quadratic graph has a minimum point.

The minimum value of $f(x) = -5$ and this occurs at $x = -1$.
$\therefore (-1, -5)$ is the minimum point as seen before.

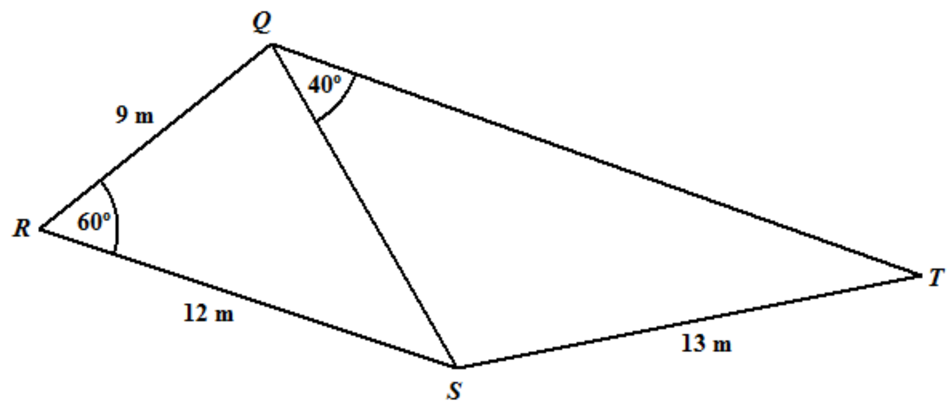
When $x = 0$
$y = 3(0)^2 + 6(0) - 2$
$= -2$

The curve cuts the $y-$axis at -2.
Hence, a sketch of $y = 3x^2 + 6x - 2$ would look like:



MEASUREMENT, GEOMETRY AND TRIGONOMETRY

10. (a) On the diagram below, not drawn to scale, $RQ = 9$ m, $RS = 12$ m, $ST = 13$ m, $\angle QRS = 60°$ and $\angle SQT = 40°$.

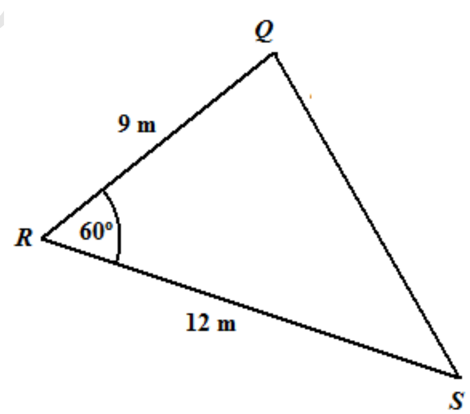


Calculate, correct to 1 decimal place,

(i) the length QS.

SOLUTION

Data: Quadrilateral PQTS with $RQ = 9cm$, $RS = 12\ cm$, $\angle QRS = 60°$ and $\angle SQT = 40°$.
Required to calculate: The length of QS, correct to 1 decimal place
Calculation:



Applying the cosine law to triangle QRS since we are given two sides and and the included angle

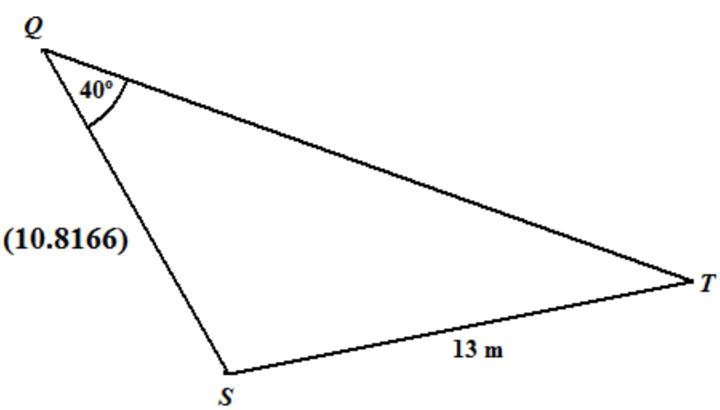
$$QS^2 = (9)^2 + (12)^2 - 2(9)(12)\cos 60°$$
$$QS^2 = 81 + 144 - 2(9)(12)\left(\frac{1}{2}\right)$$
$$QS^2 = 117$$
$$QS = 10.8\underline{\underline{1}}$$
$$= 10.8 \text{ m (correct to 1 decimal place)}$$

(ii) the measure of $\angle QTS$.

SOLUTION:

Required to calculate: The size of $Q\hat{T}S$.
Calculation:
Let us consider the triangle *QTS*.

Using the more accurate value of (10.8166)



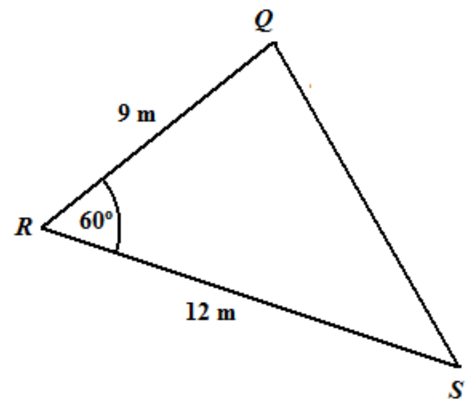
Applying the sine rule to the triangle *QTS*.

$$\frac{13}{\sin 40°} = \frac{10.8166}{\sin Q\hat{T}S}$$
$$\sin Q\hat{T}S = \frac{10.8166 \times \sin 40°}{13}$$
$$= 0.5348$$
$$Q\hat{T}S = \sin^{-1}(0.5348)$$
$$= 32.3\underline{\underline{3}}°$$
$$= 32.3° \text{ to the nearest } 0.1°$$

(iii) The area of triangle *QRS*

SOLUTION:
Required to calculate: The area of triangle *QRS*.

Calculation:



The area of a triangle can be found by the formula:

$$\text{Area} = \frac{1}{2}(\text{side})(\text{side}) \times \sin(\text{included angle})$$

$$\text{Area of } \Delta QRS = \frac{1}{2}(9)(12)\sin 60°$$

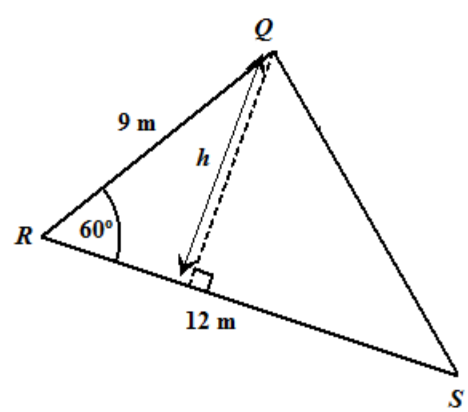
$$= 46.765 \text{ m}^2$$

$$= 46.8 \text{ m}^2 \text{ (to 1 decimal place)}$$

(iv) The perpendicular distance from Q to RS.

SOLUTION:

Required to find: The perpendicular distance from Q to RS.
Solution:



Area of the triangle $QRS = 46.765 \text{ m}^2$
For this calculation, let us use the more accurate value

Let the perpendicular distance from Q to RS be h metres, as shown on the diagram.
Consider RS as the base of the triangle.

Area of a triangle can be found by using the formula

$$\text{Area} = \frac{1}{2}(\text{base}) \times \text{perpendicular height}$$

Hence, using the area which we had found before, we now have

$$46.765 = \frac{1}{2}(12)h$$

$$h = \frac{46.765}{\frac{1}{2}(12)}$$

$$= 7.794 \text{ m}$$

$$= 7.8 \text{ m (correct to 1 decimal place)}$$

$\therefore$ The perpendicular distance from Q to $RS = 7.8$ m.

Alternative Method

Let h represent the height of triangle QRS.
From the definition of sin of an angle in the right angled triangle = $\frac{opp}{hyp}$

$$\sin 60^0 = \frac{h}{9}$$

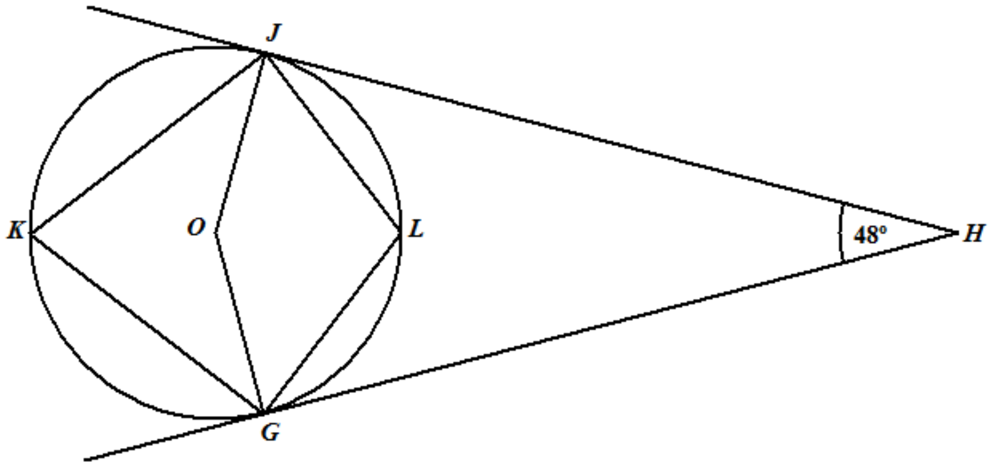
$$= 9 \times \sin 60^0$$

$$= 9 \times 0.866$$

$$= 7.794$$

$$h = 7.8 \text{ m (correct to 1 decimal place)}$$

(b) The diagram below, **not drawn to scale**, shows a circle with center O. HJ and HG are tangents to the circle and $\angle JHG = 48°$.

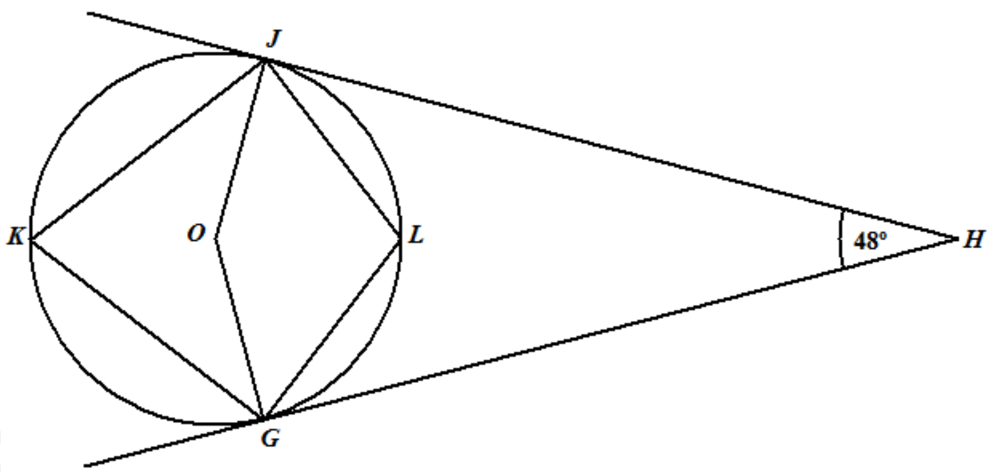


Calculate, **giving the reason for each step of your answer**, the measure of:

(i) $\angle OJH$

SOLUTION:

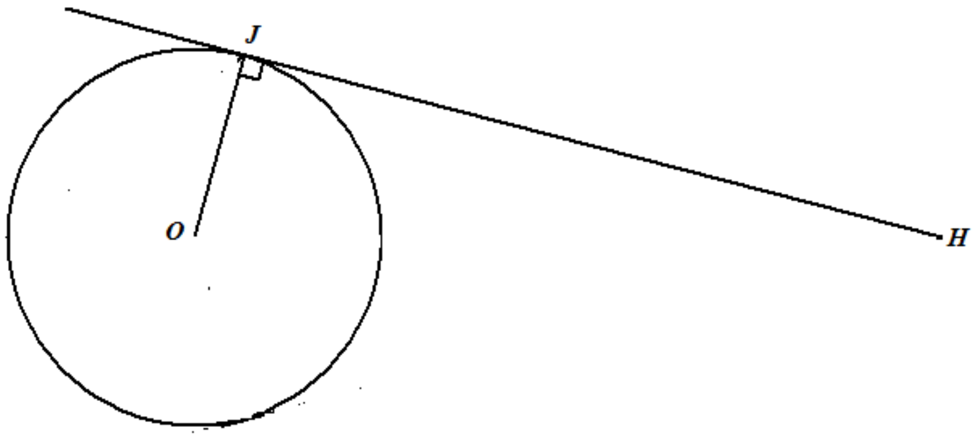
Data:



Circle, center *O*. *HJ* and *HG* are tangents to the circle. $J\hat{H}G = 48°$.

Required to calculate: $O\hat{J}H$

Calculation:

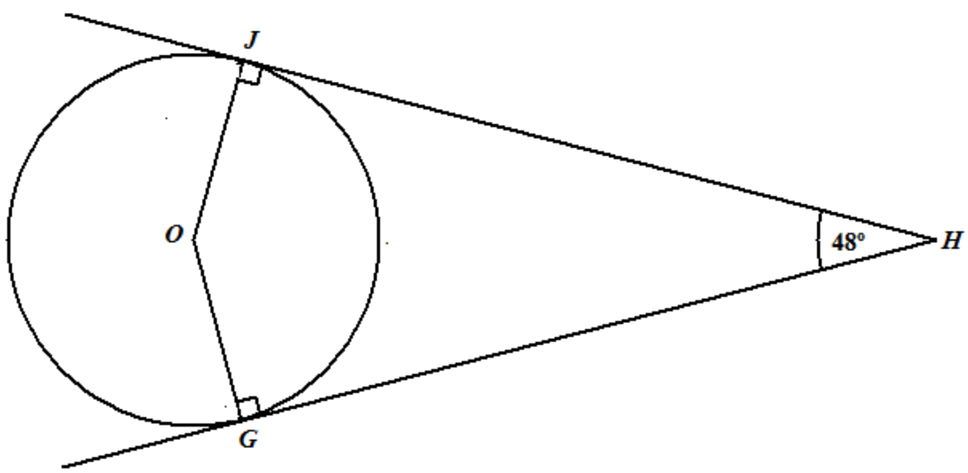


$O\hat{J}H = 90° = O\hat{G}H$ (The angle by a tangent to a circle and a radius, at the point of contact, is always a right angle).

(ii) $\angle JOG$

SOLUTION:

Required to calculate: $J\hat{O}G$
Calculation:



$O\hat{J}G = O\hat{G}H$
$= 90°$
(The angle made by the tangent to a circle and a radius, at the point of contact = 90°)
Consider the quadrilateral *JOGH* in which three of the angles are known.

$\therefore J\hat{O}G = 360° - (90° + 48° + 90°)$

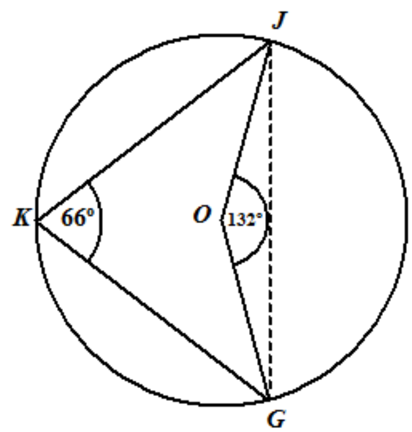
$= 132°$

(Sum of the four interior angles of any quadrilateral = 360°)

(iii) $\angle JKG$

SOLUTION

Required to calculate: $J\hat{K}G$
Calculation:



Consider the chord *JG* and which was not drawn in the original diagram, but now done so in dotted lines

$J\hat{O}G = 132°$ (from (ii))

$J\hat{K}G = \frac{1}{2}(132°)$

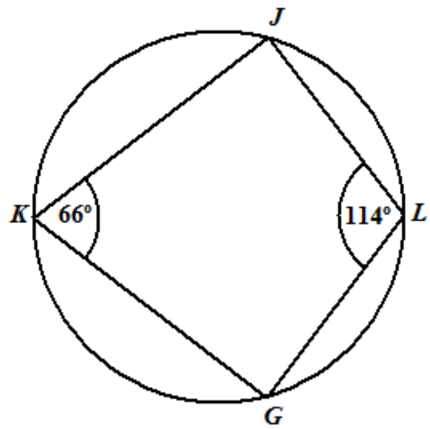
$= 66°$

(This is because the angle subtended by a chord at the center of a circle is twice the angle that the chord subtends at the circumference, standing on the same arc).

(iv) $\angle JLG$

SOLUTION:

Required to calculate: $J\hat{L}G$
Calculation:



Consider the quadrilateral *JLGK*

$J\hat{L}G = 180° - 66°$

$= 114°$

(Recall, the opposite angles of a cyclic quadrilateral are supplementary).

VECTORS AND MATRICES

11. (a) (i) Write the following simultaneous equations

$3x + 2y = -1$
$5x + 4y = 6$

in the form $AX = B$, where *A*, *X* and *B* are matrices.

SOLUTION:

Data: $3x + 2y = -1$ and $5x + 4y = 6$
Required to Express: The equations in the form of $AX = B$, where *A*, *X* and *B* are all matrices.
Solution:
Let

$\boxed{3}x + \boxed{2}y = -1 \quad \ldots(1)$

$\boxed{5}x + \boxed{4}y = 6 \quad \ldots(2)$

Looking at the coefficients of *x* and of *y* in both equations, they can be re-written as

$\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$ and is of the form $AX = B$, where $A = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$.

(ii) Use a matrix method to solve for *x* and *y*.

SOLUTION

Required to solve: for *x* and for *y*.
Solution:

$3x + 2y = -1 \quad \ldots(1)$

$5x + 4y = 6 \quad \ldots(2)$

The given equations were re-written as

$\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$ …which is a matrix equation

Let $A = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$

We now find the inverse of *A* written as A^{-1}
The first step is to find the determinant of *A*

$$\det A = (3 \times 4) - (5 \times 2)$$
$$= 2$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -(2) \\ -(5) & 3 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{4}{2} & -\frac{2}{2} \\ -\frac{5}{2} & \frac{3}{2} \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -1 \\ -2\frac{1}{2} & 1\frac{1}{2} \end{pmatrix}$$

Now we pre-multiply both sides of the matrix equation by A^{-1}

The matrix *A* multiplied by its inverse will give the identity matrix.

The identity matrix multiplied by another matrix will give the matrix, unaltered.
So,

$$\begin{pmatrix} 2 & -1 \\ -2\frac{1}{2} & 1\frac{1}{2} \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -2\frac{1}{2} & 1\frac{1}{2} \end{pmatrix}\begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

I

$\begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (2\times-1)+(-1\times6) \\ \left(-2\frac{1}{2}\times-1\right)+\left(1\frac{1}{2}\times6\right) \end{pmatrix}$$

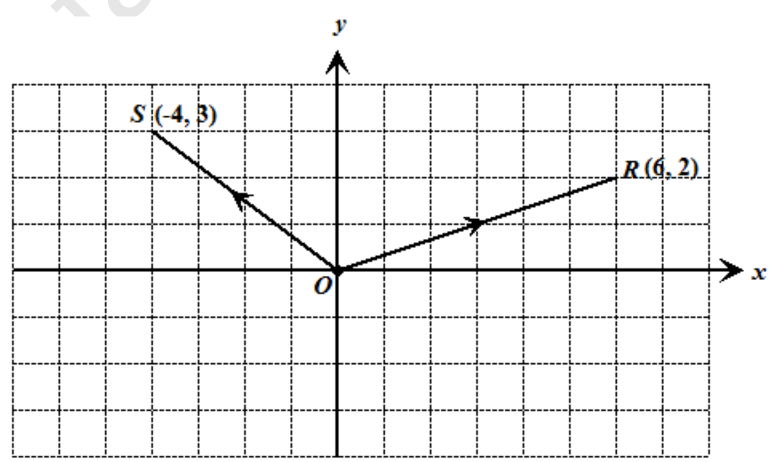
$$= \begin{pmatrix} -2-6 \\ 2\frac{1}{2}+9 \end{pmatrix}$$

$$= \begin{pmatrix} -8 \\ 11\frac{1}{2} \end{pmatrix}$$

Equating corresponding entries since both sides of the equation, the left and the right have been simplified to 2 x 1 matrices, we obtain

$x = -8$ and $y = 11\frac{1}{2}$.

(b) The diagram below shows two position vectors $\overrightarrow{OR}$ and $\overrightarrow{OS}$ such that $R\,(6, 2)$ and $S\,(-4, 3)$.

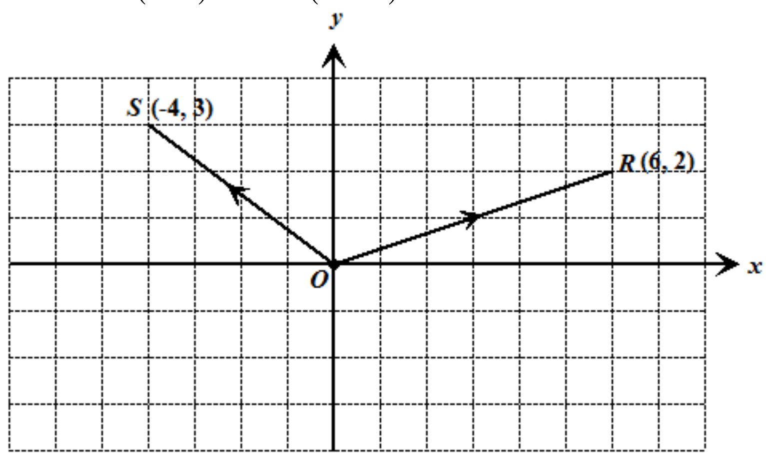


Write as a column vector in the form $\begin{pmatrix} x \\ y \end{pmatrix}$:

(i) $\overrightarrow{OR}$

SOLUTION:

Data: $R = (6, 2)$ and $S\ (-4, 3)$.



Required to write: $\overrightarrow{OR}$ as a column vector.
Solution:
The point $R\ (6, 2)$. The vector OR is measured from O

$$\therefore OR = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \text{ is of the form } \begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } x = 6 \text{ and } y = 2.$$

(ii) $\overrightarrow{OS}$

SOLUTION

Required To Write: $\overrightarrow{OS}$
Solution:
$S = (-4, 3)$ The vector OS is measured from O

$$\therefore OS = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \text{ is of the form } \begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } x = -4 \text{ and } y = 3$$

(iii) $\overrightarrow{SR}$

SOLUTION:

Required to write: $\overrightarrow{SR}$
Solution:
By the vector triangle law

$$SR = SO + OR$$
$$= -\begin{pmatrix} -4 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$
$= \begin{pmatrix} 10 \\ -1 \end{pmatrix}$ and which is of the form $\begin{pmatrix} x \\ y \end{pmatrix}$, where $x = 10$ and $y = -1$.

(iv) Find $\left|\overrightarrow{OS}\right|$.

SOLUTION:

Required to calculate: $\left|\overrightarrow{OS}\right|$

Calculation:

$$OS = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$\therefore |OS| = \sqrt{(-4)^2 + (3)^2}$ and taking the positive root only

$$= \sqrt{25}$$
$$= 5 \text{ units}$$

(v) Given that $OT = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, prove that $OSTR$ is a parallelogram.

SOLUTION:

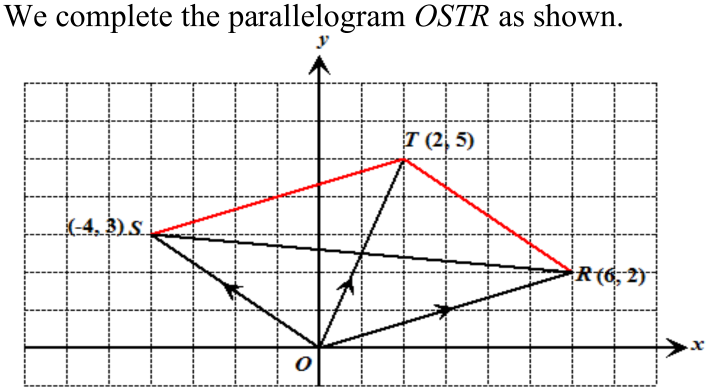
Data: $OT = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

Required to prove: $OSTR$ is a parallelogram

Proof:

Let us locate $T\,(2, 5)$ and $OT = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ on the diagram.

We complete the parallelogram $OSTR$ as shown.



We will use the geometrical fact that if one pair of opposite sides of a quadrilateral is both parallel and equal then the quadrilateral is a parallelogram. We will prove this with the sides *OS* and *RT*

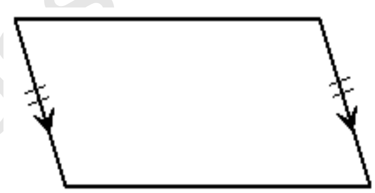
Let us now find $\overrightarrow{RT}$, by the vector triangle law.

$$RT = RO + OT$$
$$= -\begin{pmatrix} 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$
$$= \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$
$$OS = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$\overrightarrow{OS}$ is parallel to $\overrightarrow{RT}$ and $\left|\overrightarrow{OS}\right| = \left|\overrightarrow{RT}\right|$

(We could have used $\overrightarrow{OR}$ and $\overrightarrow{ST}$ instead).

So one pair of opposite sides of the quadrilateral *OSTR* is both equal and parallel, and so the quadrilateral is a parallelogram.



Hence, *OSTR* is a parallelogram.

Q.E.D

There are other feasible methods that we could have used to prove *OSTR* is a parallelogram, such as:

Prove the opposite sides of the quadrilateral are equal to each other.
Prove that the diagonal bisects each other.
Prove the opposite sides of the quadrilateral are parallel to each other.
Prove that any pair of adjacent angles is supplementary.
Prove that the angles at the opposite vertices are equal.