FAS-PASS Maths

CSEC MATHEMATICS MAY-JUNE 2012

Section I

1. (a) **Required to calculate:** $\dfrac{3\frac{1}{5}-\frac{2}{3}}{2\frac{4}{5}}$

Solution:

$$\dfrac{3\frac{1}{5}-\frac{2}{3}}{2\frac{4}{5}}$$

First we work the numerator

$$3\frac{1}{5}-\frac{2}{3}$$

$$\frac{16}{5}-\frac{2}{3}$$

$$\frac{48-10}{15}=\frac{38}{15}$$

We express the denominator as an improper fraction

$$2\frac{4}{5}=\frac{14}{5}$$

Hence,

$$\begin{aligned}\frac{\text{Numerator}}{\text{Denominator}}&=\frac{\frac{38}{15}}{\frac{14}{5}}\\&=\frac{38}{15}\div\frac{14}{5}\\&=\frac{38}{15}\times\frac{5}{14}\\&=\frac{19}{21}\text{ (in lowest terms)}\end{aligned}$$

(b) **Data:** Table showing cost price, selling price and profit or loss as a percentage.
Required to: Copy and complete the table
Solution:

(i) When the selling price of an item is less than the cost price, this indicates a loss is incurred.
The loss = The cost Price – The selling Price
= \$55.00 - \$44.00
= \$11.00

$$\text{Percentage Loss} = \frac{\text{Loss}}{\text{Cost Price}} \times 100$$
$$= \frac{11}{55} \times 100$$
$$= 20\%$$

(ii) A 25% profit implies that the selling price is 125% of the cost price.

$$\text{Therefore the cost price} = \frac{100}{125} \times 100$$
$$= \$80.00$$

The completed table will now read as:

Cost Price	Selling Price	Percentage Profit or Loss
\$55.00	\$44.00	20% loss
\$80.00	\$100.00	25% profit

(c) **Data:** US \$1.00 = EC \$2.70 and TT \$1.00 = EC \$0.40

Required to calculate:

(i) EC \$1 in TT\$

(ii) US \$80 in EC \$

(iii) TT \$648 in US \$

Solution:

(i) If EC \$0.40 = TT \$1.00

Then we find for \$0.01 EC

$$\text{EC } \$0.01 = \frac{\text{TT } \$1.00}{40}$$

And so $\text{EC } \$1.00 = \frac{\text{TT } \$1.00}{40} \times 100$

$$\text{EC } \$1.00 = \text{TT } \$2.50$$

(ii) If $\text{US } \$1.00 = \text{EC } \$2.70$

Then,

$$\text{US } \$80.00 = \text{EC } \$2.70 \times 80$$
$$= \text{EC } \$216.00$$

(iii) $\text{TT } \$1.00 = \text{EC } \$0.40$

$$\therefore \text{TT } \$648 = \text{EC } \$0.40 \times 648$$
$$= \text{EC } \$259.20$$

$$\text{EC } \$2.70 = \text{US } \$1.00$$

$$\text{EC } \$1.00 = \text{US } \frac{\$1.00}{2.70}$$

$$\text{EC } \$259.20 = \text{US } \frac{\$1.00}{2.70} \times 259.20$$
$$= \$96.00$$

$$\therefore \text{TT } \$648.00 = \text{US } \$96.00$$

2. (a) **Required to factorise:**

(i) $2x^3y + 6x^2y^2$

(ii) $9x^2 - 4$

(iii) $4x^2 + 8xy - xy - 2y^2$

Solution:

(i) $2x^3y + 6x^2y^2$

We factor out the common factor of each term, shown underlined.

$$\underline{\underline{2x^2y}}(x) + \underline{\underline{2x^2y}}(3y)$$

$$2x^2y(x + 3y)$$

(ii) $9x^2 - 4$

The expression is re-written as

$(3x)^2 - (2)^2$ and is now in a standard form

This is now expressed as a difference of two squares and hence can be factorised to

$$(3x - 2)(3x + 2)$$

(iii) $4x^2 + 8xy - xy - 2y^2$

We group the terms into two pairs and factorise each pair.

$$4x(x + 2y) - y(x + 2y)$$

$$(x + 2y)(4x - y)$$

(b) **Required to solve:** $\frac{2x-3}{3} + \frac{5-x}{2} = 3$

Solution:

$$\frac{2x-3}{3}+\frac{5-x}{2}=3$$
$$\frac{2(2x-3)+3(5-x)}{6}=3$$
$$\frac{4x-6+15-3x}{6}=3$$
$$\frac{x+9}{6}=\frac{3}{1}$$
$$1(x+9)=3(6)$$
$$x+9=18$$
$$x=18-9$$
$$x=9$$

OR

$$\frac{2x-3}{3}+\frac{5-x}{2}=\frac{3}{1}$$

We can multiply the equation by the LCM of the denominators and which are 3, 2 and 1. This is found to be 6. This is to remove the fractions all at once and have a linear equation. This is more manageable and easier to work with.

$$\frac{6(2x-3)}{3}+\frac{6(5-x)}{2}$$

This reduces to:

$2(2x-3)+3(5-x)=6\times3$

When expanded, simplified and solved we get,

$4x-6+15-3x=18$

And $x=9$

(c) **Required to solve:** $3x-2y=10$ and $2x+5y=13$ simultaneously.

Solution:

Let

$3x-2y=10 \ldots(1)$

$2x+5y=13 \ldots(2)$

Using the method of elimination

Multiply equation (1) by 2

$2(3x-2y)=2(10)$

$6x-4y=20 \ldots(3)$

Multiply equation (2) by 3

$$3(2x+5y)=3(13)$$
$$6x+15y=39...(4)$$

Equation (4) – Equation (3)

$$\begin{array}{r} 6x+15y=39 \quad - \\ \underline{6x-\ \ 4y=20} \\ \underline{19y=19} \end{array}$$
$$y=\frac{19}{19}$$
$$y=1$$

Substitute $y=1$ in equation (1) we get

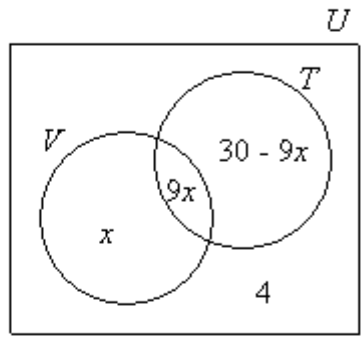
$$3x-2y=10$$
$$3x-2(1)=10$$
$$3x-2=10$$
$$3x=12$$
$$x=\frac{12}{3}$$
$$x=4$$

$\therefore x=4$ and $y=1$

We could have used the method of substitution, matrices or graphical to solve for *x* and *y*. All these methods would have yielded the exact solution.

3. (a) **Data:** In a survey of 36 students,
30 play tennis
x play volleyball only
$9x$ play both tennis and volleyball
4 do not play either tennis or volleyball
$U=\{\text{Students in survey}\}$
$V=\{\text{Students who play volleyball}\}$
$T=\{\text{Students who play tennis}\}$

(i) **Required to:** Copy and complete the Venn diagram showing the number of students in subsets marked *y* and *z*.
Solution:



(ii) **Required to write:**

a) An expression in x for the total number of students in the survey.

b) An equation in x for the total number of students in the survey and to solve for x.

Solution:

a) The number of students who play volleyball only = x (data)
The number of students who play tennis only $= 30 - 9x$ (data)
Number of students who play both tennis and volleyball $= 9x$
The number of students who do not play either tennis or volleyball = 4 (as shown on the Venn diagram above)

Therefore, the total number of students in the survey is the sum of all the students in all the subsets of the Universal set.

$$= x + (30 - 9x) + 9x + 4$$
$$= x + 30 - 9x + 9x + 4$$
$$= x + 30 + 4$$
$$= x + 34$$

b) The total no. of students = 36 (data)
The expression for the total no. of students $= x + 34$
Hence, we equate to obtain

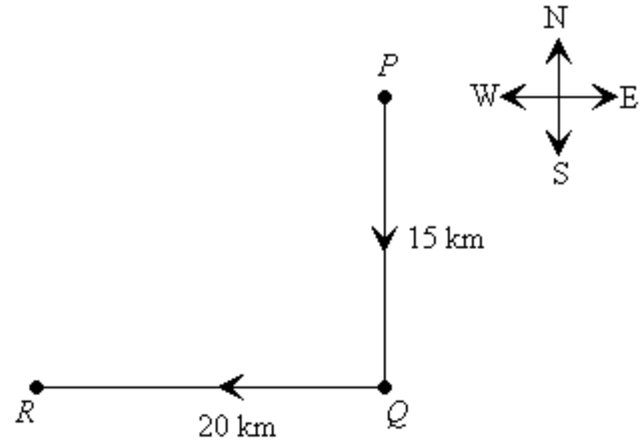
$$x + 34 = 36$$
$$x = 36 - 34$$
$$x = 2$$

(b) **Data:** Diagram showing the journey of a ship which started at port P, sailed 15 km due South to port Q and then a further 20 km due west to port R

(i) **Required to:** Copy the diagram and label it to show points Q and R and distances 20 km and 15 km.

Solution:

The direction of the ship is shown by the arrowed lines.



(ii) **Required to calculate:** The shortest distance of the ship from the port to where the journey started.

Solution:

We complete the triangle *PQR*

$PR^2 = PQ^2 + RQ^2$ (Pythagoras' Theorem)

$= (15)^2 + (20)^2$

$PR = \sqrt{(15)^2 + (20)^2}$

$= 25 \text{ km}$

(iii) **Required to calculate:** The measure of angle *QPR*, giving answer to the nearest degree.

Solution:

The triangle *PQR* is right-angled. So,

$\sin Q\hat{P}R = \frac{\text{opp}}{\text{hyp}}$

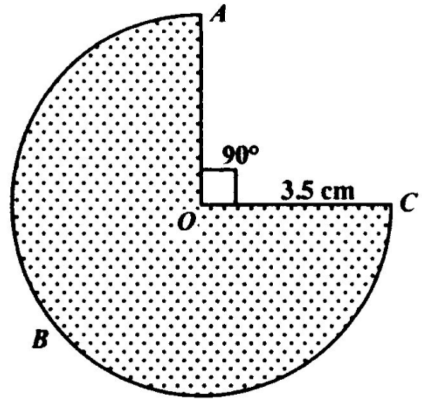
$\sin Q\hat{P}R = \frac{20}{25}$

$\therefore Q\hat{P}R = \sin^{-1}\left(\frac{20}{25}\right)$

$= 53.\underline{\underline{1}}°$

$= 53°$ (to the nearest degree)

4. **Data:** Diagram showing the cross section of a prism.



(a) **Required to calculate:**
 (i) the length of the arc *ABC*
 (ii) the perimeter of the sector *OABC*
 (iii) the area of the sector *OABC*
 Solution:
 (i) Length of the arc $ABC = \frac{270°}{360°} \times \text{Circumference of a complete circle}$

$$= \frac{3}{4} \times 2\pi \times r$$
$$= \frac{3}{4} \times 2 \times \frac{22}{7} \times 3.5$$
$$= 16.5 \text{ cm}$$

 (ii) To find the perimeter of *OABC* we need the distance all around.
 We start at say, *O* and move all around till we reach back at *O*.
 $= \text{Length of } AO + \text{Arc length } ABC + \text{Length of radius } CO$
 $= 16.5 + 3.5 + 3.5$
 $= 23.5 \text{ cm}$

 (iii) Area of the sector $OABC = \frac{270°}{360°} \times \text{Area of a complete circle}$

$$= \frac{3}{4} \times \frac{22}{7} \times (3.5)^2$$
$$= 28.87\underline{\underline{5}}$$
$$= 28.88 \text{ cm}^2 \text{ (to 2 decimal places)}$$

(b) **Data:** Prism is 20 cm long and made of tin, 1 cm^3 of tin has a mass of 7.3 g
 (This is an error in the question paper)
 Required to calculate:
 (i) the volume of the prism.

(ii) the mass of the prism, to the nearest kg.

Solution:

(i) Volume of the Prism $= \text{Cross sectional Area} \times \text{Height}$

$$= 28.875 \times 20$$

$$= 577.5 \text{ cm}^3$$

(ii) 1 cm^3 of tin weighs 7.3 g (data)

Hence, the weight, in g, of 577.5 cm^3 of tin $= 7.3 \times 577.5$ g

$$= 4\,215.75 \text{ g}$$

$$= \frac{4\,215.75}{1000} \text{ kg}$$

$$= 4.\underline{\underline{2}} \text{ kg}$$

$$= 4 \text{ kg (to the nearest kg)}$$

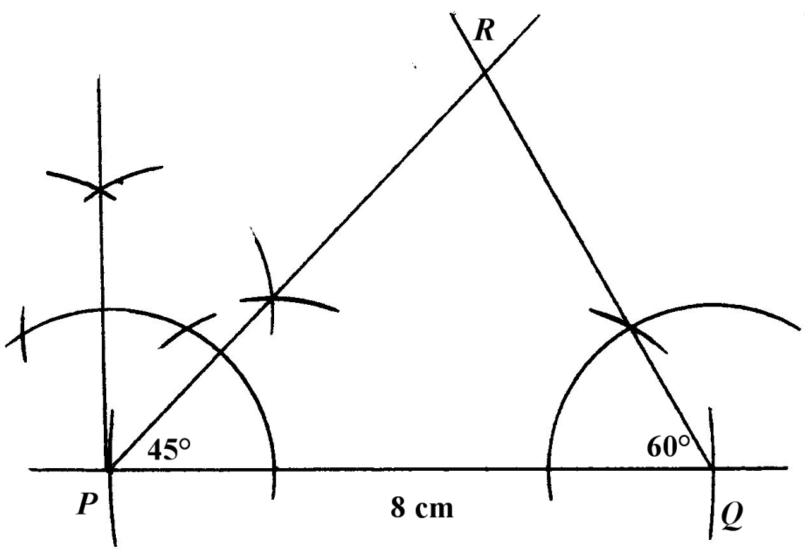
5. (a) (i) **Required to construct:** Triangle PQR with $PQ = 8$ cm, $\angle PQR = 60°$ and $\angle QPR = 45°$.

Solution:

We cut off 8 cm from a straight line drawn longer that 8 cm. This is PQ.

We construct a 45^0 angle at P and a 60^0 angle at Q.

These construction lines meet at R.



(ii) **Required to measure:** And state the length of RQ.

Solution: The length of RQ = 6.0 cm (by measurement using a ruler)

(b) **Data:** Line passes through the points $S\,(6,\,6)$ and $T\,(0,\,-2)$.

Required to find:

(i) gradient of the line
(ii) the equation of the line
(iii) the midpoint of line segment *TS*
(iv) the length of line segment *TS*

Solution:

(i) Let $(x_1, y_1) = (0, -2)$ and $(x_2, y_2) = (6, 6)$

Using the gradient formula

$$\text{Gradient } = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-(-2)}{6-0} = \frac{8}{6} = \frac{4}{3}$$

(ii) The straight line ℓ can be expressed in the form $y = mx + c$, where m is the gradient and c is the intercept on the y-axis.
Since (0, – 2) lies on the line, the y – intercept is – 2, that is, c = – 2.
We know also that the gradient, $m = \frac{4}{3}$.
Therefore, the equation of the line is $y = \frac{4}{3}x - 2$
The equation of the line, can be expressed as or in any other equivalent form, for example, multiply by 3 to get $3y = 4x - 6$.

(iii) Using the midpoint formula, the midpoint of the line segment, TS

$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{6+0}{2}, \frac{6+(-2)}{2}\right) = (3, 2)$$

(iv) The length of $TS = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(6-0)^2 + (6-(-2))^2}$$

$$= \sqrt{6^2 + 8^2}$$ (taking the positive value of the root)

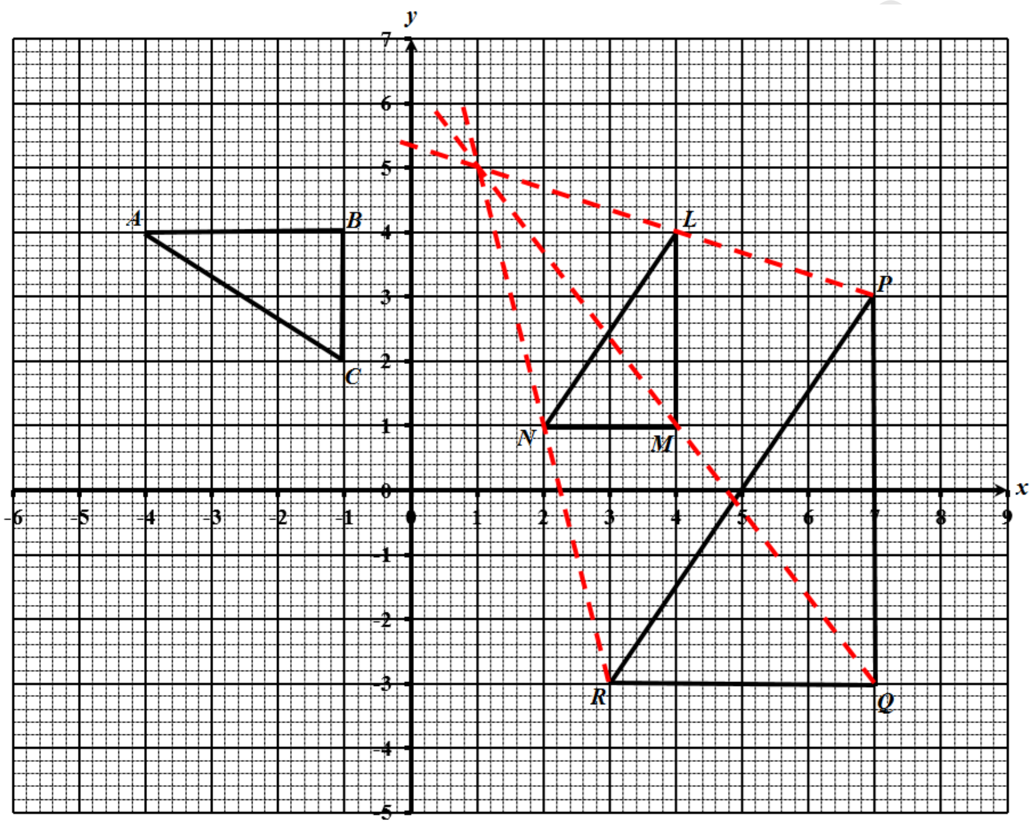
$$= 10 \text{ units}$$

6. **Data:** Graph showing triangle LMN and its image PQR after an enlargement.

(a) **Required to:** Locate the centre of enlargement.

Solution:

To find the center of enlargement we draw straight lines from each of the image points to their respective corresponding object points and extend these lines backwards so that they intersect. The common point of intersection of these lines is the center of enlargement. We only need to do this with two sets of points since these lines are all concurrent. The diagram, however, illustrates this with all three sets of points.



We can read off the center of enlargement as $=(1,\ 5)$

(b) **Required to state:** The scale factor and the center of enlargement.

Solution:

The scale factor can be found by dividing the length of any one side of the image by the length of the corresponding side of the object.

For example, we can choose the image length as PQ and so the corresponding object line will be LM. This is a good choice since the lines are vertical and their lengths can be easily found by counting the units.

The scale factor can therefore be found as $= \dfrac{\text{Length of } PQ}{\text{Length of } LM} = \frac{6}{3} = 2$

(c) **Required to determine:** The value of $\frac{\text{Area of } PQR}{\text{Area of } LMN}$.

Solution: $\frac{\text{Area of } PQR}{\text{Area of } LMN}$.

Since

$$\frac{RQ}{NM} = \frac{2}{1}$$

Then

$$\frac{\text{Area of } PQR}{\text{Area of } LMN}$$

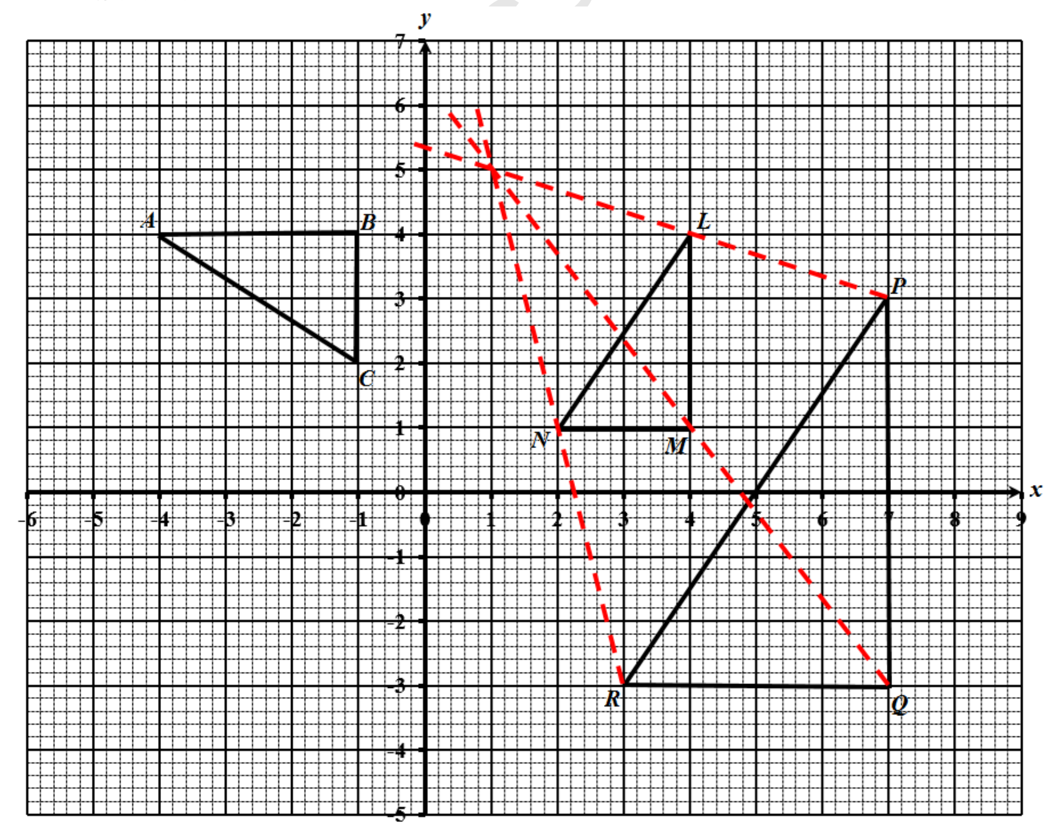
will be in the square of this ratio. This will give

$$= \frac{(2)^2}{(1)^2}$$

$$= \frac{4}{1}$$

$$= 4$$

(d) **Required to:** Draw and label triangle *ABC*.
Solution:



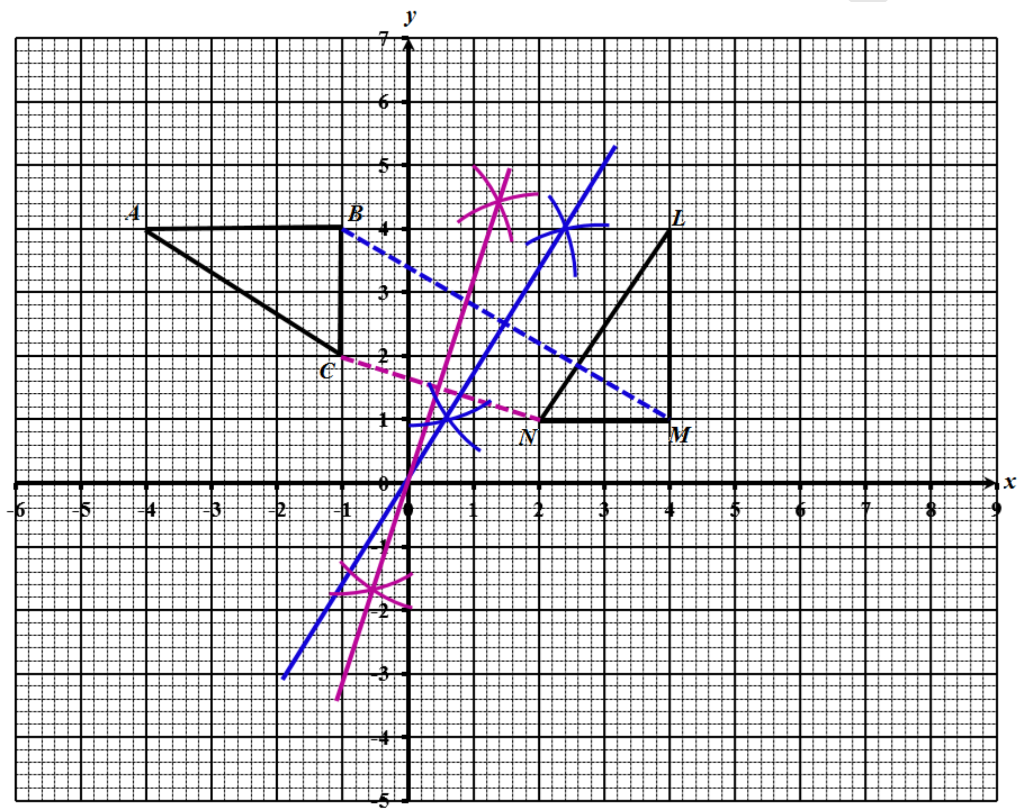
(e) **Required to:** Describe fully the transformation which maps triangle *LMN* onto triangle *ABC*.

Solution: The image, *ABC* is observed to be congruent to the object *LMN* and re-oriented with respect to the object. Hence the transformation is a rotation.

We join any object point, say *N*, to its corresponding image point, which is *C* and construct the perpendicular bisector of this line *CN*.

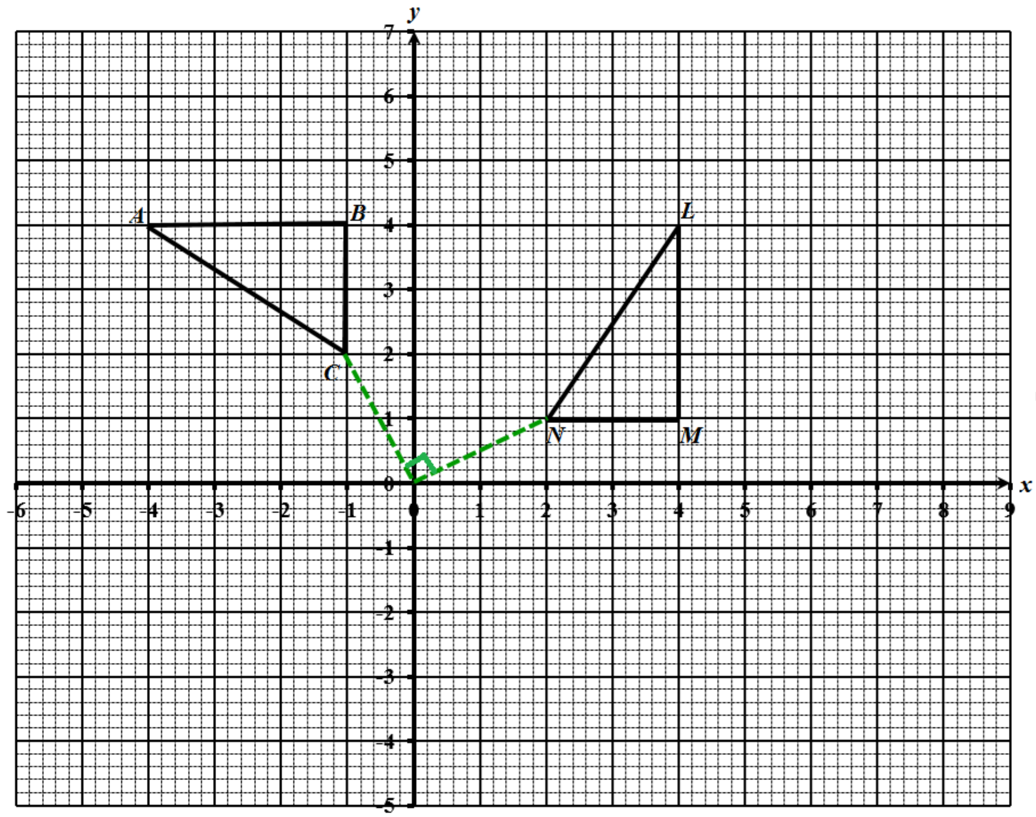
We now repeat with this procedure with a second set of points, say *M* and *B*, and construct the perpendicular bisector of the line *MP*.

The two perpendicular bisectors will meet at the center of rotation. There is no need to repeat with a third or more set of points, since these perpendicular bisectors are all concurrent.



The perpendicular bisectors of *CN* and *BM* meet at the origin. Therefore, the center of rotation is taken as $(0,\ 0)$.

AS shown below, the angle of rotation, *NOC*, is measured as 90°. The sweep of the arm from *ON* to *OC* is observed to be in an anti-clockwise direction. So the single transformation which maps ΔLMN onto ΔABC is therefore best described as a rotation of 90° in an anti-clockwise direction, about the origin, $(0,\ 0)$.



7. **Data:** Table showing the ages of persons who visited the clinic during a week.

(a) **Required to copy**: And complete the table to show cumulative frequency.

Solution:

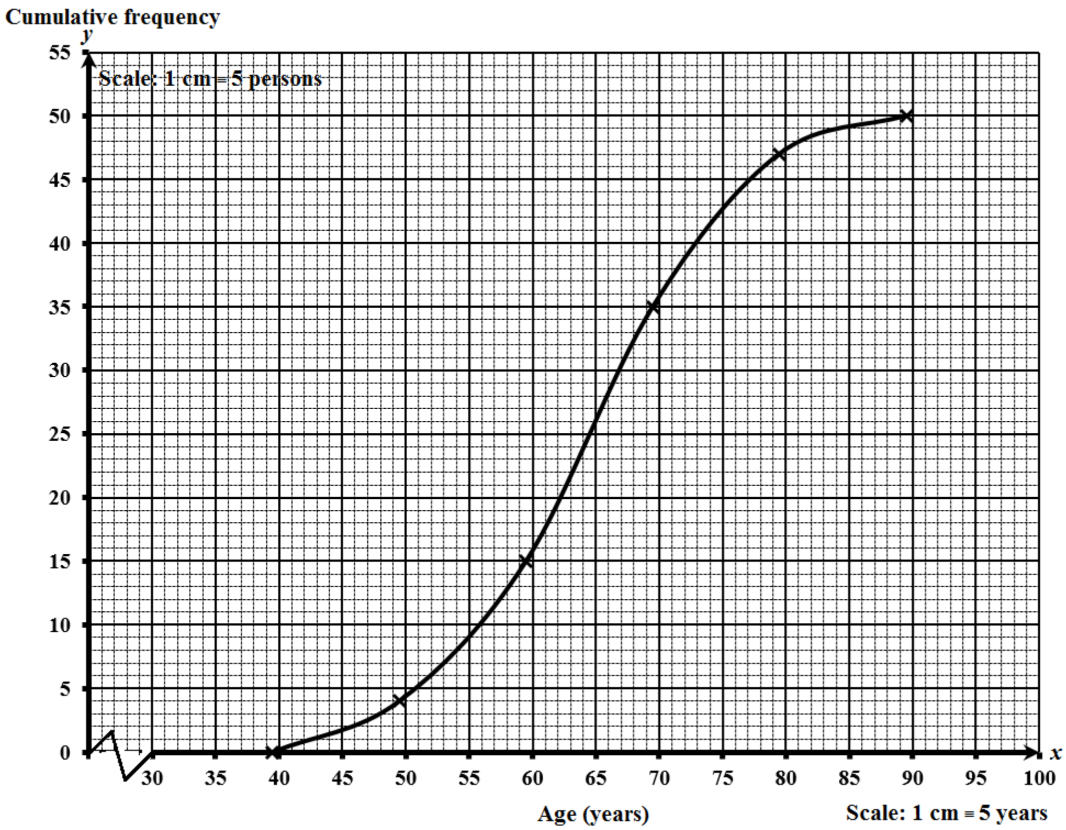
The measure of age is a continuous variable. So, the given table is modified to look like:

L.C.L-lower class limit
U.C.L-upper class limit
L.C.B-lower class boundary
U .C.B-upper class boundary

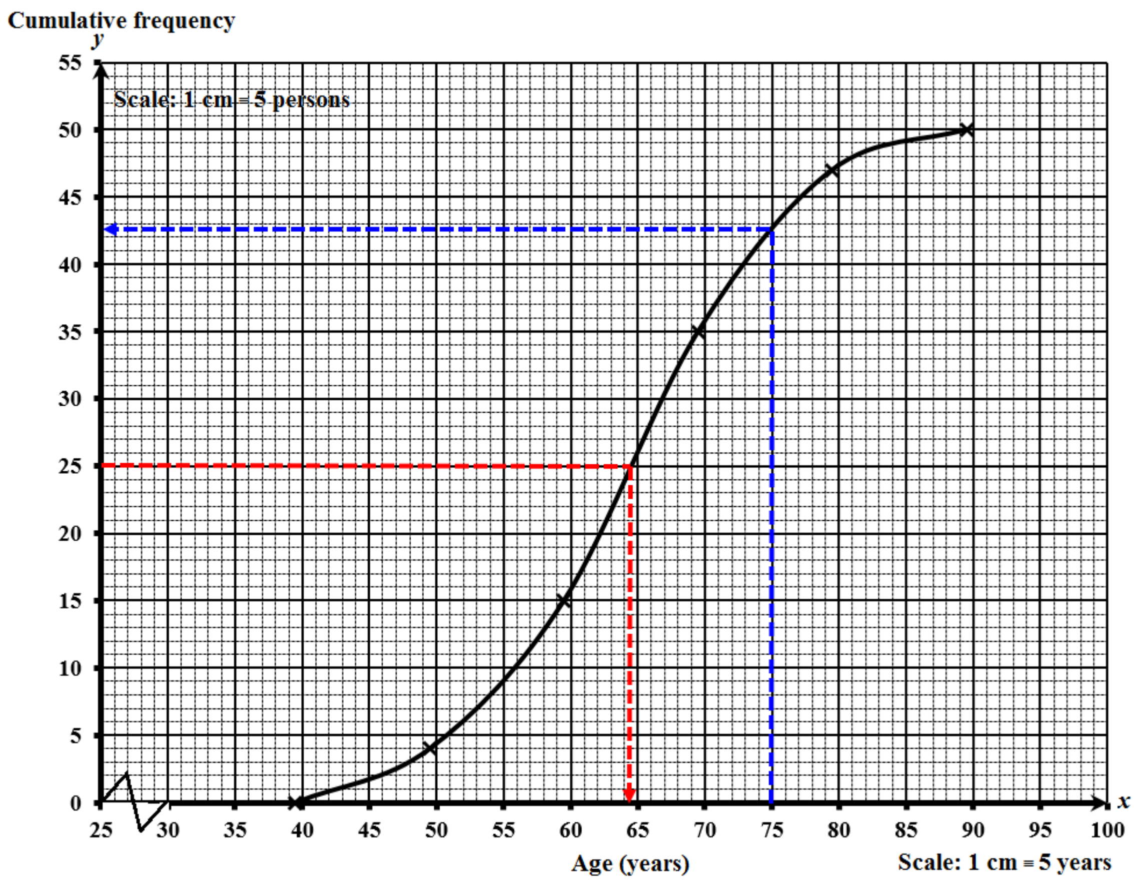
Age, x LCL -UCL	**LCB $\le x <$ UCB**	**No. of Persons**	**Cumulative Frequency**	**Points to be plotted**
				$(39.5, 0)$
40 – 49	$39.5 \le x < 49.5$	4	4	$(49.5, 4)$
50 – 59	$49.5 \le x < 59.5$	11	15	$(59.5, 15)$
60 – 69	$59.5 \le x < 69.5$	20	35	$(69.5, 35)$
70 – 79	$69.5 \le x < 79.5$	12	47	$(79.5, 47)$
80 – 89	$79.5 \le x < 89.5$	3	50	$(89.5, 50)$

The first point or starting point on the curve $(39.5, 0)$ is found by checking backwards since a cumulative frequency curve must start from the horizontal axis.

(b) **Required to draw:** The cumulative frequency curve to represent the data.
Solution:



(c) **Required to estimate:**
(i) the median age for the data.
(ii) the probability that a person who visited the clinic was 75 years or younger.
Solution:



(i) A horizontal at one half of the cumulative frequency (½ of 50 = 25) is drawn to meet the curve. At the point of meeting, a vertical is drawn to meet the horizontal axis and the reading is taken.
The median age for the data is 64.5 years (obtained by a read off from the graph).

(ii) A vertical at age 75 is drawn to meet the curve. At the point of meeting, a horizontal is drawn to meet the vertical axis and the reading is taken as 43. So, from the graph, 43 persons were seen as being 75 years or younger.

$$P(\text{person is 75 years or younger}) = \frac{\text{No. of persons 75 years or younger}}{\text{Total no. of persons}}$$

$$= \frac{42.5}{50} \text{ (Obtained from the graph as shown)}$$

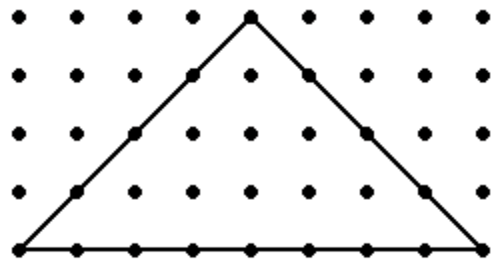
$$= \frac{17}{20}$$

8. **Data:** Diagram showing three figures in a sequence of figures.

(a) **Required to draw:** The fourth figure in the sequence of figures.

Solution:

This has been done, as shown**.**



(b) **Required to copy,** and complete the table given.

Solution:

Figure	**Area of Triangle**	**No. of Pins on Base**
1	1	$(2\times 1)+1=3$
2	4	$(2\times 2)+1=5$
3	9	$(2\times 3)+1=7$
4	$4^2=16$	(2 x 4) + 1 = 9
$\sqrt{100}=10$	100	(2 x 10) + 1 = 21
20	$20^2=400$	(2 x 20) + 1 = 41
n	n^2	$2n+1$

Section II

9. (a) (i) **Required to solve:** $y = 8 - x$ and $2x^2 + xy = -16$ simultaneously.

Solution:

Let

$y = 8 - x$ ...(1)

$2x^2 + xy = -16$ ...(2)

We are going to solve by the method of substitution as the equations do not contain terms only expressed in *x* and in *y*.

Substitute (1) into (2)

$$2x^2 + x(8 - x) = -16$$
$$2x^2 + 8x - x^2 = -16$$
$$x^2 + 8x = -16$$
$$x^2 + 8x + 16 = 0$$
$$(x+4)(x+4) = 0$$
$$\therefore x = -4 \text{ only}$$

(the equation has a repeated root)

When $x = -4$

$$y = 8 - (-4)$$
$$= 8 + 4$$
$$= 12$$

$\therefore x = -4$ and $y = 12$.

(ii) **Required to state:** With reason whether or not $y = 8 - x$ is a tangent to the curve with equation $2x^2 + xy = -16$.

Solution: When the equations $y = 8 - x$ and $2x^2 + xy = -16$ are solved simultaneously, a quadratic is obtained.
The solution of the quadratic is $x = -4$ only, that is only one solution s the roots were real and equal. Therefore, the straight line does not intersect the curve, but only touches it at that one point i.e at the point where $x = -4$. Therefore, the straight line is a tangent to the curve at the point where *x* = -4 and at the point with coordinates $(-4, 12)$.

(b) **Data:** *x* roses and *y* orchids in each bouquet with the following constraints.
The number of orchids must be at least half the number of roses.
There must be at least 2 roses.
There must be no more than 12 flowers in the bouquet.

(i) **Required to write:** The three inequalities for the constraints given.
Solution:
We group the words to deduce the inequation from the data.

No. of orchids must be at least half the no. of roses
$\quad y \qquad\qquad\qquad\qquad \geq \qquad \frac{1}{2} \qquad\qquad x$

$y \geq \dfrac{1}{2}x$

We group the words to deduce the inequation from the data.

No. of roses must be at least 2
$\quad x \qquad\qquad\qquad \geq \quad 2$

$x \geq 2$

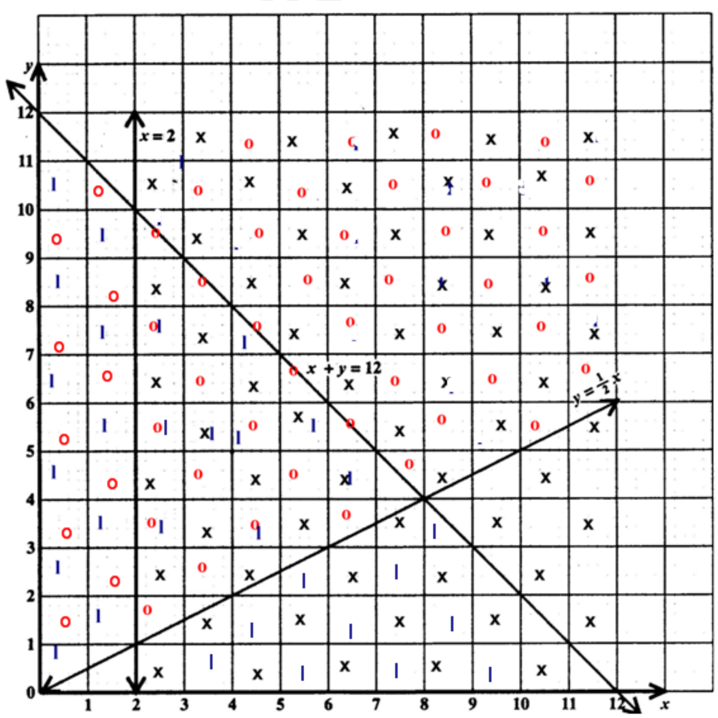
We group the words to deduce the inequation from the data.

Total no. of flowers must be no more than 12
$\quad x+y \qquad\qquad\qquad\qquad \leq \qquad 12$

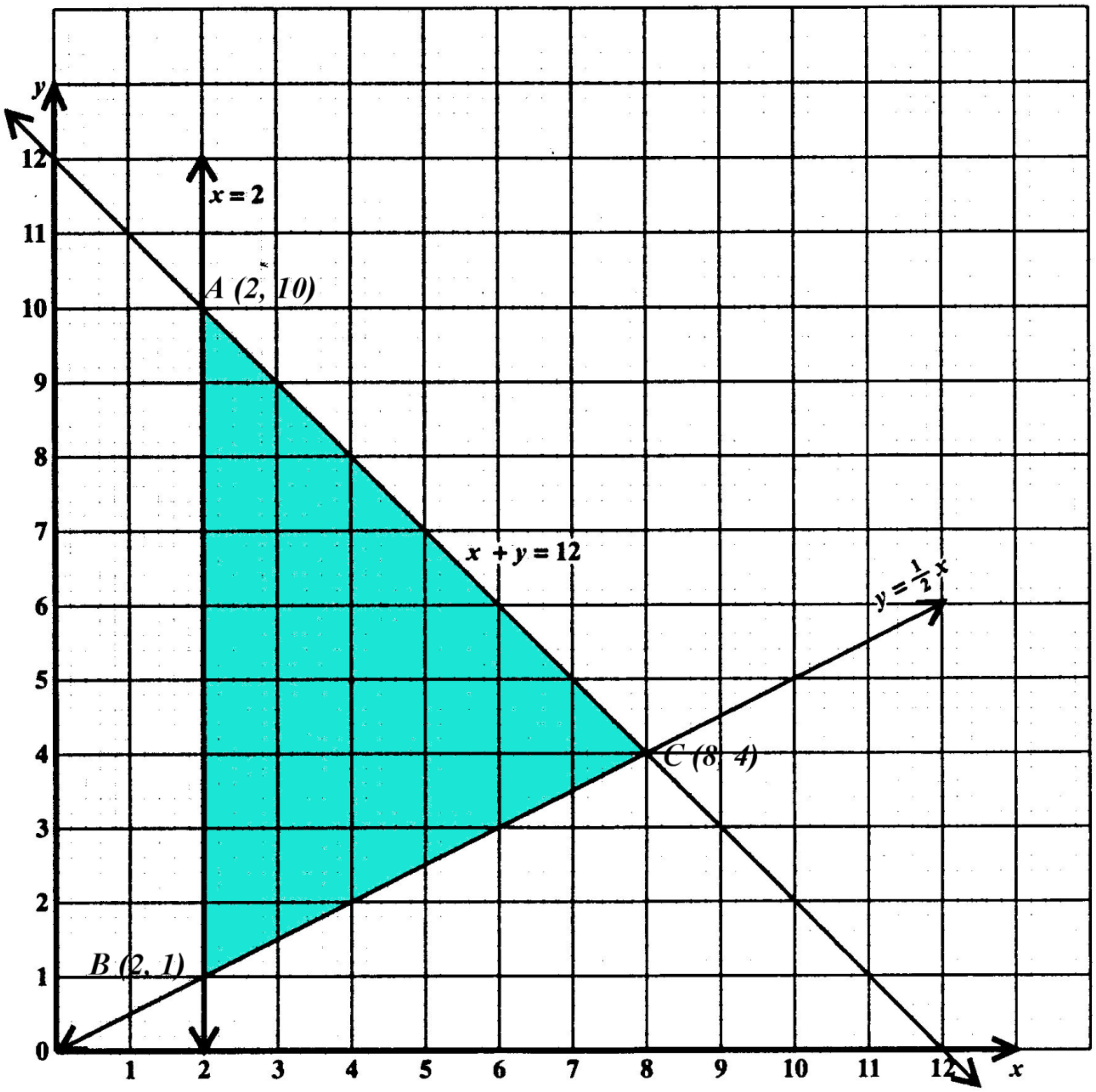
$x + y \leq 12$

$\therefore$ The three inequalities are: $y \geq \dfrac{1}{2}x$, $x \geq 2$ and $x + y \leq 12$.

(ii) **Required to shade:** The region that satisfies all three inequalities.
Solution:



The common region that is satisfied by all three inequalities, the feasible region, is re-drawn in the diagram below.



(iii) **Required to state:** The coordinates of the points which represent the vertices of the region showing the solution set.
Solution:
The feasible region is shown shaded as ΔABC where, the coordinates of A, B and C are $A=(2, 10)$, $B=(2, 1)$ and $C=(8, 4)$.

(iv) **Data:** A profit of \$3 is made on each rose and \$4 on each orchid.
Required to determine: The maximum possible profit on the sale of a bouquet
Solution:
The maximum profit occurs at a set of values given at one of the vertices.

The total profit that is made on x roses at \$3 each and on y orchids at \$4 each $=(x\times 3)+(y\times 4)$

Let P be the profit. So,

$$P=3x+4y$$

We test the three points at the three vertices of the feasible region.

Testing the point $A=(2,\ 10)$

$x=2,\ y=10$

Total profit $=(2\times 3)+(10\times 4)$

$=\$46.00$

Testing the point $B=(2,\ 1)$

Total the profit $=(2\times 3)+(1\times 4)$

$=\$10.00$

(For obvious reason, this point, B, need not have been tested)

Testing point $C=(8,\ 4)$

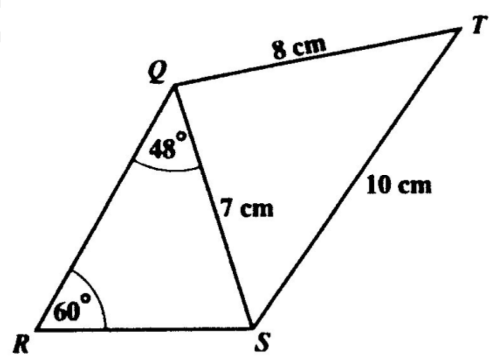
$x=8,\ y=4$

Total profit $=(8\times 3)+(4\times 4)$

$=\$40.00$

$\therefore$ The maximum profit occurs at point A with a value of \$46.00 and when the number of roses, $x=2$ and the number of orchids, $y=10$.

10. (a) **Data:** Diagram showing quadrilateral *QRST*.



Required to calculate:

(i) the length of *RS*.

(ii) the measure of $\angle QTS$.

Solution:

(i)

Applying the sine rule to triangle QRS

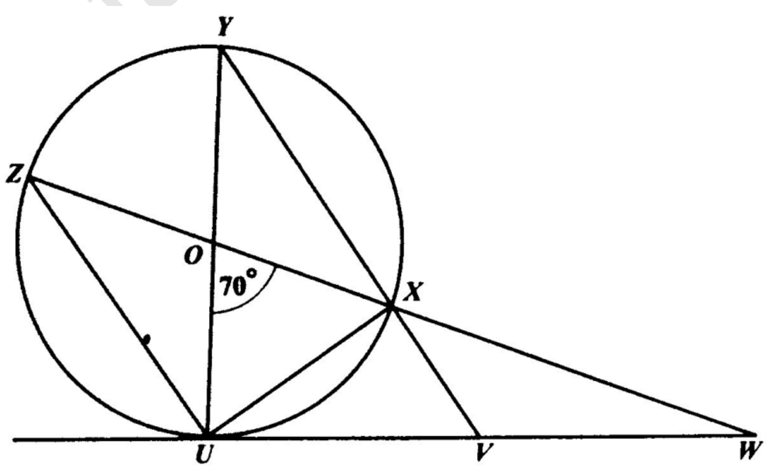
$$\frac{QS}{\sin \hat{R}} = \frac{RS}{\sin \hat{Q}} \quad \text{(Sine Rule)}$$

$$\frac{7}{\sin 60^\circ} = \frac{RS}{\sin 48^\circ}$$
$$7 \sin 48^\circ = RS \sin 60^\circ$$
$$RS = \frac{7 \sin 48^\circ}{\sin 60^\circ}$$
$$= 6.00\underline{\underline{6}}$$
$$= 6.01 \text{ (to 2 decimal places)}$$

(ii) Considering triangle *QTS* and applying the cosine rule to this triangle as we have two sides and the included angle.

$$QS^2 = QT^2 + TS^2 - 2(QT)(TS)\cos \hat{T}$$
$$7^2 = 8^2 + 10^2 - 2(8)(10)\cos \hat{T}$$
$$\cos \hat{T} = \frac{7^2 - 8^2 - 10^2}{-2(8)(10)}$$
$$= \frac{-115}{-160}$$
$$\hat{T} = \cos^{-1}(0.7187)$$
$$= 44.0\underline{\underline{2}}$$
$$= 44.0^\circ \text{ (to the nearest } 0.1^\circ)$$

(b) **Data:**



(i) **Required to calculate:** The measure of angle:

a) *OUZ*

b) UVY
c) UWO

Solution:

a) $Z\hat{O}U = 180° - 70°$
$= 110°$

The sum of angles at a pint on a straight line = 180°

In ΔZOU , $OZ = OU$ (radii of the same circle)
Therefore the triangle is isosceles.

$\therefore O\hat{U}Z = \frac{(180° - 110°)}{2}$
$= 35°$

Using the fact that the base angles of an isosceles triangle are equal and sum of the three interior angles of a triangle = 180°.

b) $U\hat{Y}V = \frac{70°}{2}$
$= 35°$

The angle subtended by a chord at the circumference of circle is half that of the angle that chord subtends at the center of the circle, and standing on the same arc.

$O\hat{U}V$ or $Y\hat{U}V = 90°$

(The angle made by the tangent *UW* to a circle and a radius *OU*, at the point of contact, *O*, is a right angle.)

$U\hat{V}Y = 180° - (90° + 35°)$
$= 55°$

The sum of the three interior angles of a triangle = 180°.

c) $W\hat{O}U = 70°$
$O\hat{U}W = 90°$

The sum of the three interior angles of a triangle = 180°.

$\therefore U\hat{W}O = 180° - (90° + 70°)$
$= 20°$

(ii) **Required to name:** The triangle which is congruent to triangle:

a) ZOU
b) YXU

Solution:

a) $OZ = OY$ (radii of the same circle)
$OU = OX$ (radii of the same circle)

$Z\hat{O}U = Y\hat{O}X$ (vertically opposite angles are equal when two straight lines intersect)

$\therefore \Delta ZOU$ is congruent to ΔYOX (by reason of two sides and the included angle).

b) Consider ΔYXU and ΔZUX

$YU = ZX$ (diameter of the same circle)

$Y\hat{X}U = Z\hat{U}X = 90°$ (angle in a semi-circle)

UX is a common side to both triangles.

Therefore, both right angled triangles, *YXU* and *ZUX* have the same hypotenuse and share a common side,

ΔYXU is congruent to ΔZUX (Reason-Right angle, hypotenuse and one side)

11. (a) **Data:** $OA = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$, $OB = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $OC = \begin{pmatrix} 12 \\ -2 \end{pmatrix}$.

(i) **Required to express:** In the form $\begin{pmatrix} x \\ y \end{pmatrix}$ the vector:

a) BA

b) BC

Solution:

Applying the vector triangle law to get

a) $BA = BO + OA$

$= -\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is of the form $\begin{pmatrix} x \\ y \end{pmatrix}$, where $x = 3$ and $y = -2$.

b) Applying the vector triangle law to get

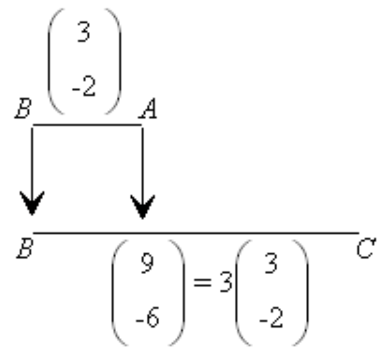
$BC = BO + OC$

$= -\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 12 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 9 \\ -6 \end{pmatrix}$ is of the form $\begin{pmatrix} x \\ y \end{pmatrix}$, where $x = 9$ and $y = -6$.

(ii) **Required to state:** One geometrical relationship between *BA* and *BC*.

Solution:



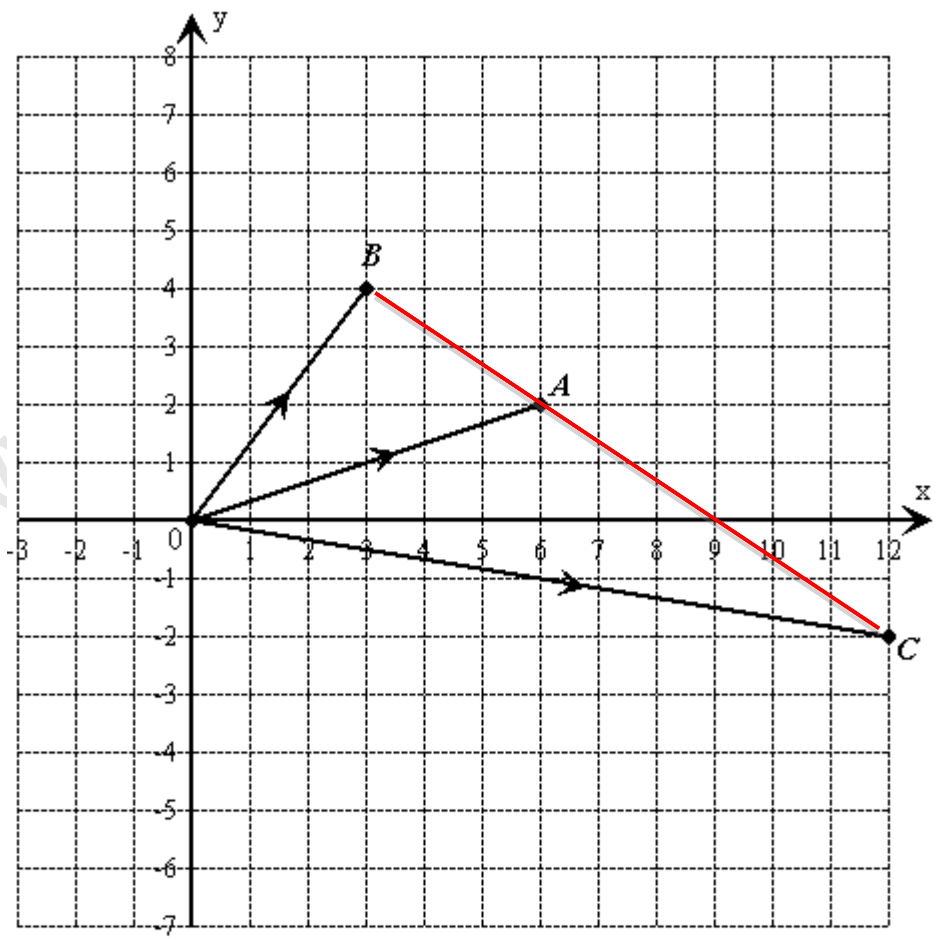
$$\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 9 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$|BC| = 3|BA|$$

The vector *BA* is a scalar multiple of the vector *BC* and *B* is a common point on both vectors so the vectors *BA* and *BC* are parallel.
Therefore, *A*, *B* and *C* are collinear and *BC* is three times (since the scalar multiple is 3) the length of *BA*.

(iii) **Required to draw:** A sketch to show the relative positions of *A*, *B* and *C*.
Solution:



(b) (i) **Data:** $\begin{pmatrix} a & -4 \\ 1 & b \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Required to calculate: The value of a and of b.

Solution:

We check to see the conformability of the matrices to matrix multiplication.

$$\underset{2\times 2}{\begin{pmatrix} a & -4 \\ 1 & b \end{pmatrix}}\underset{2\times 2}{\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}}$$

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

The matrices are conformable to multiplication and we proceed to calculate each entry in turn.

$$\begin{aligned} e_{11} &= (2\times a)+(-4\times 1) \\ &= 2a-4 \end{aligned}$$

$$\begin{aligned} e_{12} &= (-4\times a)+(-4\times -3) \\ &= 12-4a \end{aligned}$$

$$\begin{aligned} e_{21} &= (1\times 2)+(b\times 1) \\ &= b+2 \end{aligned}$$

$$\begin{aligned} e_{22} &= (1\times -4)+(b\times -3) \\ &= -4-3b \end{aligned}$$

$$\begin{pmatrix} 2a-4 & 12-4a \\ b+2 & -4-3b \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

The two matrices are 2 x 2 and are equal. So equating corresponding entries

$$\begin{aligned} 2a-4 &= 2 \\ 2a &= 6 \\ a &= 3 \end{aligned} \qquad \begin{aligned} b+2 &= 0 \\ b &= -2 \end{aligned}$$

$$\begin{aligned} 12-4a &= 0 \\ -4a &= -12 \\ a &= 3 \end{aligned} \qquad \begin{aligned} -4-3b &= 2 \\ -3b &= 6 \\ b &= -2 \end{aligned}$$

$$\therefore a = 3 \text{ and } b = -2$$

(ii) **Required to find:** The inverse of $\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}$.

Solution:

Let $A = \begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}$

$$\begin{pmatrix} 3 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\times \frac{1}{2}$

$$AA^{-1} = I$$

$$\therefore \frac{1}{2}\begin{pmatrix} 3 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{3}{2} & -2 \\ \frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{3}{2} & -2 \\ \frac{1}{2} & -1 \end{pmatrix}$$

(iii) **Required to solve**: For x and y in $\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$.

Solution:

$$\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$$

We multiply the matrix equation by A^{-1}, the inverse of the 2 x 2 matrix A.

$$\begin{pmatrix} \frac{3}{2} & -2 \\ \frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -2 \\ \frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} 12 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -2 \\ \frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} 12 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix}$$

$$\begin{aligned} e_{11} &= \left(\frac{3}{2} \times 12\right) + (-2 \times 7) \\ &= 18 + (-14) \\ &= 4 \end{aligned}$$

$$\begin{aligned} e_{12} &= \left(\frac{1}{2} \times 12\right) + (-1 \times 7) \\ &= 6 + (-7) \\ &= -1 \end{aligned}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Both sides have a 2 x 1 matrix and are they are equal
Equating corresponding entries to get
$x = 4$ and $y = -1$