CSEC ADDITIONAL MATHEMATICS SPECIMEN PAPER 2

SECTION I

1. (a) **Data:** $f(x) = x^2$ $f : R \to R$

(i) **Required to state:** Whether *f* is one to one or many one.

Solution:

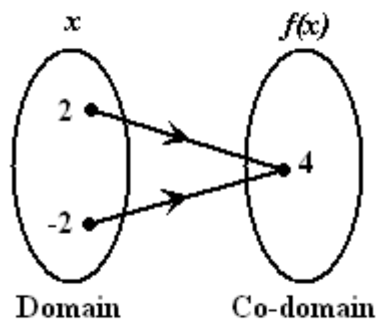
f is not one to one since, with the exception of 0, there are always two elements of the domain that are mapped onto the same element of the co – domain.

For example, when

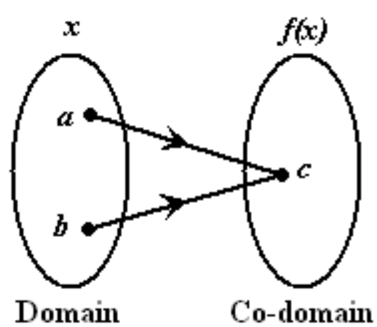
$f(x) = x^2$

$f(2) = (2)^2$
$= 4$

$f(-2) = (-2)^2$
$= 4$

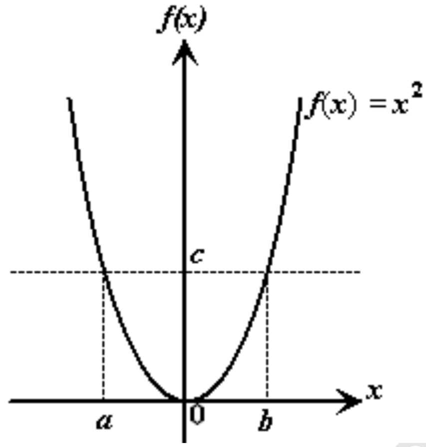


Hence, *f* is many to one.



OR

This can be illustrated by the 'horizontal line test' as illustrated below.



(ii) **Data:** The domain of *f* is the set of non-negative real numbers.

a) **Required to determine:** A function *g* such that $g\left[f\left(x\right)\right]=x$ for all values of *x* in the domain.

Solution:

$f \xrightarrow[y=x]{\text{Reflection in}} f^{-1}$

Hence,

$gf\left(x\right)=x \Rightarrow g=f^{-1}$

Let

$y=x^2$

$x=\sqrt{y}$

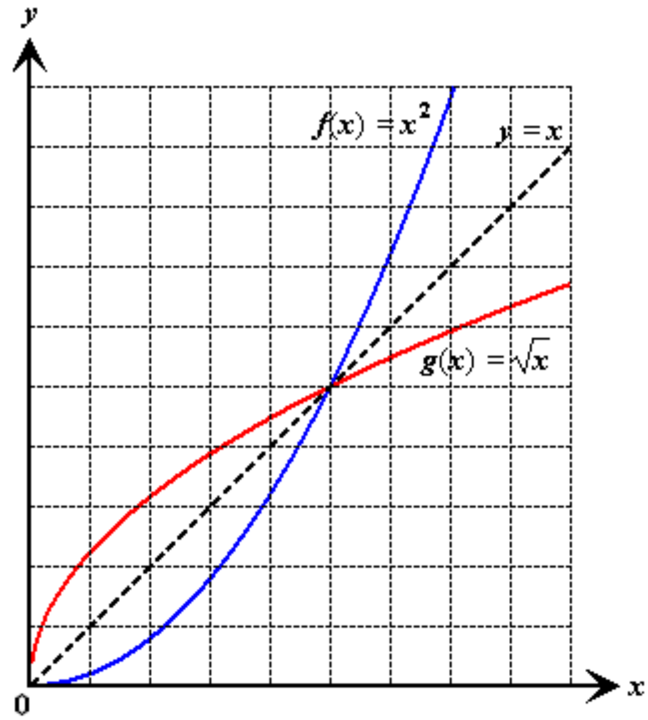
Replace *y* by *x* to obtain:

$f^{-1}\left(x\right)=\sqrt{x}\ ,\ \ x\geq 0$

$\therefore g\left(x\right)=\sqrt{x}\ ,\ \ x\geq 0$

b) **Required to sketch:** The graphs of *f* and *g* on the same axes.

Solution:



(b) **Data:** $x^3 - 2x^2 + 4x - 21$ is divided by $x - 3$.
Required to calculate: The remainder
Calculation:
Recall the remainder and factor theorem:
If $f(x)$ is a polynomial and $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$.
If $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$.

Let $f(x) = x^3 - 2x^2 + 4x - 21$
When $f(x)$ is divided by $(x-3)$ the remainder is $f(3)$.

$$
\begin{aligned}
f(3) &= (3)^3 - 2(3)^2 + 4(3) - 21 \\
&= 27 - 18 + 12 - 21 \\
&= 0
\end{aligned}
$$

The remainder is therefore 0 and this indicates that $(x-3)$ is a factor of $f(x)$.

OR
We may divide by the algebraic method

$$
\begin{array}{rl}
 & x^2+x+7 \\
x-3 \,\big) & \overline{x^3-2x^2+4x-21} \\
- & \underline{x^3-3x^2} \\
 & \quad\; x^2+4x-21 \\
- & \quad\; \underline{x^2-3x} \\
 & \qquad\quad 7x-21 \\
- & \qquad\quad \underline{7x-21} \\
 & \qquad\qquad\;\; \underline{0} \quad \text{(Remainder)}
\end{array}
$$

$\therefore$ The remainder is 0.

(c) **Data:** Table showing laboratory data for two variables p and q.
$p = aq^n$, where a and n are constants.

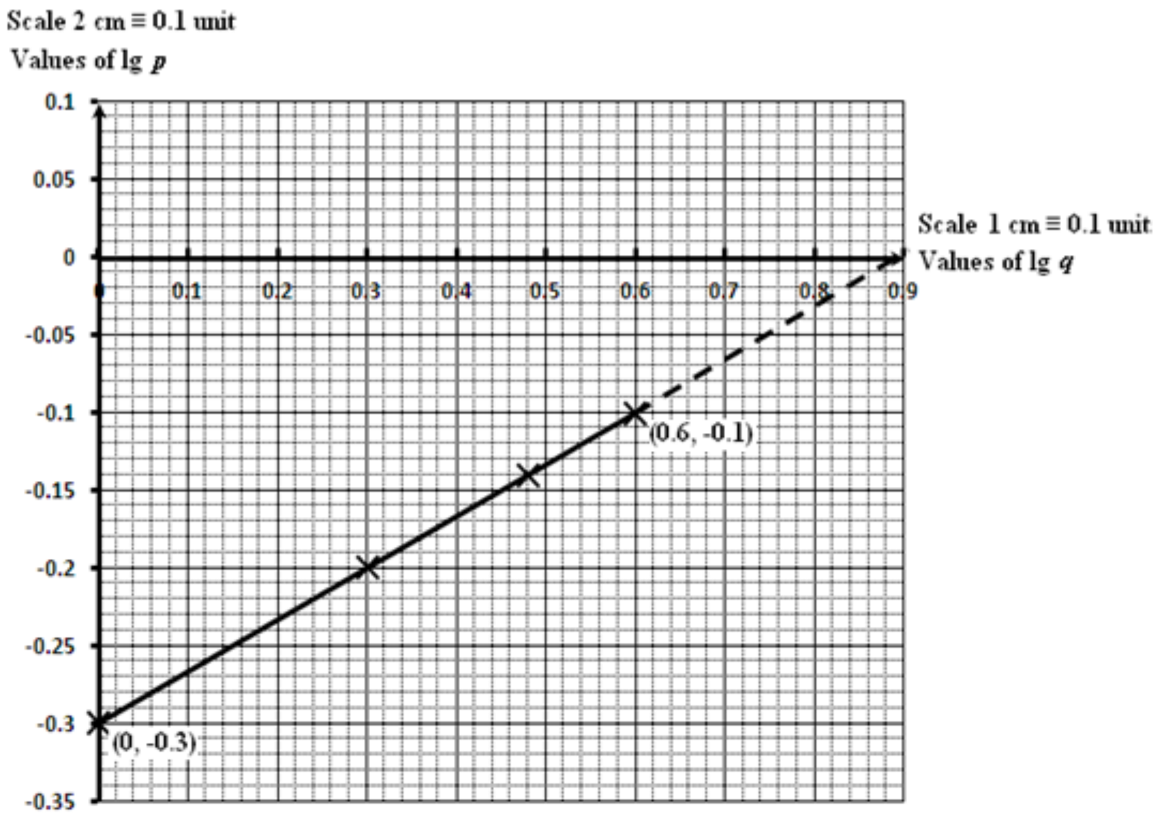
q	1	2	3	4
p	0.50	0.63	0.72	0.80

(i) **Required to express:** The relation in linear form.
Solution:
$p = aq^n$
Take lg
$\lg p = \lg\left(aq^n\right)$
$\lg p = \lg a + \lg q^n$
$\lg p = n\lg q + \lg a$ and is of the form of a straight line $y = mx + c$, where $y = \lg p$ is a variable, $x = \lg q$ is a variable, $c = \lg a$ is a constant and $m = n$ is a constant.
The graph of $\lg p$ vs $\lg q$ would be a straight line with gradient n and intercept on the vertical axis of $\lg a$.

(ii) **Required to plot:** A suitable straight line graph and obtain the value of a and n.
Solution:

p	0.50	0.63	0.72	0.80
$\lg p$	–0.30	–0.20	–0.14	–0.10
q	1	2	3	4
$\lg q$	0.00	0.30	0.48	0.60

Using the scales provided, the graph of $\lg p$ vs $\lg q$ is drawn.



Choose any two points on the straight line to find the gradient.
Using the points $(0, -0.3)$ and $(0.6, -0.1)$.

$$\text{Gradient } = \frac{-0.1-(-0.3)}{0.6-0}$$
$$= \frac{0.2}{0.6}$$
$$= \frac{1}{3}$$
$$\therefore n = \frac{1}{3}$$

The intercept on the vertical axis is -0.3.
$\therefore \lg a = -0.3$

$$a = \text{antilog}(-0.3)$$
$$a = 0.50\underline{\underline{1}}$$
$$a = 0.50 \text{ (to 2 decimal places)}$$

2. (a) **Data:** $f(x) = 3x^2 + 12x - 18$

(i) **Required to express:** $f(x)$ in the form $a(x+b)^2 + c$.

Solution:

$$f(x) = 3x^2 + 12x - 18$$
$$= 3(x^2 + 4x) - 18$$

One half the coefficient of $x = \frac{1}{2}(4)$
$= 2$

$\therefore 3(x^2 + 4x) - 18 = 3(x+2)^2 + *$, where * is a number to be found.

$$3(x+2)^2 = 3(x+2)(x+2)$$
$$= 3(x^2 + 4x + 4)$$
$$= 3x^2 + 12x + 12$$

Hence, $12 + * = -18$
$\therefore * = -30$

$\therefore 3x^2 + 12x - 18 = 3(x+2)^2 - 30$ and is of the form $a(x+b)^2 + c$, where $a = 3 \in \Re$, $b = 2 \in \Re$ and $c = -30 \in \Re$.

Alternative Method:

$$f(x) = 3x^2 + 12x - 18$$
$$= a(x+b)^2 + c$$
$$= a\{x^2 + 2bx + b^2\} + c$$
$$= ax^2 + 2abx + ab^2 + c$$

Equating coefficients:

$a = 3$

$$2ab = 12$$
$$\therefore 2(3)b = 12$$
$$b = 2$$

$$ab^2 + c = -18$$
$$3(2)^2 + c = -18$$
$$\therefore c = -18 - 12$$
$$c = -30$$

$\therefore 3x^2+12x-18=3(x+2)^2-30$ and is of the form $a(x+b)^2+c$, where $a=3\in\Re$, $b=2\in\Re$ and $c=-30\in\Re$.

(ii) **Required to state:** The minimum value of $f(x)$.

Solution:

$$f(x)=3x^2+12x-18$$
$$=3\underbrace{(x+2)^2}_{\geq 0\ \forall x}-30$$

$\therefore$ Minimum value of $f(x)=0-30$
$$=-30$$

(iii) **Required to determine:** The value of x at which $f(x)$ is a minimum.

Solution:

$$f(x)=3\underbrace{(x+2)^2}_{\geq 0\ \forall x}-30$$

$\therefore f(x)$ minimum occurs when

$$3(x+2)^2=0$$
$$(x+2)^2=0$$
$$x+2=0$$
$$x=-2$$

Alternative Method:

$$f(x)=3x^2+12-18$$
$$f'(x)=3(2x)+12$$
$$=6x+12$$

At a stationary value, $f'(x)=0$

Let

$$6x+12=0$$
$$\therefore x=-2$$

$$f''(x)=6\ (>0)$$

Hence, $f(x)$ is a minimum at $x = -2$.

$$f(-2) = 3(-2)^2 + 12(-2) - 18$$
$$= -30$$

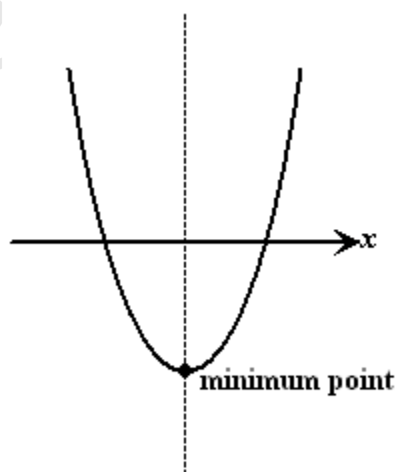
$\therefore f(x)$ has a minimum value of – 30 at $x = -2$.

Alternative Method:

$f(x) = 3x^2 + 12x - 18$

Coefficient of $x^2 > 0 \Rightarrow f(x)$ has a minimum value and the graph of $f(x)$ should look like:

$(12)^2 > 4(3)(-18) \Rightarrow$ Roots are real and distinct.



The axis of symmetry passes through the minimum point and has equation:

$$x = \frac{-(12)}{2(3)}$$
$$= -2$$

$\therefore$ At the minimum point $x = -2$.

$$f(-2) = 3(-2)^2 + 12(-2) - 18$$
$$= -30$$

$\therefore$ Minimum point is $(-2, -30)$.

$\therefore f(x)$ has a minimum value of -30 at $x = -2$.

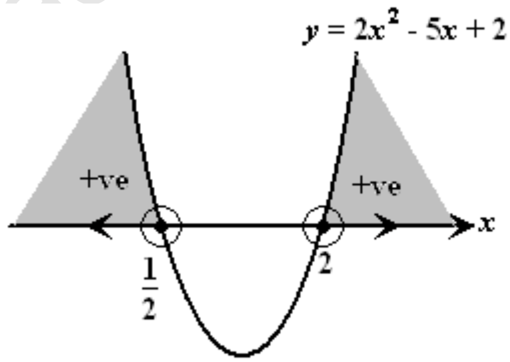
(b) **Data:** $2x^2 + 2 > 5x$
Required to solve: For *x*.
Solution:

$$2x^2 + 2 > 5x$$
$$2x^2 - 5x + 2 > 0$$

Let
$$2x^2 - 5x + 2 = 0$$
$$(2x-1)(x-2) = 0$$

$\therefore y = 2x^2 - 5x + 2$ cuts the x – axis at $\frac{1}{2}$ and 2.

Coefficient of $x^2 > 0 \Rightarrow$ the quadratic curve has a minimum point.
Sketching $y = 2x^2 - 5x + 2$ we obtain $2x^2 + 2 > 5x$



For $\{x : x > 2\} \cup \left\{x : x < \frac{1}{2}\right\}$ as shown shaded in the above diagram

(c) **Data:** The series, $\frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^{10}} + \ldots$

(i) **Required to prove:** The series is geometric.

Proof:

No. of term	1	2	3	4
Term	$\frac{1}{2}$	$\frac{1}{2^4}$ $=\frac{1}{2}\times\frac{1}{2^3}$	$\frac{1}{2^7}$ $=\frac{1}{2}\times\frac{1}{2^6}$ $=\frac{1}{2}\times\left(\frac{1}{2^3}\right)^2$	$\frac{1}{2^{10}}$ $=\frac{1}{2}\times\frac{1}{2^9}$ $=\frac{1}{2}\times\left(\frac{1}{2^3}\right)^3$
Form	a	ar	ar^2	ar^3

Hence, the n^{th} term $= ar^{n-1}$, which is in a geometric progression where the first term, $a=\frac{1}{2}$ and the common ratio, $r=\frac{1}{2^3}$.

(ii) **Required to calculate:** The sum to infinity of the series.
Calculation:
For a geometric progression,

$$S_\infty = \frac{a}{1-r} \qquad ,|r|<1$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2^3}}$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{8}}$$

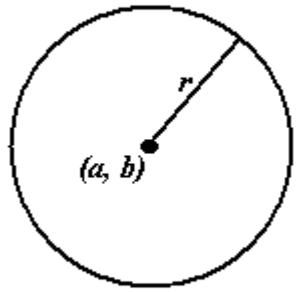
$$= \frac{\frac{1}{2}}{\frac{7}{8}}$$

$$= \frac{4}{7}$$

SECTION II

3. (a) **Data:** A circle, C, has center $(-1, 2)$ and radius $\sqrt{13}$ units.

(i) **Required to find:** The equation of the circle, C.
Solution:

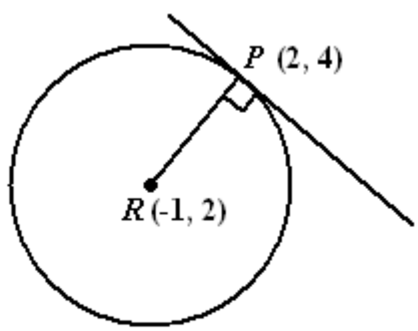


The equation of a circle with center (a, b) and radius r is
$(x-a)^2+(y-b)=r^2$.

$\therefore$ The equation of C is

$$(x-(-1))^2+(y-2)^2=(\sqrt{13})^2$$
$$(x+1)^2+(y-2)^2=13$$
$$x^2+2x+1+y^2-4y+4=13$$
$$x^2+y^2+2x-4y-8=0$$

(ii) **Required to find:** The equation of the tangent to the circle at $P\,(2, 4)$.
Solution:
Let R be the center of the circle.



$$\text{Gradient of } RP=\frac{4-2}{2-(-1)}$$
$$=\frac{2}{3}$$

The angle made by the tangent to a circle and a radius at the point of contact is a right angle.

$\therefore$ Gradient of the tangent at $P = \dfrac{-1}{\frac{2}{3}}$.

$$= -\frac{3}{2} \text{ (Product of the gradients of perpendicular lines } = -1\text{).}$$

$\therefore$ Equation of the tangent to P is

$$\frac{y-4}{x-2} = \frac{-3}{2}$$
$$2(y-4) = -3(x-2)$$
$$2y - 8 = -3x + 6$$
$$2y = 14 - 3x \text{ or any other equivalent form.}$$

OR

The general equation of a circle with center $(-g, -f)$ is
$x^2 + y^2 + 2gx + 2fy + c = 0$.

Differentiating with respect to x:

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-g-x}{f+y}$$
$$= \frac{-(g+x)}{f+y}$$

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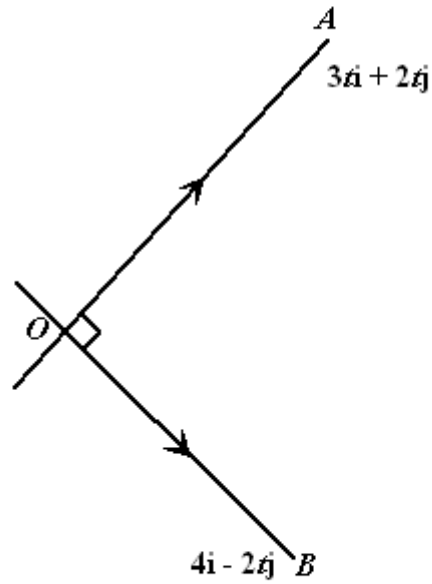
$\therefore$ Gradient of the circle with center $(-1, 2)$ at $(2, 4)$ is $\dfrac{-(1+2)}{-2+4} = -\dfrac{3}{2}$.

The equation of the tangent at $P\,(2, 4)$ is

$$\frac{y-2}{x-(-1)} = \frac{-3}{2}$$
$$2y = 14 - 3x$$

(b) **Data:** $OA = 3t\mathbf{i} + 2t\mathbf{j}$ and $OB = 4\mathbf{i} - 2t\mathbf{j}$, where $t > 0$.

Required to find: The value of t such that OA and OB are perpendicular.
Solution:



If OA is perpendicular to OB, then the dot product is 0. That is,

$a.b = |a||b|\cos\theta$ if $\theta = 90°$

$a.b = 0$

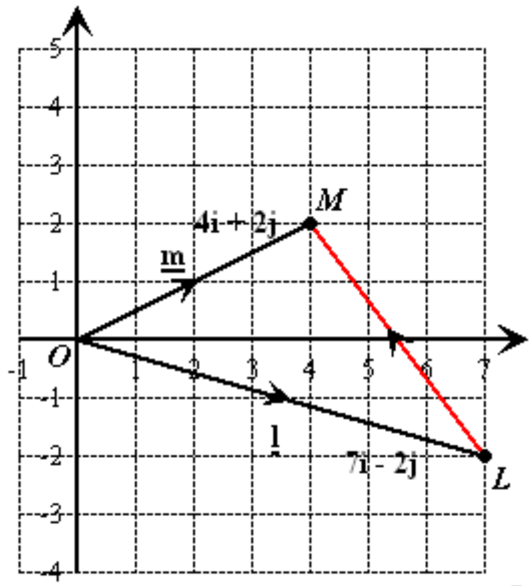
Hence,

$$(3t \times 4) + (2t \times -2t) = 0$$
$$12t - 4t^2 = 0$$
$$4t(3 - t) = 0$$
$$t = 0 \text{ or } 3$$
$$t > 0 \text{ (data)}$$
$$\therefore t = 3$$

(c) **Data:** $OL = 7\mathbf{i} - 2\mathbf{j} = \underset{\sim}{l}$ and $OM = 4\mathbf{i} + 2\mathbf{j} = \underset{\sim}{m}$.

Required To Find: The unit vector in the direction of LM.

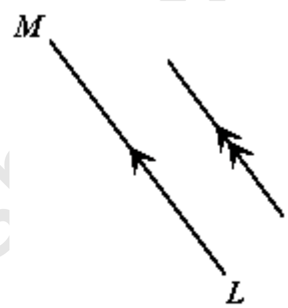
Solution:



$LM = LO + OM$

$\quad = -(7\mathbf{i} - 2\mathbf{j}) + 4\mathbf{i} + 2\mathbf{j}$

$\quad = -3\mathbf{i} + 4\mathbf{j}$



Any vector in the direction of LM will be of the form $\alpha(-3\mathbf{i} + 4\mathbf{j})$, where α is a scalar.

$= -3\alpha\,\mathbf{i} + 4\alpha\,\mathbf{j}$

Since the required vector is a unit vector, then

$$\left|-3\alpha\,\mathbf{i} + 4\alpha\,\mathbf{j}\right| = 1$$

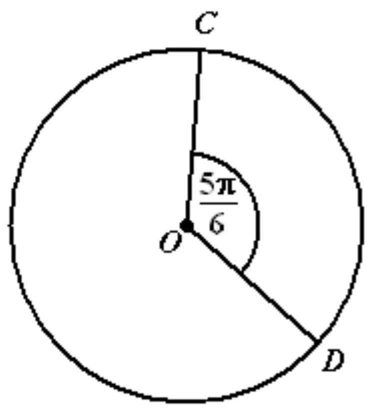
$$\sqrt{(-3\alpha)^2 + (4\alpha)^2} = 1$$

$$\alpha = \frac{1}{5}$$

$\therefore$ The unit vector in the direction of $LM = \frac{1}{5}(-3\mathbf{i} + 4\mathbf{j})$

$$= -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

4. (a) **Data:** A circle with center O and radius 6 cm. $C\hat{O}D = \frac{5\pi}{6}$ in sector COD.



(i) **Required to calculate:** The length of the minor arc CD.
Calculation:

Length of minor arc $CD = 6 \times \frac{5\pi}{6}$

$= 5\pi$ cm

(ii) **Required to calculate:** The area of the minor sector OCD.
Calculation:

Area of the minor sector $OCD = \frac{1}{2} \times (6)^2 \times \frac{5\pi}{6}$

$= 15\pi \text{ cm}^2$

(b) (i) **Data:** $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Required to prove: $\sin\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\sin x - \cos x)$

Proof:

$$\sin\left(x - \frac{\pi}{4}\right) = \sin x \cos\left(\frac{\pi}{4}\right) - \cos x \sin\left(\frac{\pi}{4}\right)$$ (Compound angle formula)

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\therefore \sin\left(x - \frac{\pi}{4}\right) = (\sin x)\frac{\sqrt{2}}{2} - (\cos x)\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{2}(\sin x - \cos x)$$

Q.E.D.

(ii) **Data:** $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

Required to calculate: $\sin\left(\frac{\pi}{12}\right)$.

Calculation:

$$\frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

$$\therefore \sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$ (Compound angle formula)

$$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}}{2}.\frac{\sqrt{2}}{2} - \frac{1}{2}.\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6} - 2}{4}$$

$$= \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} - \frac{1}{2}.\frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}} \text{ in exact form.}$$

(c) **Required to prove:** $\left(\tan\theta - \frac{1}{\cos\theta}\right)^2 \equiv -\frac{\sin\theta - 1}{\sin\theta + 1}$

Proof:
Working with the L.H.S

$$\tan\theta - \frac{1}{\cos\theta} = \frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}$$

$$= \frac{\sin\theta - 1}{\cos\theta}$$

The left hand side becomes:

$$\left(\frac{\sin\theta - 1}{\cos\theta}\right)^2 = \frac{(\sin\theta - 1)(\sin\theta - 1)}{\cos^2\theta}$$

Recall: $\sin^2\theta+\cos^2\theta=1$

$$\therefore \cos^2\theta = 1-\sin^2\theta$$
$$=(1-\sin\theta)(1+\sin\theta) \quad \text{(Difference of 2 squares)}$$

$$\left(\frac{\sin\theta-1}{\cos\theta}\right)^2=\frac{(\sin\theta-1)(\sin\theta-1)}{(1-\sin\theta)(1+\sin\theta)}$$
$$=-\frac{\sin\theta-1}{\sin\theta+1} \quad \text{(Right hand side)}$$

Q.E.D.

SECTION III

5. (a) **Required to differentiate:** $(5-2x)(1+x)^4$ with respect to *x*.

Solution:

Let $y=(5-2x)(1+x)^4$

y is of the form uv, where:

$u=5-2x \qquad \frac{du}{dx}=-2$ and

$v=(1+x)^4$

Let $t=1+x \qquad \frac{dt}{dx}=1$

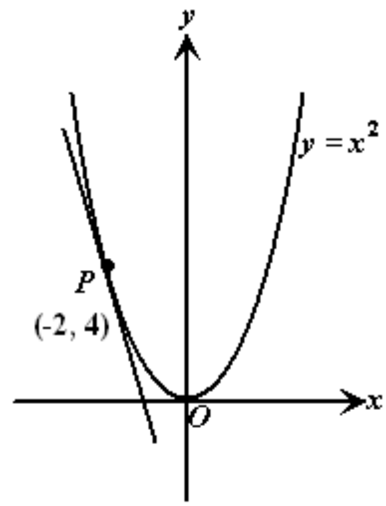
$v=t^4 \qquad \frac{dv}{dt}=4t^3$

$$\frac{dv}{dx}=\frac{dv}{dt}\times\frac{dt}{dx} \quad \text{(Chain rule)}$$
$$=4t^3\times 1$$
$$=4t^3$$
$$=4(1+x)^3$$

$$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx} \quad \text{(Product law)}$$
$$=(1+x)^4\times -2+(5-2x)(1+x)^3$$
$$=-2(1+x)^4+4(5-2x)(1+x)^3$$
$$=(1+x)^3\{-2(1+x)+4(5-2x)\}$$

$$= (1+x)^3 \{-2-2x+20-8x\}$$
$$= (1+x)^3 (18-10x)$$
$$= 2(9-5x)(1+x)^3$$

(b) **Data:** P lies on $y = x^2$ and the x – coordinate of P is – 2.
Required to find: The equation of the tangent to the curve at P
Solution:



When $x = -2$
$$y = (-2)^2$$
$$= 4$$
$$\therefore P = (-2, 4)$$

Gradient function, $\dfrac{dy}{dx} = 2x$

$\therefore$ The equation of the tangent at P is
$$\frac{y-4}{x-(-2)} = -4$$
$$y-4 = -4(x+2)$$
$$y-4 = -4x-8$$
$$y = -4x-4$$
$$4x+y+4 = 0$$

(c) **Data:** $f(x) = 2x^3 - 9x^2 + 12x.$

Required to find: The stationary points of $f(x)$ and their nature

Solution:

At a stationary point, $f'(x)=0$

$$f'(x)=2(3x^2)-9(2x)+12$$
$$=6x^2-18x+12$$

Let $f'(x)=0$

$$\therefore 6(x^2-3x+2)=0$$
$$(x-2)(x-1)=0$$

$\therefore x=1$ and 2 are the $x-$ coordinates of the stationary points.

$$f(1)=2(1)^3-9(1)^2+12(1)$$
$$=2-9+12$$
$$=5$$

$\therefore (1,\ 5)$ is a stationary point.

$$f(2)=2(2)^3-9(2)^2+12(2)$$
$$=16-36+24$$
$$=4$$

$\therefore (2,\ 4)$ is a stationary point.

$$f''(x)=6(2x)-18$$
$$=12x-18$$

$$f''(1)=12(1)-18$$
$$=-6 \text{ (negative)}$$

$\therefore (1,\ 5)$ is a maximum point.

$$f''(2)=12(2)-18$$
$$=6 \text{ (positive)}$$

$\therefore (2,\ 4)$ is a minimum point.

Alternative method to determine the nature of the stationary points:

x	1.9	1	1.1	$\therefore (1,\ 5)$ is a maximum point.
$f(x)$	+	0	−	
x	1.9	2	2.1	$\therefore (2,\ 4)$ is a minimum point.
$f(x)$	−	0	+	

6. (a) **Required to evaluate:** $\int_1^2 (3x-1)^2\ dx$

Solution:

$$\int_1^2 (3x-1)^2\ dx = \int_1^2 \left(9x^2-6x+1\right) dx$$

$$= \left[\frac{9x^3}{3} - \frac{6x^2}{2} + x\right]_1^2$$

(The constant of integration is omitted because it cancels off in a definite integral).

$$\int_1^2 (3x-1)^2\ dx = \left[3x^3 - 2x^2 + x\right]_1^2$$

$$= \left\{3(2)^3 - 2(2)^2 + (2)\right\} - \left\{3(1)^3 - 2(1)^2 + (1)\right\}$$

$$= (24-12+2) - (3-3+1)$$

$$= 14 - 1$$

$$= 13$$

(b) **Required to evaluate:** $\int_0^{\frac{\pi}{2}} (5\sin x - 3\cos x)\ dx$

Solution:

$$\int_0^{\frac{\pi}{2}} (5\sin x - 3\cos x)\ dx = \left[5(-\cos x) - 3(\sin x)\right]_0^{\frac{\pi}{2}}$$

$$= \left[-5\cos x - 3\sin x\right]_0^{\frac{\pi}{2}}$$

$$= \left\{-5\cos\left(\frac{\pi}{2}\right) - 3\sin\left(\frac{\pi}{2}\right)\right\} - \left\{-5\cos(0) - 3\sin(0)\right\}$$

$$= \left\{-5(0) - 3(1)\right\} - \left\{-5(1) - 3(0)\right\}$$

$$= -3 - (-5)$$

$$= 2$$

(c) **Data:** $\frac{dy}{dx}$ for a curve is $2-x$ and $P\left(0,\ \frac{7}{2}\right)$ lies on the curve.

(i) **Required to find:** The equation of the curve.

Solution:

$y = \int (2 - x)\ dx$

$y = 2x - \dfrac{x^2}{2} + C$ (where C is the constant integration)

$\left(0, \dfrac{7}{2}\right)$ lies on the curve.

$\therefore \dfrac{7}{2} = 2(0) - \dfrac{(0)^2}{2} + C$

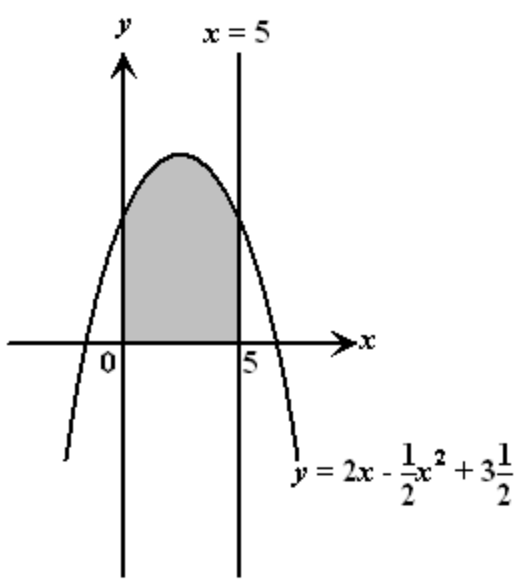
$C = \dfrac{7}{2}$

$\therefore$ Equation of the curve is $y = 2x - \dfrac{1}{2}x^2 + 3\dfrac{1}{2}$.

(ii) **Required to find:** The area of the region bounded by the curve and the x – axis, the y – axis and the line $x = 5$.

Solution:

The region required lies entirely above the x – axis.



Area $= \int_0^5 \left(2x - \dfrac{1}{2}x^2 + 3\dfrac{1}{2}\right) dx$

$$= \left[x^2 - \frac{x^3}{6} + 3\frac{1}{2}x \right]_0^5$$

$$= \left\{ (5)^2 - \frac{(5)^3}{6} + 3\frac{1}{2}(5) \right\} - \left\{ (0)^2 - \frac{(0)^3}{6} + 3\frac{1}{2}(0) \right\}$$

$$= 25 - \frac{125}{6} - 17\frac{1}{2}$$

$$= 42\frac{1}{2} - 20\frac{5}{6}$$

$$= 21\frac{2}{3} \text{ square units}$$

SECTION IV

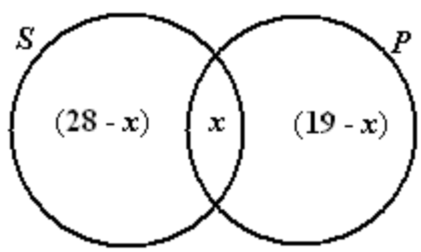
7. (a) **Data:** A class of 43 students, all studying Statistics (*S*) or Physics (*P*) or both. 28 study *S* and 19 study *P*.

(i) **Required to calculate:** The probability that a student selected at random studies both Statistics and Physics.

Calculation:

Creating a Venn diagram to illustrate the data given

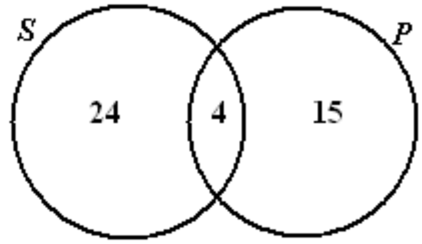
Let the number of students who study Statistics and Physics be x.



$$\therefore (28 - x) + x + (19 - x) = 43$$

$$28 + 19 - x = 43$$

$$x = 4$$



$$P\left(\text{Students studies both } S \text{ and } P\right)$$
$$= \frac{\text{No. of students who study both } S \text{ and } P}{\text{Total no. of students}}$$
$$= \frac{4}{43}$$

(ii) **Required to calculate:** The probability that a student chosen at random studies Physics only.
Calculation:
$$P\left(\text{Student studies } P \text{ only}\right)$$
$$= \frac{\text{No. of students who studies } P \text{ only}}{\text{Total no. of students}}$$
$$= \frac{15}{43}$$

(b) **Data:** Two tetrahedral die (one red and one blue) have their faces numbered 0, 1, 3 and 4. The score on the red die is *R* and the score on the blue is *B*.

(i) **Required to calculate:** The red die scores 3 and the blue die scores 0.
Calculation:
$$P\left(R = 3 \text{ and } B = 0\right) = P\left(R = 3\right) \times P\left(B = 0\right)$$
$$= \frac{1}{4} \times \frac{1}{4}$$
$$= \frac{1}{16}$$

(ii) **Data:** The random variable *T* is $R \times B$, that is the product of the scores.
Required to complete: The sample space diagram given shown for possible values of *T*
Solution:

3	$0 \times 3 = 0$	$1 \times 3 = 3$	$2 \times 3 = 6$	$3 \times 3 = 9$

2	$0\times2=0$	$1\times2=2$	$2\times2=4$	$3\times2=6$
1	$0\times1=0$	$1\times1=1$	$2\times1=2$	$3\times1=3$
0	$0\times0=0$	$1\times0=0$	$2\times0=0$	$3\times0=3$
B / ***R***	**0**	**1**	**2**	**3**

$\therefore$ Sample space diagram of T is

3	0	3	6	9
2	0	2	4	6
1	0	1	2	3
0	0	0	0	0
B / ***R***	**0**	**1**	**2**	**3**

(iii) **Data:** An incomplete probability table.

t	0	1	2	3	4	6	9
$P(T=t)$	a	$\frac{1}{16}$	$\frac{1}{8}$	b	c		

Required to calculate: The values of a, b and c and complete the table.
Calculation:

$$P(T=0)=\frac{\text{No. of scores of } 0}{\text{No. of possible scores}}$$

$$=\frac{7}{16}$$

$$\therefore a=\frac{7}{16}$$

$$P(T=3)=\frac{\text{No. of scores of 3}}{\text{No. of possible scores}}$$
$$=\frac{2}{16}$$
$$=\frac{1}{8}$$
$$\therefore b=\frac{1}{8}$$

$$P(T=4)=\frac{\text{No. of scores of 4}}{\text{No. of possible scores}}$$
$$=\frac{1}{16}$$
$$\therefore c=\frac{1}{16}$$

$$P(T=6)=\frac{\text{No. of scores of 6}}{\text{No. of possible scores}}$$
$$=\frac{2}{16}$$
$$=\frac{1}{8}$$

$$P(T=9)=\frac{\text{No. of scores of 9}}{\text{No. of possible scores}}$$
$$=\frac{1}{16}$$

$\therefore$ The completed table looks like:

t	0	1	2	3	4	6	9
$P(T=t)$	$\frac{7}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

(c) **Data:** A stem and leaf diagram showing the number of cars parked on each night in August.

1	0	5							
2	1	2	4	8					
3	0	3	3	3	4	7	8	8	
4	1	1	3	5	8	8	8	9	9
5	2	3	6	6	7				

6		2	3	4

Key: 1|0 means 10

(i) **Required to calculate:** The median and the quartiles for the data.
Calculation:
The number of values in the set of data = 31
The middle value is the 16th value from the diagram. The 16th value is 41.
$\therefore$ Median is 41.

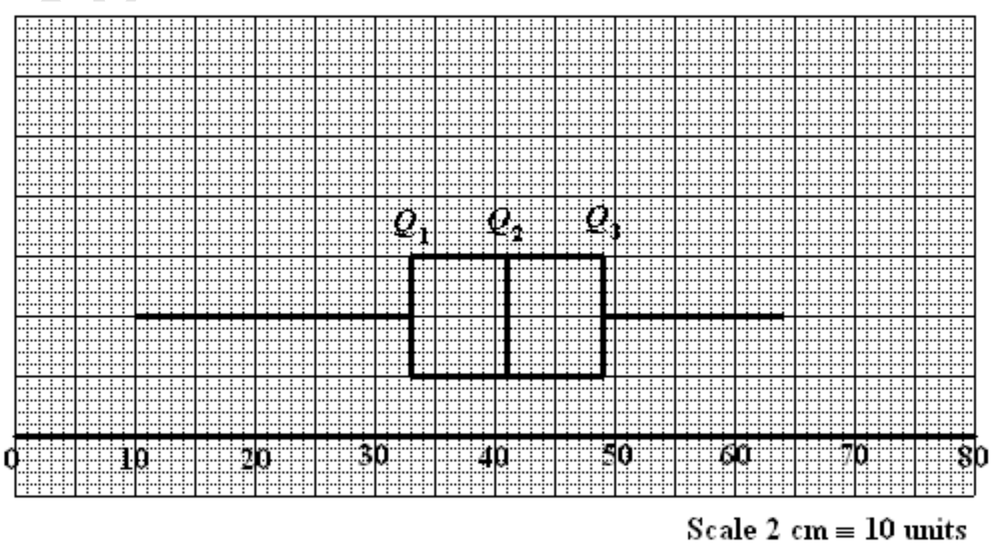
There are 15 values before the 16th value.
$\therefore$ The lower quartile is the 8th value which is 33.
$\therefore Q_1 = 33$

There are 15 values after the median.
$15 + 8 = 23$
$\therefore$ The 23rd value is the upper quartile which is 49.
$\therefore Q_3 = 49$

The interquartile range is $Q_3 - Q_1 = 49 - 33$
$= 16$

The semi-interquartile range $= \frac{1}{2}(16)$
$= 8$

(ii) **Required to construct:** A box and whisker plot to illustrate the data and comment on the shape of the distribution.
Solution:



In the box and whisker plot, the whisker to the left is longer than the whisker to the right. Hence, there are more extreme values to the negative end. The distribution is negatively skewed.

$$\begin{aligned} Q_3 + 1.5\text{I.Q.R} &= 49 + 1.5(16) \\ &= 49 + 24 \\ &= 73 \end{aligned}$$

$$\begin{aligned} Q_1 - 1.5\text{I.Q.R.} &= 33 - 1.5(16) \\ &= 33 - 24 \\ &= 11 \end{aligned}$$

There are no values more than 73 but there is one value less than 11. Hence, there is one outlier (10).

8. (a) **Data:** A car moves along a horizontal straight road, passing two points A and B. The speed of the car at A is 15 ms^{-1}. When the driver passes A, he sees a warning sign W ahead of him, 120 m away. He immediately applies the brakes and the car decelerates uniformly, reaching W at a speed of 5 ms^{-1}. At W, the driver sees that the road is clear. He then immediately accelerates the car with uniform acceleration for 16 seconds to reach a speed of V ms^{-1}, where $V > 15$. He then maintains a constant speed of V ms^{-1} for 22 seconds, passing B.

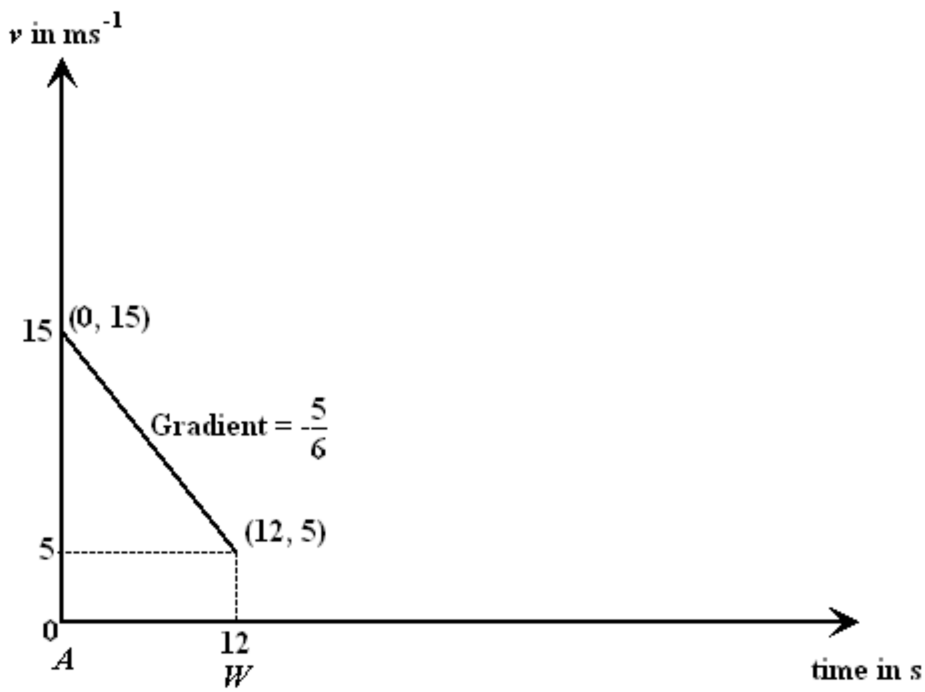
(i) **Required to sketch:** A velocity – time graph to illustrate the motion of the car as it moves from A to B

Solution:
From A to W:
Initial velocity of the car, u = 15 ms^{-1}.
Final velocity of the car, v = 5 ms^{-1}.
Distance covered = 120 m

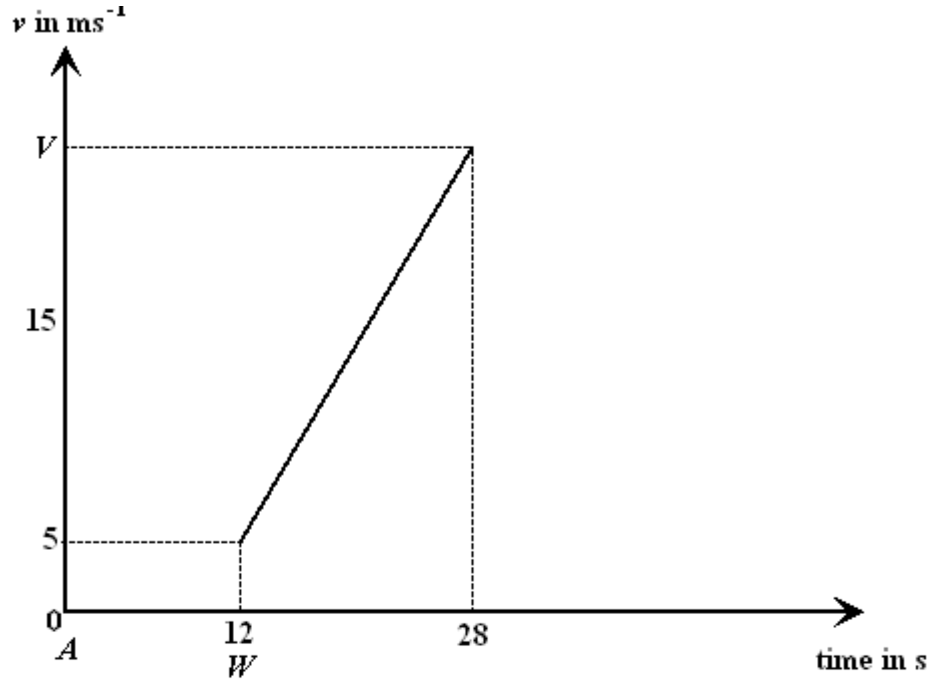
$$\begin{aligned} v^2 &= u^2 + 2as \\ (5)^2 &= (15)^2 + 2(a)(120) \\ 25 &= 225 + 240a \\ -200 &= 240a \\ a &= -\frac{200}{240} \\ a &= -\frac{5}{6}\ ms^{-2} \end{aligned}$$

$\therefore$ The deceleration is $\frac{5}{6}$ ms^{-2}.

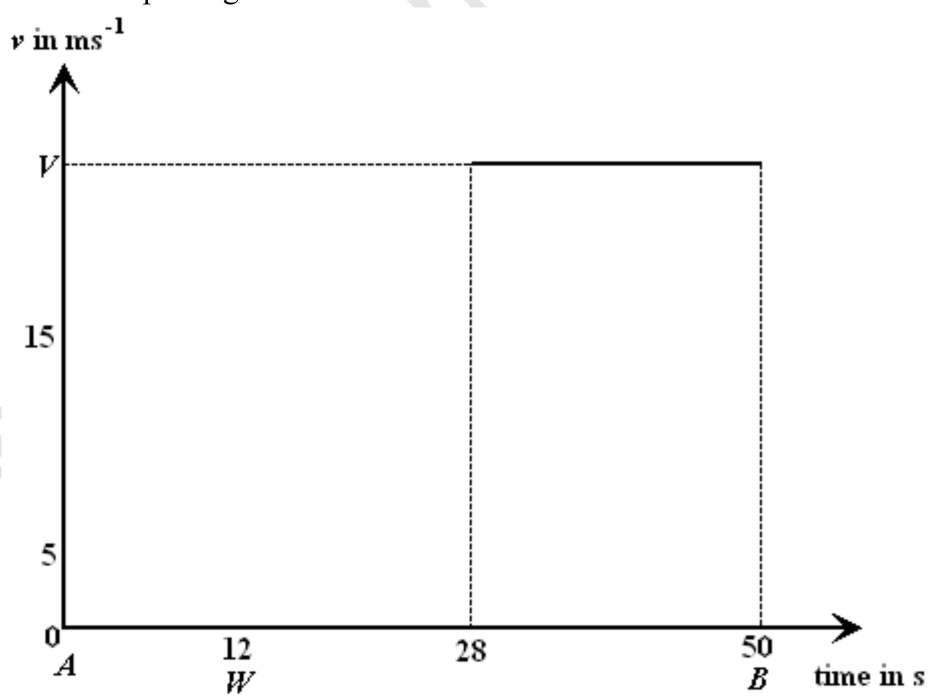
$$v = u + at$$
$$5 = 15 + \left(-\frac{5}{6}\right)t$$
$$-10 = -\frac{5}{6}t$$
$$t = 12$$



From W until the car is at V ms^{-1} ($V > 15$):
Initial velocity of the car, u = 5 ms^{-1}.
Final velocity of the car, v, = V ms^{-1}.
Time, t = 16 s.
12 + 16 = 28

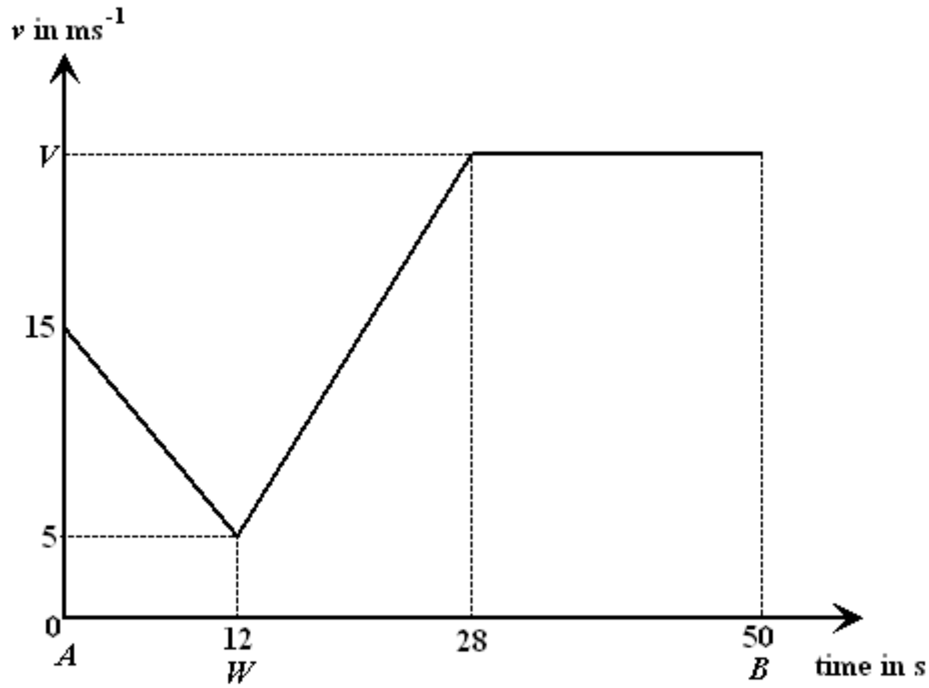


Upon reaching the speed of V ms^{-1}, the car continues at V ms^{-1} for 22 seconds passing B.



$28 + 22 = 50$

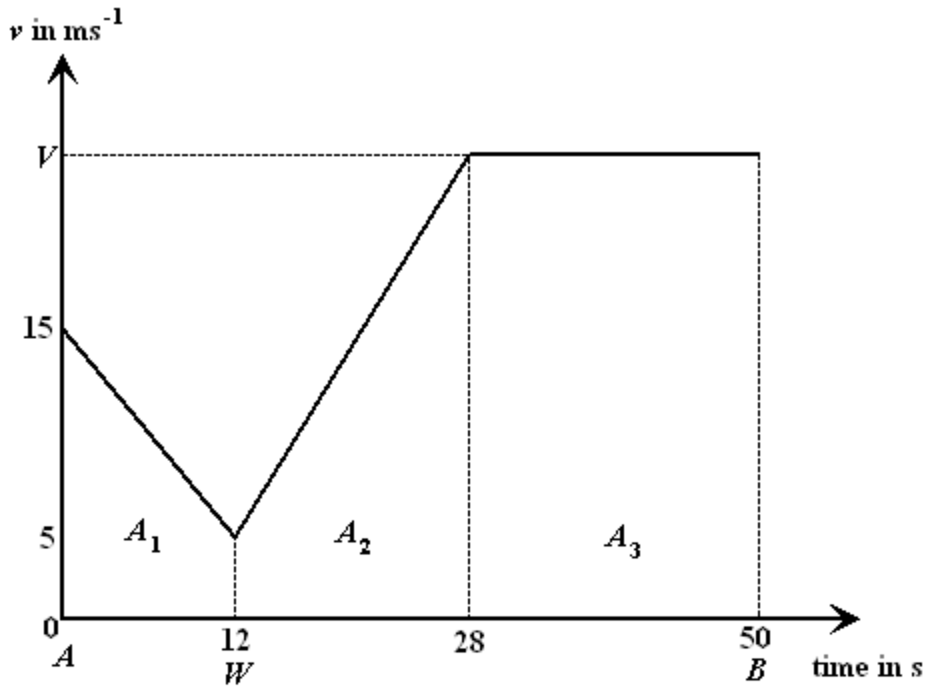
The velocity – time graph looks like:



(ii) **Required to calculate:** The time taken for the car to move from A to B.
Calculation:
Time taken to move from A to B

= Time from A to W + Time taken from W till speed of V is attained + Time taken when speed V is attained till B.
$=12+16+22$
$=50$ seconds

(iii) **Data:** The distance from A to B = 1 km = 1 000 m.
Required to calculate: V
Calculation:



The area under the graph = 1 000 m (Distance from A to B)

The total region is divided into A_1, A_2 and A_3 as shown.

Area of $A_1 = \frac{1}{2}(5+15)\times 12$
$= 120$ m

Area of $A_2 = \frac{1}{2}(5+V)\times 16$
$= (40+8V)$ m

Area of $A_3 = V \times 22$
$= (22V)$ m

Hence,
$$120+40+8V+22V=1000$$
$$30V=1000-160$$
$$V=\frac{840}{30}$$
$$V=28$$

(b) **Data:** A particle moving in a straight line passes through O with velocity v ms^{-1} such that $v = 3t^2 - 30t + 72$.

(i) **Required to calculate:** The value of t when the particle is at instantaneous rest.

Calculation:

At instantaneous rest, $v = 0$.

Let

$$3t^2 - 30t + 72 = 0$$

$\div 3$

$$t^2 - 10t + 24 = 0$$

$$(t-6)(t-4) = 0$$

$$\therefore t = 4 \text{ and } 6$$

Hence, at $t = 4$ and $t = 6$, the particle is at instantaneous rest.

(ii) **Required to calculate:** The distance moved by the particle between $t = 4$ and $t = 6$.

Calculation:

$$s = \int \left(3t^2 - 30t + 72\right) dt$$

$$= \frac{3t^3}{3} - \frac{30t^2}{2} + 72t + C \quad \text{(where } C \text{ is the constant of integration)}$$

$$s = t^3 - 15t^2 + 72t + C$$

$s = 0$ at $t = 0$

$$\therefore 0 = (0)^3 - 15(0)^2 + 72(0) + C$$

$$C = 0$$

$$\therefore s = t^3 - 15t^2 + 72t$$

When $t = 4$

$$s = (4)^3 - 15(4)^2 + 72(4)$$

$$= 64 - 240 + 288$$

$$= 112$$

When $t = 6$

$$s = (6)^3 - 15(6)^2 + 72(6)$$

$$= 216 - 540 + 432$$

$$= 108$$

The particle stops at 112 m from O when $t=4$, changes direction and moves $112-108=4$ m in the opposite direction, stopping again at $t=6$.
$\therefore$ Distance = 4 m
OR

Since there is no stopping during the period $t=4$ and $t=6$:

$$s=\int_4^6 v\ dt$$
$$=\int_4^6 \left(3t^2-30t+72\right)\ dt$$
$$=\left[t^3-15t^2+72t\right]_4^6$$
$$=\left\{(6)^3-15(6)^2+72(6)\right\}-\left\{(4)^3-15(4)^2+72(4)\right\}$$
$$=4\text{ m}$$

(iii) **Required to calculate:** The distance moved by the particle between $t=0$ and $t=7$.
Calculation:
Recall: The particle stopped at $t=4$ and at $t=6$.

The particle moved 112 m from *O* after 4 seconds.

Between $t=4$ and $t=6$, the particle moved 4 m in the opposite direction and was 108 m from *O*. That is, the particle would have covered a total distance of $112+4=116$ m covered so far.

When $t=7$

$$s=(7)^3-15(7)^2+72(7)$$
$$=343-735+504$$
$$=112$$

$\therefore$ When $t=7$, the particle is 112 m from *O*.

Hence from $t=6$ to $t=7$, the particle moved 4 m in the opposite direction to be at 112 m from *O*.

The total distance now covered after 7 seconds $=116+4$
$=120$ m

$\therefore$ Total distance covered from $t=0$ to $t=7$ is 120 m.