FAS-PASS Maths

# CSEC ADDITIONAL MATHEMATICS MAY 2023 PAPER 2

## SECTION I

## ALGEBRA, SEQUENCES AND SERIES

**ALL working must be clearly shown.**

**1. (a)** Solve the equation, $3^{2x+1} - 5\left(3^x\right) - 2 = 0$, giving your answer to 3 decimal places.

**SOLUTION:**
**Data:** $3^{2x+1} - 5\left(3^x\right) - 2 = 0$
**Required to solve:** For $x$
**Solution:**

$$3^{2x+1} - 5\left(3^x\right) - 2 = 0$$

$$3\left(3^{2x}\right) - 5\left(3^x\right) - 2 = 0$$

$$3\left(3^x\right)^2 - 5\left(3^x\right) - 2 = 0$$

For convenience and simplicity, we let $t = 3^x$
The equation can now be re-written as a simple quadratic of the form

$$3t^2 - 5t - 2 = 0$$

$$(3t+1)(t-2) = 0$$

$$\therefore t = -\frac{1}{3} \quad \text{OR} \quad t = 2$$

When $3^x = -\frac{1}{3}$

$$x \lg 3 = \lg\left(-\frac{1}{3}\right)$$

However, $\nexists \lg(-\text{ve})$

Hence, in this case, $x$ has no real solutions.

When $3^x = 2$

$$x \lg 3 = \lg 2$$

$$x = \frac{\lg 2}{\lg 3} \text{ (in exact form)}$$

$$x = 0.6309$$

$$x \approx 0.631 \text{ (correct to 3 d p)}$$

**(b) (i)** Given that $3x + 2$ is a factor of $3x^3 + bx^2 - 3x - 2$, find the value of $b$.

**SOLUTION:**
**Data:** $3x + 2$ is a factor of $3x^3 + bx^2 - 3x - 2$

**Required to find:** The value of $b$




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**Solution:**
Recall: If $(ax+b)$ is a factor of any polynomial $f(x)$, then, according to the remainder and factor theorem, $f\left(-\frac{b}{a}\right)=0$.

Let $f(x)=3x^3+bx^2-3x-2$

Hence, $f\left(\frac{-(2)}{3}\right)=0$

Substituting, we get

$$3\left(-\frac{2}{3}\right)^3+b\left(-\frac{2}{3}\right)^2-3\left(-\frac{2}{3}\right)-2=0$$

$$-\frac{8}{9}+\frac{4b}{9}+2-2=0$$

$$\frac{4b}{9}=\frac{8}{9}$$

$$b=2$$

**(ii)** Hence, factorise completely $3x^3+bx^2-3x-2$.

**SOLUTION:**
**Required to factorise:** $3x^3+bx^2-3x-2$
**Solution:**
From (i) $b=2$
So, $f(x)=3x^3+2x^2-3x-2$

$$\begin{array}{r}
x^2-1 \\
3x+2\overline{)\,3x^3+2x^2-3x-2} \\
-\quad \underline{3x^3+2x^2} \\
0x^2-0x \\
-\quad \underline{-3x-2} \\
-\quad \underline{-3x-2} \\
\underline{0}
\end{array}$$

$(x^2 -1)$ is a difference of two squares and factorises to $(x-1)(x+1)$

Hence, taking the given linear factor, the cubic polynomial $3x^3+2x^2-3x-2$ factorises completely to $(3x+2)(x-1)(x+1)$

**(c)** Determine the value(s) of $p$ for which the function $px^2+3x+2p$ has two real distinct roots, giving your answer in its simplest form.

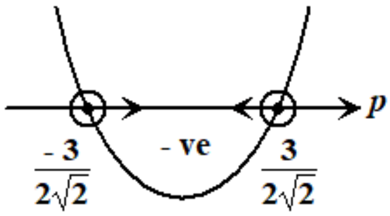
**SOLUTION:**
**Data:** $px^2+3x+2p$

**Required to determine:** Value(s) of $p$ if $px^2+3x+2p=0$ has two real roots.
**Solution:**
Recall, $ax^2+bx+c=0$ has real and distinct roots when $b^2>4ac$.
Hence, if $px^2+3x+2p=0$ has real and distinct roots, then

$$(3)^2>4(p)(2p)$$
$$9>8p^2$$
$$8p^2<9$$
$$p^2<\frac{9}{8}$$

Let $p^2=\frac{9}{8}$
$$\therefore p=\sqrt{\frac{9}{8}}$$
$$p=\pm\frac{3}{2\sqrt{2}}$$

So, for $p^2<\frac{9}{8}$ we have:




So, $9>8\,p^2$ for
$$\left\{p:\frac{-3}{2\sqrt{2}}<p<\frac{3}{2\sqrt{2}}\right\}$$

**NOTE:** A function does NOT have roots but an equation does. The question should have required the candidate to find the values of $p$ when the **EQUATION** $px^2+3x+2p=0$ has real and distinct roots.

**2.** Given that $f(x)=3x^2-9x+1$,

**(a)** **(i)** Express $f(x)$ in the form $a(x+b)^2+c$, where $a$, $b$ and $c$ are real numbers.

**SOLUTION:**
**Data:** $f(x)=3x^2-9x+1$
**Required to express:** $f(x)$ in the form $a(x+b)^2+c$, where $a$, $b$ and $c$ are real numbers.
**Solution:**

$$f(x) = a(x+b)^2 + c$$
$$= a(x^2 + 2bx + b^2) + c$$

$3x^2 - 9x + 1 = ax^2 + 2abx + ab^2 + c$

Equating coefficients of $x^2$ we get

$a = 3$

Equating coefficients of $x$ and using $a = 3$

$2(3)b = -9$

$$b = \frac{-9}{6}$$

$$b = -1\frac{1}{2}$$

Equating constants and using $a = 3$ and $b = -1\frac{1}{2}$

$$3\left(-1\frac{1}{2}\right)^2 + c = 1$$

$$c = -5\frac{3}{4}$$

Hence, $f(x) = 3\left(x - 1\frac{1}{2}\right)^2 - 5\frac{3}{4}$ and which is of the form $a(x+b)^2 + c$, where $a = 3$, $b = -1\frac{1}{2}$ and $c = -5\frac{3}{4}$ and where $a$, $b$ and $c$ are real numbers.

***Alternative Method:***

$f(x) = 3x^2 - 9x + 1$

$f(x) = 3(x^2 - 3x) + 1$

($\frac{1}{2}$ of the coefficient of $x$ is $\frac{1}{2}(-3) = -\frac{3}{2}$)

So, $f(x)$ *can be expressed as* $3\left(x - 1\frac{1}{2}\right)^2 + ?$

$$\uparrow$$

$$3\left(x^2 - 3x + \frac{9}{4}\right)$$

$$\textit{which is } 3x^2 - 9x + \frac{27}{4}$$

$$\textit{or } 3x^2 - 9x + 6\frac{3}{4}$$

To obtain $f(x) = 3x^2 - 9x + 1$ we must therefore re-write the equation of $f(x)$ as $3x^2 - 9x + 6\frac{3}{4}$ $\mathbf{-5\frac{3}{4}}$

Hence, in $3\left(x - 1\frac{1}{2}\right)^2 + ?$ the value of the unknown ? *is* $-5\frac{3}{4}$





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So, $f(x)=3\left(x-1\frac{1}{2}\right)^2-5\frac{3}{4}$ and is of the form $a(x+b)^2+c$, where $a=3$, $b=-1\frac{1}{2}$ and $c=-5\frac{3}{4}$ where $a$, $b$ and $c$ are real numbers.

**(ii)** State the coordinates of the minimum point of $f(x)$.

**SOLUTION:**
**Required to state:** The coordinates of the minimum point of $f(x)$
**Solution:**

$$f(x)=3\left(x-1\frac{1}{2}\right)^2-5\frac{3}{4}$$

$$\uparrow$$

$$\geq 0\ \forall x$$

$\therefore f(x)_{\min}=-5\frac{3}{4}$. This occurs at $3\left(x-1\frac{1}{2}\right)^2=0$, that is, at $x=1\frac{1}{2}$.

Hence, the minimum point on the graph $f(x)$ will be at the point with coordinates $\left(1\frac{1}{2},\ -5\frac{3}{4}\right)$.

***Alternative Method:***

$$f(x)=3x^2-9x+1$$

$$f'(x)=3\left(2x^{2-1}\right)-9$$

$$=6x-9$$

At a stationary point, $f'(x)=0$

When $6x-9=0$

$$x=1\frac{1}{2}$$

$$f\left(1\frac{1}{2}\right)=3\left(1\frac{1}{2}\right)^2-9\left(1\frac{1}{2}\right)+1$$

$$=-5\frac{3}{4}$$

$$f''(x)=6$$

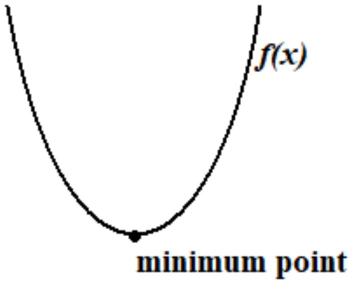
$$>0$$

Hence, $\left(1\frac{1}{2},\ -5\frac{3}{4}\right)$ is the minimum point on $f(x)$.

***Alternative Method:***

$f(x) = 3x^2 - 9x + 1$

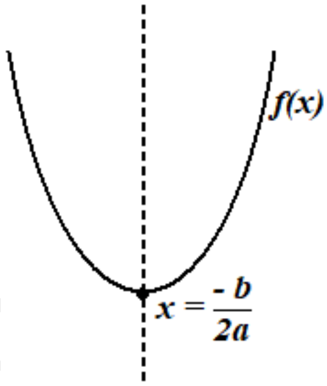
Coefficient of $x^2 > 0 \Rightarrow f(x)$ has a minimum point.




The minimum point occurs at $x = \frac{-(-9)}{2(3)}$

$= 1\frac{1}{2}$

This is the equation of the axis of symmetry and the minimum point on $f(x)$ has the same $x$-coordinate as does the axis of symmetry.




$f\left(1\frac{1}{2}\right) = 3\left(1\frac{1}{2}\right)^2 - 9\left(1\frac{1}{2}\right) + 1$

$= -5\frac{3}{4}$

$\therefore$Minimum point of $f(x)$ is $\left(1\frac{1}{2}, -5\frac{3}{4}\right)$.

**(b)** The equation $3x^2 - 6x - 2 = 0$ has roots $\alpha$ and $\beta$. Find the value of $\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$.

**SOLUTION:**

**Data:** $3x^2 - 6x - 2 = 0$ has roots $\alpha$ and $\beta$

**Required to find:** The value of $\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$

**Solution:**

If $\alpha$ and $\beta$ are the roots of $ax^2+bx+c=0$, then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

$\alpha$ and $\beta$ are the roots of $3x^2-6x-2=0$

$$\therefore \alpha+\beta=\frac{-(-6)}{3}$$
$$=2$$

and $\alpha\beta=\frac{-2}{3}$

$$\frac{1}{\alpha}+\frac{1}{\beta}$$
$$\frac{\beta+\alpha}{\alpha\beta}=\frac{2}{-\frac{2}{3}}$$
$$=\frac{2}{1}\times\frac{-3}{2}$$
$$=-3$$

**(c)** John's grandparents started a university fund for him at a bank, with \$4000. The bank offered two options for interest.

Option 1 – \$240 per annum
Option 2 – 5% of the current balance per annum

Determine the sum of money in the university fund at the beginning of the ninth year for both options.

**SOLUTION:**
**Data:** A university fund was started with \$4000 with two options for interest. Option 1 – \$240 per annum, Option 2 – 5% of the current balance per annum.

**Required to determine:** The sum of money in the fund at the beginning of the ninth year for both options.

**Solution:**
Option 1:
From the start to the beginning of the 9th year means the constant interest of \$240 will be earned for 8 years.
Interest earned $=\$240\times 8$
$=\$1920$

Hence, the sum of the money in the fund at the start of the 9th year
$=\$4000+\$1920$
$=\$5920$

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Option 2:

| Amount at start | End of year 1 | End of year 2 |
|---|---|---|
| \$4 000 | $4000\left(\frac{5}{100}\times 4000\right)$ <br> $=4000\times 1.05$ | $(4000\times 1.05)+\frac{5}{100}(4000\times 1.05)$ <br> $=4000\times 1.05^2$ |

This pattern continues.

Hence, at the end of year 8 and which is the beginning of year 9, the amount will be

$=4000\times 1.05^8$

$=\$5909.82$

***Alternatively:***

Using the formula for compound interest and where
*A* = Amount after the period, *P* = Principal invested, *R* = rate per annum and *n* = time in years.

$$A=P\left(1+\frac{R}{100}\right)^n$$

$$=4000\left(1+\frac{5}{100}\right)^8$$

$$=\$5909.82 \text{ (correct to the nearest cent)}$$

## SECTION II

## COORDINATE GEOMETRY, VECTORS AND TRIGONOMETRY

**3. (a)** The equation of a circle is $x^2+y^2-8x-18y+93=0$.

**(i)** Determine the coordinates of the centre of the circle.

**SOLUTON:**
**Data:** $x^2+y^2-8x-18y+93=0$
**Required to determine:** The coordinates of the centre of the circle
**Solution:**

$$x^2+y^2-8x-18y+93=0$$

$x^2+y^2+2(-4)x+2(-9)y-93=0$ is of the form

$x^2+y^2+2gx+2fy+c=0$ where $g=-4,\ f=-9$ and $c=93$

Centre $=(-g,-f)$

 

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$$=(-(-4), -(-9))$$
$$=(4, 9)$$

**(ii)** Find the length of the radius.

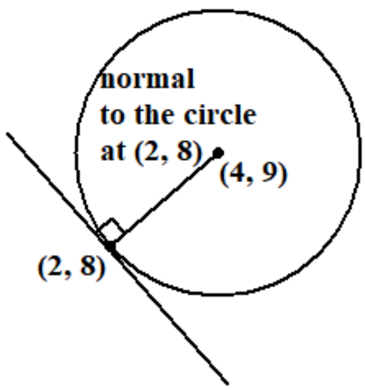
**SOLUTION:**
**Required to find:** The length of the radius
**Solution:**

Length of the radius $=\sqrt{g^2+f^2-c}$

$$=\sqrt{(-4)^2+(-9)^2-93}$$
$$=\sqrt{16+81-93}$$
$$=\sqrt{4}$$
$$=2 \text{ units}$$

**(iii)** Find the equation of the normal to the circle at the point $(2, 8)$.

**SOLUTION:**
**Required to find:** The equation of the normal to the circle at the point $(2, 8)$.
**Solution:**




Gradient of the normal:

$$\frac{9-8}{4-2}=\frac{1}{2}$$

Equation of the normal using the point (4, 9) and gradient $=\frac{1}{2}$

$$y-9=\frac{1}{2}(x-4)$$
$$2y-18=x-4$$
$$2y=x+14$$

(As a point of interest, the coordinates (2 8) do NOT satisfy the equation of the circle and so there can NEVER be a normal or a tangent to the circle at this point)

**(b)** The position vectors of two points, $A$ and $B$, relative to an origin $O$, are such that $\overrightarrow{OA} = 3\mathbf{i} - \mathbf{j}$ and $\overrightarrow{OB} = 5\mathbf{i} - 4\mathbf{j}$. Determine

(i) the unit vector $AB$

**SOLUTION:**
**Data:** $\overrightarrow{OA} = 3\mathbf{i} - \mathbf{j}$ and $\overrightarrow{OB} = 5\mathbf{i} - 4\mathbf{j}$ are the position vectors of two points $A$ and $B$, relative to an origin, $O$.
**Required to determine:** The unit vector $AB$
**Solution:**

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -(3\mathbf{i} - \mathbf{j}) + 5\mathbf{i} - 4\mathbf{j} \\ &= 2\mathbf{i} - 3\mathbf{j}\end{aligned}$$

Any vector parallel to $\overrightarrow{AB}$ is $\alpha(2\mathbf{i} - 3\mathbf{j}) = 2\alpha\,\mathbf{i} - 3\alpha\,\mathbf{j}$, where $\alpha$ is a scalar.

Since the vector is a unit vector, its modulus or magnitude $= 1$

Hence, $\sqrt{(2\alpha)^2 + (-3\alpha)^2} = 1$

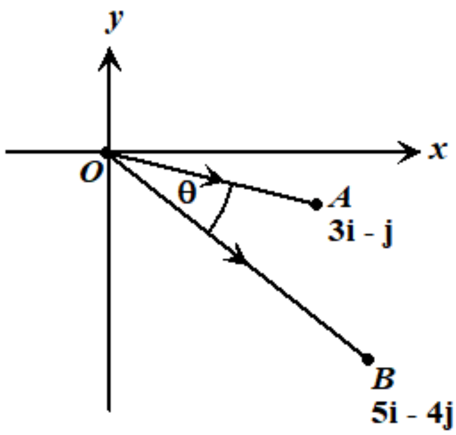
$$\alpha = \frac{1}{\sqrt{13}}$$

So, the unit vector in the direction of $\overrightarrow{AB} = \frac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{j})$

**NOTE:** $\overrightarrow{AB}$ itself is not a unit vector. The question should be restated as: "Determine the unit vector in the direction of $\overrightarrow{AB}$."

**(ii)** the acute angle $AOB$, in degrees, to 1 decimal place.

**SOLUTION:**
**Required to determine:** The acute angle, $AOB$, to 1 decimal place
**Solution:**




Let $A\hat{O}B$ be $\theta$.

$$\overrightarrow{OA}.\overrightarrow{OB} = \left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\cos\theta$$

$$(3\times5)+(-1\times-4)=\sqrt{(3)^2+(-1)^2}\ \sqrt{(5)^2+(-4)^2}\ \cos\theta$$

$$19=\sqrt{10}\sqrt{41}\cos\theta$$

$$\cos\theta=\frac{19}{\sqrt{410}}$$

$$\theta = 20.2\underset{=}{3}°$$

$$\theta \approx 20.2° \text{ (correct to 1 decimal place)}$$

**(c)** Solve the equation $2\sin^2\theta = 3\cos\theta$ where $0° < \theta < 180°$.

**SOLUTION:**
**Data:** $2\sin^2\theta = 3\cos\theta$
**Required to solve:** For $\theta$, where $0° < \theta < 180°$
**Solution:**

$$2\sin^2\theta = 3\cos\theta$$

Recall: $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2\theta = 1-\cos^2\theta$

Hence, $2\left(1-\cos^2\theta\right)-3\cos\theta=0$

$$2-2\cos^2\theta-3\cos\theta=0$$

$$2\cos^2+3\cos\theta-2=0$$

$$(2\cos\theta-1)(\cos\theta+2)=0$$

$$2\cos\theta-1=0 \qquad \text{OR} \qquad \cos\theta+2=0$$

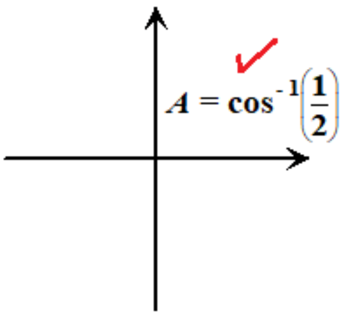
$$\cos\theta=\frac{1}{2} \qquad\qquad \cos\theta=-2$$

Since $-1 \le cos\theta \le 1$
$\theta$ has no real solutions.

$$\cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 60°$$




$\theta = 60^0$, for $0° < \theta < 180°$

**(d)** Prove the identity $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = \frac{2\tan x}{\cos x}$.

**SOLUTION:**

**Required to prove:** $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = \frac{2\tan x}{\cos x}$

**Proof:**

Consider the left-hand side of the identity:

$$\frac{1}{1-\sin x} - \frac{1}{1+\sin x}$$

$$\frac{1(1+\sin x) - 1(1-\sin x)}{(1-\sin x)(1+\sin x)} = \frac{2\sin x}{1-\sin^2 x}$$

$\text{Re}: \sin^2 x + \cos^2 x = 1 \Rightarrow 1 - \sin^2 x = \cos^2 x$

$$= \frac{2\sin x}{\cos^2 x}$$

$$= 2\frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$= \frac{2\tan x}{\cos x}$$

$= \text{Right hand side}$

**Q.E.D.**

## SECTION III

## INTRODUCTORY CALCULUS

**4. (a)** A function given by $y = ax^2 + bx + c$ has a gradient of $9 - \frac{1}{2}x$ at a stationary value of 5.

**(i)** Determine the values of $a$, $b$ and $c$ in the function.

**SOLUTION:**

**Data:** by $y = ax^2 + bx + c$ has a gradient of $9 - \frac{1}{2}x$ at a stationary value of 5.
**Required to determine:** The values of $a$, $b$ and $c$
**Solution:**

Gradient function, $\frac{dy}{dx} = a\left(2x^{2-1}\right) + b(1) + 0$

$= 2ax + b$

Hence, $-\frac{1}{2}x + 9 = 2ax + b$

Equating coefficients of $x$:

$2a = -\frac{1}{2}$

$a = -\frac{1}{4}$

Equating constants:
$b = 9$

$\therefore y = -\frac{1}{4}x^2 + 9x + c$

$\frac{dy}{dx} = 0$ at a stationary value

$\therefore 9 - \frac{1}{2}x = 0$

$x = 18$

When $x = 18$ $\quad 5 = -\frac{1}{4}(18)^2 + 9(18) + c$

$c = -76$

$\therefore a = -\frac{1}{4}$, $b = 9$ and $c = -76$

***Alternative Method***

$$\frac{dy}{dx} = 9 - \frac{1}{2}x$$
$$dy = \left(9 - \frac{1}{2}x\right)dx$$
Integrating both sides w.r.t. $x$
$$y = \int (9 - \frac{1}{2}x)dx$$
$$y = 9x - \frac{1}{4}x^2 + c$$

At a stationary point,
$$\frac{dy}{dx} = 0$$
$$9 - \frac{1}{2}x = 0$$
$$\frac{1}{2}x = 9$$
$$x = 18$$

When $x = 18, y = 5$

$$y = 9x - \frac{1}{4}x^2 + c$$
$$5 = 9(18) - \frac{1}{4}(18)^2 + c$$
$$c = 5 - 162 + 81 = -76$$
$$y = -\frac{1}{4}x^2 + 9x - 76$$
$$\therefore a = -\frac{1}{4},\ b = 9 \text{ and } c = -76$$

**(ii)** Determine the nature of the stationary point.

**SOLUTION:**
**Required to determine:** The nature of the stationary point
**Solution:**

$$\frac{d^2y}{dx^2} = 0 - \frac{1}{2}(1)$$
$$= -\frac{1}{2}$$
$$< 0$$

Hence, the stationary point, (18, 5) is a maximum point.

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**(b)** A drone tracks the movement of an object in motion on the ground. The following movements are recorded.

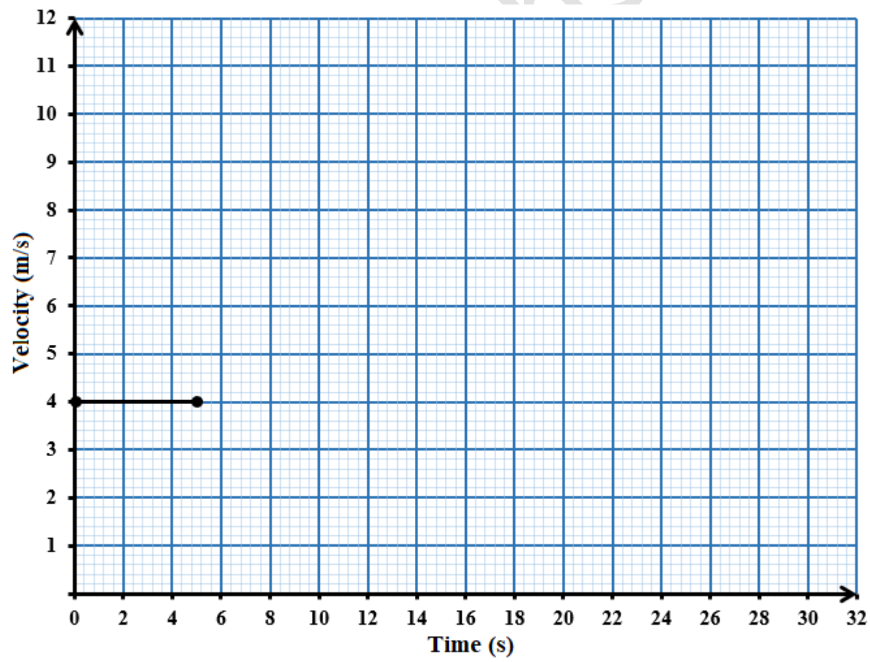
- It moves at a constant velocity of 4 m/s for 5 seconds.
- Its velocity increases uniformly for 3 seconds to 10 m/s.
- It moves at that velocity for 7 seconds.
- It slows uniformly until it comes to rest after 4 seconds.

**(i)** Sketch the graph of the movement of the object.

**SOLUTION:**

**Required to sketch:** A graph to illustrate the movement of the object.
**Solution:**

Branch 1: A constant velocity of 4 m/s for 5 seconds
A horizontal branch indicates constant velocity.




Branch 2: Velocity increases uniformly for 3 seconds to 10 m/s.
Straight line indicates uniform acceleration.

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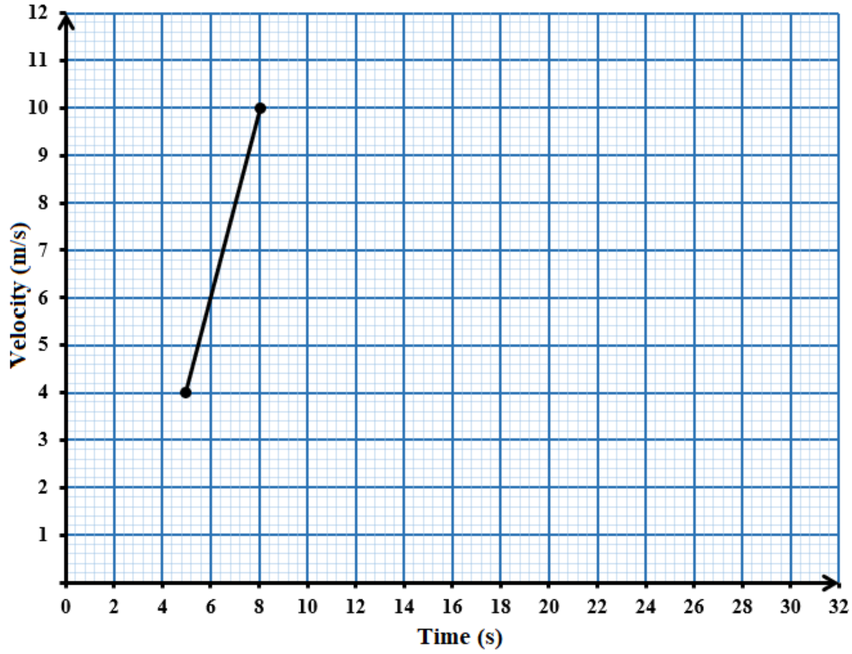
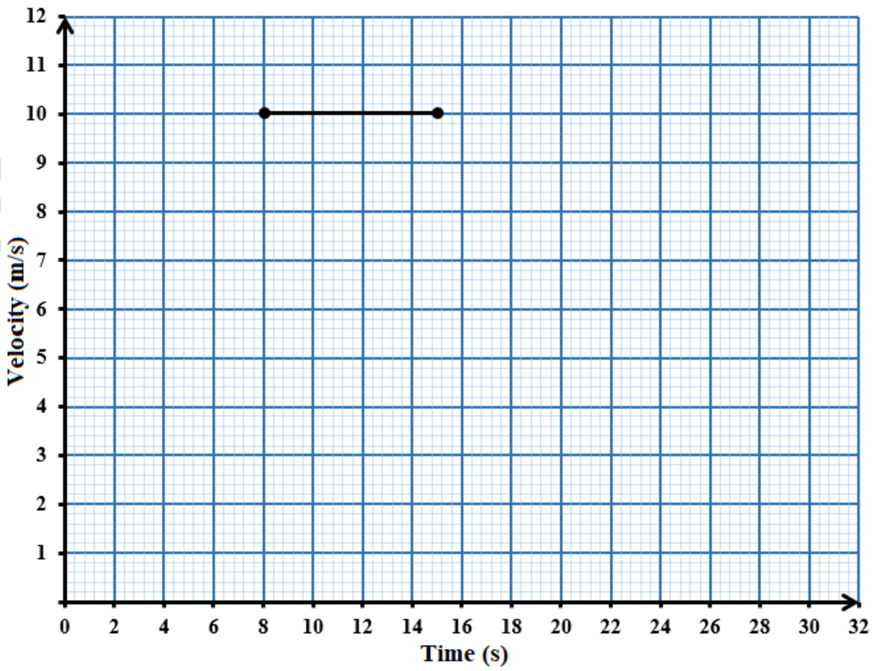
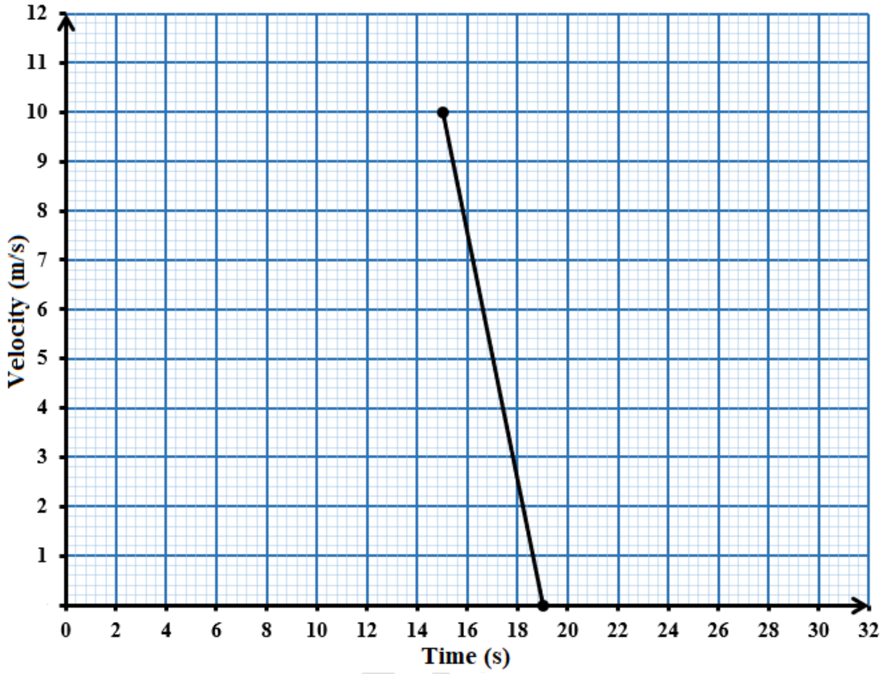


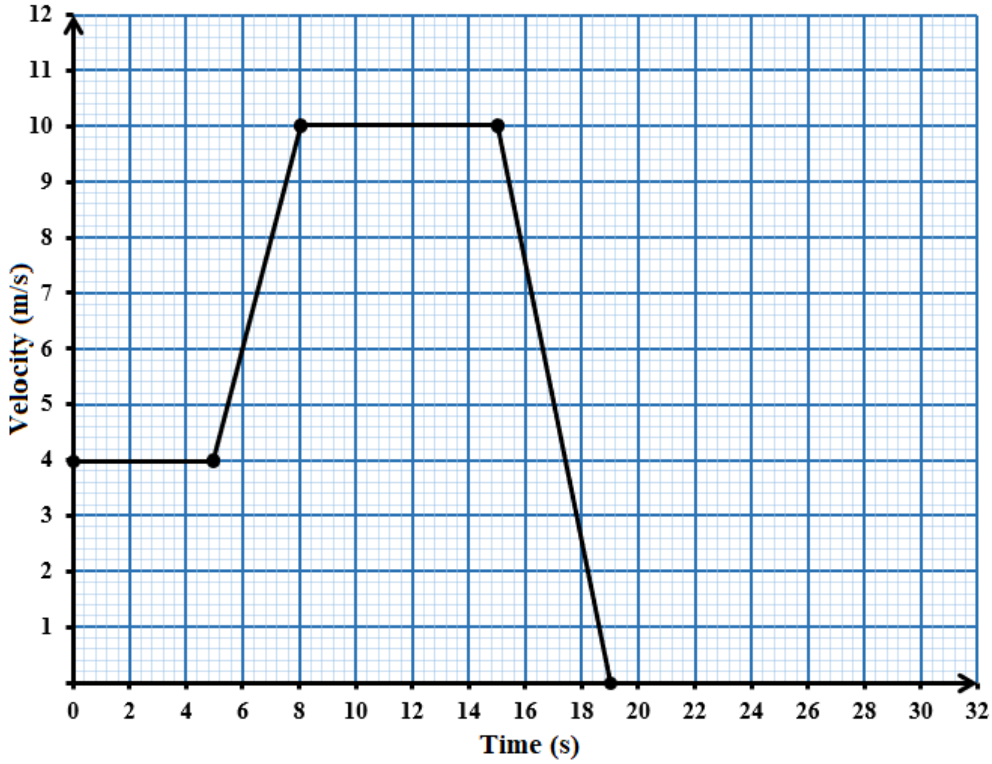

Branch 3: Moves at a constant velocity for 7 seconds.
Horizontal branch indicates constant velocity.



Branch 4: It slows uniformly until it comes to rest after 4 seconds
Straight line indicates uniform deceleration.




The completed graph will now look like:




**(ii)** Calculate the distance travelled by the object in the second part of the journey.

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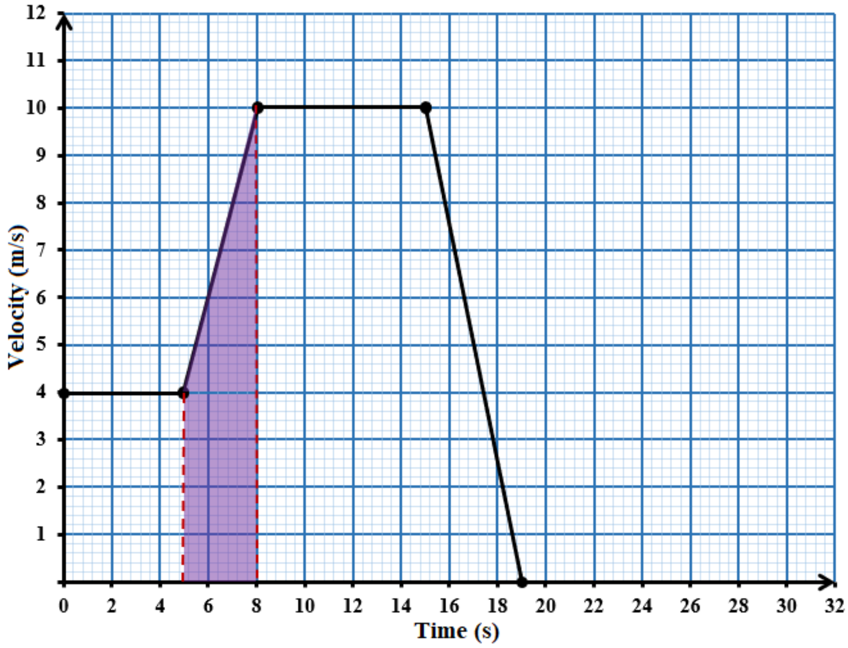
**SOLUTION:**
**Required to calculate:** The distance travelled by the object in the second part of the journey.
**Calculation:**
The shaded region in the diagram below shows the second part of the journey.




Area of the shaded region $=$ distance covered

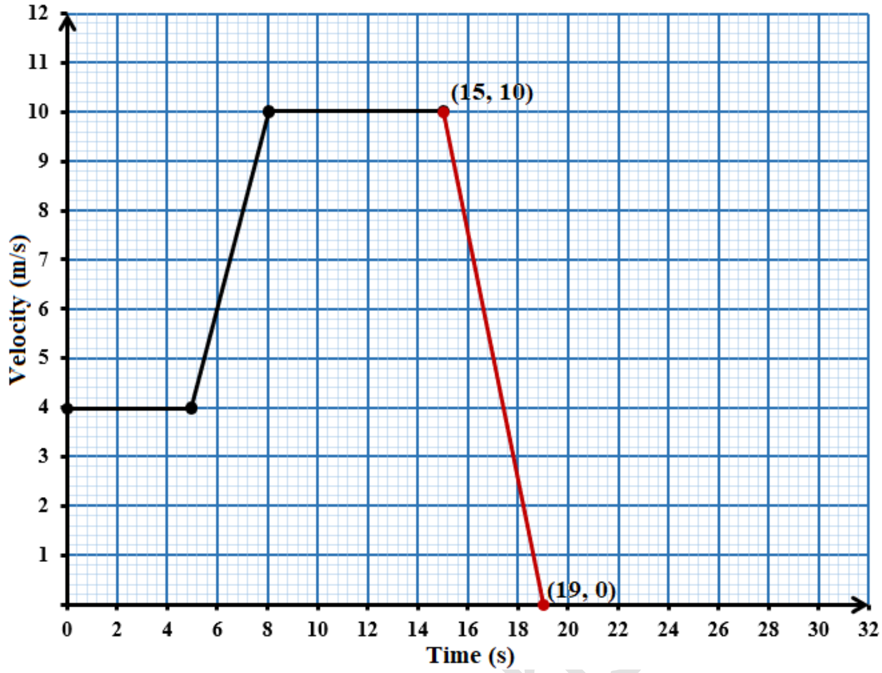
Area of the shaded trapezium $=\frac{1}{2}(4+10)(8-5)$

$$=\frac{1}{2}(14)(3)\text{ m}$$

$$=21\text{ m}$$

**(iii)** Determine the object's acceleration in the final part of the journey.

**SOLUTION:**
**Required to determine:** The object's acceleration in the final part of the journey.
**Solution:**
The branch in red shows the final part of the journey.




We determine the acceleration by finding the gradient line that shows the final part of the journey and which is the branch shown in red.

$$\text{Gradient } = \frac{10-0}{15-19}$$
$$= \frac{10}{-4}$$
$$= -2\frac{1}{2}$$

Hence, the acceleration is $-2\frac{1}{2}$ ms$^{-2}$ or the deceleration is $2\frac{1}{2}$ ms$^{-2}$.

**5.** **(a)** Consider a toy truck moving along a miniaturised race course with an acceleration of $a(t)=9t^2+2t-1$, where $t$ is the time in seconds. Assume that the speed of the toy truck is measured in cm/second.

**(i)** Outline how an equation for the speed of the truck would be found, using the given equation for the acceleration.

**SOLUTION:**
**Data:** The acceleration of a toy truck is given by $a(t)=9t^2+2t-1$, where $t$ is the time in seconds. Assume that the speed of the toy truck is measured in cm/second.

**Required to outline:** How an equation for the speed of the truck would be found.
**Solution:**
If the speed of the truck is expressed as $v(t)$, then $v(t)=\int a(t)\,dt$.

**(ii)** Find the speed of the truck at $t=10$ seconds.

**SOLUTION:**
**Required to find:** The speed of the truck at $t = 10$ seconds.
**Solution:**

$$v(t) = \int (9t^2 + 2t - 1)\ dt$$

$$v = \frac{9t^{2+1}}{2+1} + \frac{2t^{1+1}}{1+1} - t + C, \text{ where } C \text{ is constant of integration}$$

$$v = 3t^3 + t^2 - t + C$$

Assuming the truck starts from rest.
$v = 0$ when $t = 0$:

$$\therefore 0 = 3(0)^3 + (0)^2 - (0) + C$$

$$C = 0$$

$$v = 3t^3 + t^2 - t$$

When $t = 10$:

$$v = 3(10)^3 + (10)^2 - (10)$$

$$= 3000 + 100 - 10$$

$$= 3090 \text{ cms}^{-1}$$

**(iii)** Find the distance covered by the truck between 5 and 10 seconds. Express your answer to the nearest whole number.

**SOLUTION:**
**Required to find:** The distance covered by the truck between 5 and 10 seconds, correct to the nearest whole number.
**Solution:**
Let the distance at time, *t*, be expressed as $s(t)$.

$$s(t) = \int v(t)\ dt$$

$$s(t) = \int (3t^3 + t^2 - t)\ dt$$

Distance covered between $t = 5$ and $t = 10$

$$= \int_5^{10} \left(3t^3 + t^2 - t\right)\, dt$$

$$= \left[\frac{3t^{3+1}}{3+1} + \frac{t^{2+1}}{2+1} - \frac{t^{1+1}}{1+1} + k\right]_5^{10} \text{, where } k \text{ is a constant}$$

$$= \left[\frac{3t^4}{4} + \frac{t^3}{3} - \frac{t^2}{2} + k\right]_5^{10}$$

$$= \left[\frac{3(10)^4}{4} + \frac{(10)^3}{3} - \frac{(10)^2}{2}\right] - \left[\frac{3(5)^4}{4} + \frac{(5)^3}{3} - \frac{(5)^2}{2}\right]$$

$$= \left(7500 + 333\frac{1}{3} - 50\right) - \left(468\frac{3}{4} + 41\frac{2}{3} - 12\frac{1}{2}\right)$$

$$= 7783\frac{1}{3} - 497\frac{11}{12}$$

$$= 7285\frac{5}{12} \text{ cm}$$

$\approx 7\,285$ cm (correct to the nearest whole number)

**(b)** Find

**(i)** $\int (2x+3)^2\, dx$

**SOLUTION:**

**Required to find:** $\int (2x+3)^2\, dx$

**Solution:**

$$\int (2x+3)^2\, dx = \int (2x+3)(2x+3)\, dx$$

$$= \int \left(4x^2 + 12x + 9\right) dx$$

$$= \frac{4x^{2+1}}{2+1} + \frac{12x^{1+1}}{1+1} + 9(x) + C, \text{ where } C \text{ is a constant}$$

$$= \frac{4}{3}x^3 + 6x^2 + 9x + C$$

***Alternative Method***

We can use:

$$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$$

$$\int (2x+3)^2\, dx = \frac{(2x+3)^{2+1}}{2(2+1)} + C$$

$$= \frac{(2x+3)^3}{6} + C$$

**(ii)** $\int 5\cos 2x\ dx$

**SOLUTION:**
**Required to find:** $\int 5\cos 2x\ dx$
**Solution:**
$\int 5\cos 2x\ dx$

Let $t = 2x$
$$\frac{dt}{dx} = 2$$
$$\therefore dx = \frac{dt}{2}$$

Hence, $\int 5\cos 2x\ dx = \int 5\cos t\ \frac{dt}{2}$
$$= \frac{5}{2}\int \cos t\ dt$$
$$= \frac{5}{2}\sin t + C, \text{ where } C \text{ is a constant}$$

Re-substituting $t = 2x$ to get:
$$\int 5\cos 2x\ dx = \frac{5}{2}\sin 2x + C$$

**(c)** If $\frac{dy}{dx} = 6x - 10$ and $y = 12$ when $x = 0$, find the equation for $y$ in terms of $x$.

**SOLUTION:**
**Data:** $\frac{dy}{dx} = 6x - 10$
**Required to find:** $y$ in terms of $x$, when $x = 0$ and $y = 12$
**Solution:**
$$y = \int \frac{dy}{dx}\ dx$$
$$y = \int (6x - 10)\ dx$$
$$y = \frac{6x^{1+1}}{1+1} - 10(x) + C, \text{ where } C \text{ is a constant}$$
$$y = 3x^2 - 10x + C$$

When $x = 0,\ y = 12$:
$$12 = 3(0)^2 - 10(0) + C$$
$$C = 12$$
$$\therefore y = 3x^2 - 10x + 12$$



By Fayad W. Ali and Shereen A. Khan.

## SECTION IV

## PROBABILITY AND STATISTICS

**ALL working must be clearly shown.**

**6.** The following stem and leaf diagram represents the scores, out of 80, of students in an Additional Mathematics exam.

| | | | | | |
|---|---|---|---|---|---|
| 3 | 0 | 3 | 7 | | |
| 4 | 2 | 4 | 6 | 7 | 9 |
| 5 | 1 | 3 | 3 | 6 | |
| 6 | 0 | 7 | 7 | | |
| 7 | 1 | 9 | | | |

**(a)** Write the raw data set that was used to construct the diagram above.

**SOLUTION:**
**Data:** Stem and leaf diagram showing scores, out of 80, of students in an Additional Mathematics exam.
**Required to write:** The raw data set that was used to construct the stem and leaf diagram
**Solution:**
30, 33, 37, 42, 44, 46, 47, 49, 51, 53, 53, 56, 60, 67, 67, 71, 79

**(b)** Determine the following measures for the data set.

**(i)** The median exam score

**SOLUTION:**
**Required to determine:** The median exam score.
**Solution:**
There are 17 scores. The middle score when arranged in either ascending or descending order of magnitude, and which in this case is the 9$^{th}$ score, is the median.

When arranged in order of magnitude, the 9$^{th}$ score is 51.

**(ii)** The mean exam score

**SOLUTION:**
**Required to determine:** The mean exam score
**Solution:**

$$\bar{x} = \frac{\sum x}{n}, \text{ where } \bar{x} = \text{mean}, x = \text{scores}, n = \text{no. of scores}$$

$$= \frac{885}{17}$$




By Fayad W. Ali and Shereen A. Khan.

$= 52.06$

$\approx 52.1$ (correct to 1 decimal place)

**(iii)** The modal score(s)

**SOLUTION:**
**Required to determine:** The modal score
**Solution:**
Two scores occurred twice and the other scores only occurred once. Hence, the modal scores are 53 and 67. The distribution is bi-modal.

**(iv)** Given that for the data set $\sum x^2 = 48999$ and $\sum x = 885$, find the standard deviation of the data set using the formula

$$S = \sqrt{\frac{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}{n-1}}.$$

**SOLUTION:**

**Data:** $\sum x^2 = 48999$, $\sum x = 885$ and $S = \sqrt{\frac{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}{n-1}}$
**Required to find:** The standard deviation for the data set.
**Solution:**

$$S = \sqrt{\frac{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}{n-1}}$$

$$= \sqrt{\frac{48999 - \frac{(885)^2}{17}}{17-1}}$$

$$= \sqrt{\frac{48999 - 46072.05}{16}}$$

$$= \sqrt{\frac{2926.95}{16}}$$

$$= \sqrt{182.93}$$

$= 13.53$ (correct to 2 decimal places)

**(c)** If a student needed to score at least half the total marks possible to pass the exam, determine the probability of a student failing the exam. Give your answer to **2 decimal places**.

**SOLUTION:**
**Data:** A student needs to score at least half the marks to possible to pass the exam.
**Required to determine:** The probability that a student fails the exam.
**Solution:**

Half the maximum score $= \frac{1}{2}(80)$
$= 40$

Number of students who got less than 40 $= 3$

$$\therefore P(\text{student failed}) = \frac{\text{Number of students who scored less than 40}}{\text{Total number of students}}$$
$$= \frac{3}{17} \text{ (exact)}$$
$$= 0.176$$
$$= 0.18 \text{ (correct to decimal places)}$$

**(d)** Given that a randomly selected student has passed the exam, what is the probability that the student scored over 60? Give your answer to **2 decimal places**.

**SOLUTION:**
**Required to find:** The probability that student scored over 60, given that they passed the exam.
**Solution:**
Let $A$ be the event that a randomly selected student passes the exam.
Let $B$ be the event that a student scored over 60.

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$
$$= \frac{\frac{4}{17}}{\frac{14}{17}} \text{ (4 students scored more than 60)}$$
$$= \frac{2}{7}$$
$$= 0.285$$
$$\approx 0.29 \text{ (correct to 2 decimal places)}$$

**(e)** Based on the given stem and leaf diagram, describe the distribution of the data set.

**SOLUTION:**
**Required to describe:** The distribution of the data based on the stem and leaf diagram.
**Solution:**
From the data, the distribution is positively skewed or is a right-skewed distribution.

**(f)** Your friend suggested that a bar graph or histogram could be used to represent the data. Advise your friend on which graph is the better option giving ONE reason to support your answer.

**SOLUTION:**
**Data:** Your friend suggested that a bar graph or histogram could be used to represent the data.
**Required to advise:** Which graph is the better option and give a reason for the answer.
**Solution:**

Test scores are classified as discrete data as they can only take integer values. Bar Graphs are more appropriate for representing discrete data while histograms are used to represent continuous data. Hence, the better option is the bar graph.



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