FAS-PASS Maths

CSEC ADDITIONAL MATHEMATICS JUNE 2022 PAPER 2

SECTION I

ALGEBRA, SEQUENCES AND SERIES

ALL working must be clearly shown.

1. (a) Consider the quadratic equation $qx^2-(4p)x+pq^2=0$, where p and q are both positive integers.

(i) Express the sum AND product of the roots of the equation in terms of p and q.

SOLUTION:

Data: $qx^2-(4p)x+pq^2=0$

Required to express: The sum AND product of the roots of the equation in terms of p and q.

Solution:

If $ax^2+bx+c=0$ has roots α and β, then
$(x-\alpha)(x-\beta)=0$
i.e. $x^2-(\alpha+\beta)x+\alpha\beta=0$
Equating coefficients, we have
The sum of the roots, $\alpha+\beta=-\frac{b}{a}$ and the product of the roots $\alpha\beta=\frac{c}{a}$

In the given equation $qx^2-(4p)x+pq^2=0$, we have $a = q$, $b = -4p$ and $c = pq^2$

$$\text{Sum of the roots } =\frac{-(-4p)}{q}$$
$$=\frac{4p}{q}$$

$$\text{Product of the roots } =\frac{pq^2}{q}$$

$$= pq$$

(ii) Determine the value of q such that the sum of the roots is equal to the product of the roots.

SOLUTION:

Required to determine: The value of q such that the sum of the roots is equal to the product of the roots

Solution:

If the sum of the roots $=$the product of the roots, then

$$\frac{4p}{q} = pq$$

$$(\div p)$$

$$q^2 = 4$$

$$q = \pm 2$$

$$q > 0$$

$$\therefore q = 2 \text{ only}$$

(iii) If the sum of the roots of the equation is 20, use your answer from (a) (ii) to determine a value for p.

SOLUTION:

Data: The sum of the roots is 20.

Required to determine: The value of p

Solution:

$$\frac{4p}{q} = 20 \qquad \text{(Data)}$$

$$q = 2$$

$$\therefore \frac{4p}{2} = 20$$

$$p = 10$$

(iv) Hence, express the given quadratic equation in terms of its numerical coefficients.

SOLUTION:

Required to express: The given quadratic equation in terms of its numerical coefficients.

Solution:

$p = 10$ and $q = 2$

$$qx^2 - 4px + pq^2 = 0$$
$$2x^2 - 4(10)x + 10(2)^2 = 0$$
$$2x^2 - 40x + 40 = 0$$

$\therefore$ The given quadratic equation is $2x^2 - 40x + 40 = 0$.

(b) A series is given by

$$25 - 5 + 1 - \frac{1}{5} + \frac{1}{25} \ldots$$

(i) Show that the series is geometric.

SOLUTION:

Data: A series is given as $25 - 5 + 1 - \frac{1}{5} + \frac{1}{25} \ldots$

Required to show: The series is geometric

Proof:

Term	1^{st} $=25$	2^{nd} $=-5$	3^{rd} $=1$	4^{th} $=-\frac{1}{5}$	5^{th} $=\frac{1}{25}$
		$25 \times -\frac{1}{5}$	$25 \times \left(-\frac{1}{5}\right)^2$	$25 \times \left(-\frac{1}{5}\right)^3$	$25 \times \left(-\frac{1}{5}\right)^4$
Form	a	ar	ar^2	ar^3	ar^4

By generalising, we see the n^{th} term, T_n, can be expressed in the form where $T_n = ar^{n-1}$, where the first term, $a = 25$ and the common ratio, $r = -\frac{1}{5}$.

This is in the general form for the n^{th} term of a geometric progression and hence the series is geometric.

Q.E.D.

(ii) Calculate the sum to infinity of the series, giving the answer to **2 decimal places**.

SOLUTION:

Required to calculate: The sum to infinity of the series.
Calculation:

The sum to infinity, $S_\infty = \frac{a}{1-r},\ |r|<1$

Hence, $S_\infty = \frac{25}{1-\left(-\frac{1}{5}\right)}$

$= \frac{25}{\frac{6}{5}}$

$= \frac{125}{6}$

$= 20.83\underline{\underline{3}}$

≈ 20.83
(expressed to 2 d.p.)

(c) A recent university graduate was offered a starting salary of $720 000 for the first year, with increases of $5 000 at the start of every year thereafter. Determine the number of years (to the nearest whole number) that it would take for her annual salary to be 20% greater than her salary in the first year.

SOLUTION:

Data: A university graduate is given a starting salary of $720 000 for the first year, with increases of $5 000 at the start of every year thereafter.

Required to determine: The number of years (to the nearest whole number) that it would take for her annual salary to be 20% greater than her salary in the first year.

Solution:

Year	1	2	3	…
Salary	$720\,000$	$720\,000+5\,000$	$720\,000+2(5\,000)$	…
Form	a	$a+d$	$a+2d$	

We see the n th term is of the form $T_n = a + (n-1)\,d$

This is in the form of an arithmetic progression with first term, $a = 720\,000$ and common difference, $d = 5\,000$.

20% more than the initial salary $= \frac{120}{100} \times \$720\,000$
$= \$864\,000$

We need to find the n^{th} term of the arithmetic progression where $T_n = 864\,000$

Recall: $T_n = a + (n-1)d$

$$\therefore 864\,000 = 720\,000 + (n-1) \times 5\,000$$
$$144\,000 = (n-1)5\,000$$
$$n - 1 = 28.8$$
$$n = 29.8$$
$$n \approx 30 \text{ (correct to the nearest whole number)}$$

So, the number of years required for her salary to be 20% greater than her initial salary is 30 (correct to the nearest whole number).

2. (a) When the polynomial expression $2x^3 - 3x^2 - cx + d$ is divided by $(x+1)$ and $(x-2)$, the same remainder of 64 is obtained.

Determine the value of *c* and *d*.

SOLUTION:

Data: The polynomial expression $2x^3 - 3x^2 - cx + d$ gives a remainder of 64 when divided by $(x+1)$ and $(x-2)$.

Required to determine: The value of *c* and of *d*

Solution:
If $f(x)$ is any polynomial and $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$. If $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$.

Let $f(x) = 2x^3 - 3x^2 - cx + d$

Hence, $f(-1) = 64$ and $f(2) = 64$ (data)

$$2(-1)^3 - 3(-1)^2 - c(-1) + d = 64$$
$$-2 - 3 + c + d = 64$$
$$c + d = 69 \quad \ldots❶$$

And

$$2(2)^3 - 3(2)^2 - c(2) + d = 64$$
$$16 - 12 - 2c + d = 64$$
$$-2c + d = 60 \quad \ldots❷$$

We solve simultaneously to obtain the value of *c* and of *d*.
Equation ❶ – Equation ❷:

$$\begin{array}{rrcl} & c + d & = & 69 \quad ...(1) \\ - & -2c + d & = & 60 \quad ...(2) \\ \hline & 3c & = & 9 \\ \hline & c & = & 3 \end{array}$$

Substitute $c=3$ into Equation ❶:

$$c+d=69$$
$$3+d=69$$
$$d=66$$

Hence, $c=3$ and $d=66$.

(b) Show that the expression $\dfrac{\sqrt{25}}{\sqrt{45}}$ is the same as $\dfrac{\sqrt{5}}{3}$.

SOLUTION:

Required to show: $\dfrac{\sqrt{25}}{\sqrt{45}}$ is the same as $\dfrac{\sqrt{5}}{3}$

Proof:

$$\begin{aligned} \frac{\sqrt{25}}{\sqrt{45}} &= \frac{\sqrt{25}}{\sqrt{9\times 5}} \\ &= \frac{5}{\sqrt{9}\sqrt{5}} \\ &= \frac{5}{3\sqrt{5}} \\ &= \frac{\sqrt{5}\sqrt{5}}{3\sqrt{5}} \\ &= \frac{\sqrt{5}}{3} \end{aligned}$$

Q.E.D.

(c) **(i)** Given $g(x)=6x^2+12x-18$, express $g(x)$ in the form $a(x+h)^2+k$.

SOLUTION:

Data: $g(x)=6x^2+12x-18$

Required to express: $g(x)$ in the form $a(x+h)^2+k$

Solution:

$6x^2+12x-18=6(x^2+2x)-18$

$\frac{1}{2}$ coefficient of $2x=\frac{1}{2}(2)$
$=+1$

$6x^2+12x-18=6(x+1)^2+?$

$$\begin{aligned}6(x+1)^2&=6(x+1)(x+1)\\&=6(x^2+2x+1)\\&=6x^2+12x+6\end{aligned}$$

Hence, $6x^2+12x+6+\ ?\equiv 6x^2+12x-18$

$$\begin{array}{r}6\\ \underline{-24}\\ \underline{-18}\end{array}\qquad \therefore ?=-24$$

Hence, $g(x)=6(x+1)^2-24$, where $a=6$, $h=1$ and $k=-24$.

Alternative Method:

$$\begin{aligned}a(x+h)^2+k&=a(x^2+2h+h^2)+k\\&=ax^2+2ahx+ah^2+k\end{aligned}$$

$\Rightarrow 6x^2+12x-18=ax^2+2ahx+ah^2+k$

Equating coefficient of x^2
$a=6$

Equating coefficient of x

$2(6)h=12$
$h=1$

Equating the constant
$6(1)^2+k=-18$
$k=-24$

Hence, $g(x)=6(x+1)^2-24$, where $a=6$, $h=1$ and $k=-24$.

(ii) Using the expression derived in (c) (i), determine the roots of $g(x)$.

SOLUTION:

Required to determine: The roots of $g(x)$, using the expression derived in (c) (i).

Solution:

If $g(x)=0$ (an assumption, since only equations have roots)

Then $6x^2+12x-18=0$

$$6(x+1)^2-24=0$$

$$(x+1)^2=\frac{24}{6}$$

$$(x+1)^2=4$$

$$x+1=\pm 2$$

$x=+2-1$ or $x=-2-1$

$x=1$ or $x=-3$

$\therefore$ The roots of $g(x)$ are 1 or – 3.

NOTE: If the method was not specified, we could also have found the roots by (i) Drawing the graph of the function OR (ii) factorising OR (iii) using the quadratic equation formula)

(iii) Hence, sketch the graph of $g(x)$ on the following grid.

SOLUTION:

Required to sketch: The graph of $g(x)$

Solution:

$g(x) = 6(x+1)^2 - 24$

$g(x)_{\text{min}} = -24$ at $x = -1$

$\therefore$ Minimum point on $g(x) = (-1, -24)$.

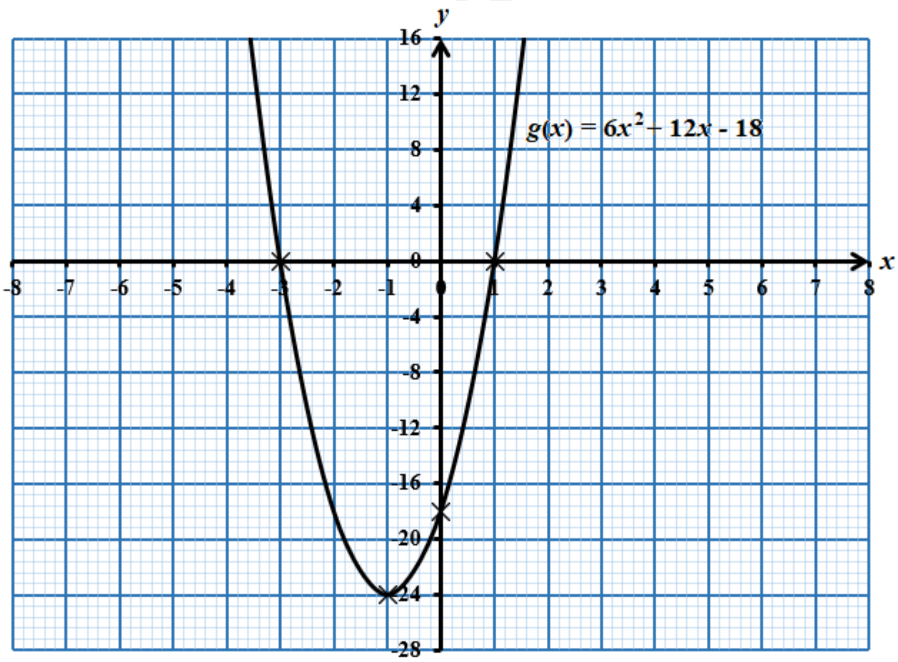
$g(0) = -18$

$\therefore g(x)$ cuts the vertical axis at $(0, -18)$.

Roots of $g(x) = -3$ and 1

$\therefore g(x)$ cuts the horizontal axis at $(-3, 0)$ and $(1, 0)$.

A sketch of $g(x)$ is shown on the grid below.



SECTION II

COORDINATE GEOMETRY, VECTORS AND TRIGONOMETRY

ALL working must be clearly shown.

3. (a) The equation of a circle is $x^2+y^2+4x-8y+10=0$.

(i) Determine the coordinates of its centre AND the length of the radius of the circle.

SOLUTION:

Data: The equation of a circle is $x^2+y^2+4x-8y+10=0$.

Required to determine: The coordinates of its centre AND the length of the radius of the circle.

Solution:

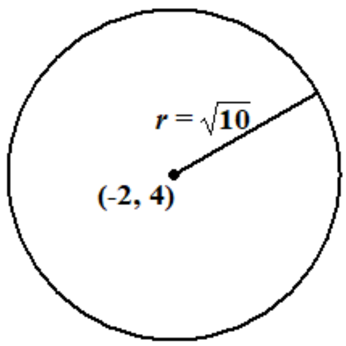
$$x^2+y^2+4x-8y+10=0$$
$$x^2+y^2+2(2)y+2(-4)y+10=0$$

This is of the form $x^2+y^2+2gx+2fy+c=0$, where $g=2,\ f=-4$ and $c=10$

Centre of the circle is $(-g,\ -f)=(-(2),\ -(-4))$
$=(-2,\ 4)$

Radius $=\sqrt{g^2+f^2-c}$
$=\sqrt{(2)^2+(-4)^2-10}$
$=\sqrt{4+16-10}$
$=\sqrt{10}$ units

A sketch of the circle is shown below.



Alternative Method:

$$x^2+y^2+4x-8y+10=0$$

$$\underbrace{x^2+4x+4}+\underbrace{y^2-8y+16}=-10+4+16$$

$$(x+2)^2+(y-4)^2=10$$

This is of the form $(x+a)^2+(y+b)=r^2$ which is the equation of a circle, centre (a,b) and radius r

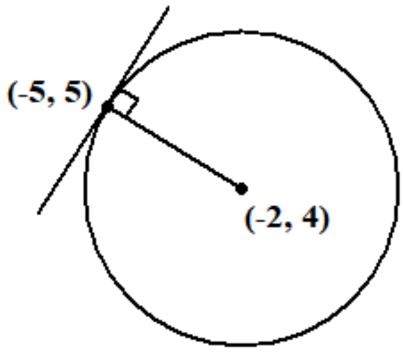
Hence, centre $=(-2,\ 4)$ and radius $=\sqrt{10}$ units

(ii) Determine the equation of the tangent to the circle at the point $P(-5,\ 5)$.

SOLUTION:

Required to determine: The equation of the tangent to the circle at the point $P(-5,\ 5)$.

Solution:



The angle made by a tangent to a circle and a radius at the point of contact is a right angle.

The gradient of the radius $=\dfrac{5-4}{-5-(-2)}$

$$=-\frac{1}{3}$$

$\therefore$ Gradient of the tangent at (-5, 5) $=-\dfrac{1}{-\frac{1}{3}}$

$$=3$$

(Product of the gradients of perpendicular lines is equal to -1)

OR ***Alternatively***,

Gradient of the tangent $= \frac{-(x+g)}{y+f}$

((x, y) is the point where the tangent is drawn and (-g, -f) is the centre of the circle)

$$= \frac{-(-5+2)}{5+(-4)}$$

$$= 3$$

$\therefore$ The equation of the tangent at $(-5, 5)$ is

$$\frac{y-5}{x-(-5)} = 3$$

$$y-5 = 3(x+5)$$

$$y-5 = 3x+15$$

$$y = 3x+20$$

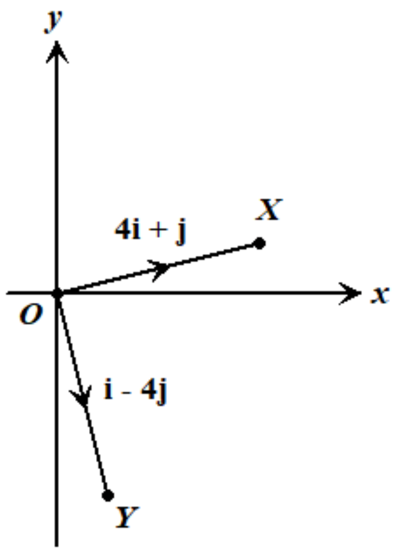
(b) The vectors $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are such that $\overrightarrow{OX} = 4\mathbf{i}+\mathbf{j}$ and $\overrightarrow{OY} = \mathbf{i}-4\mathbf{j}$. Show that the vectors $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are perpendicular.

SOLUTION:

Data: $\overrightarrow{OX} = 4\mathbf{i}+\mathbf{j}$ and $\overrightarrow{OY} = \mathbf{i}-4\mathbf{j}$

Required to show: $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are perpendicular to each other.

Proof:



Recall: If a and b are vectors and θ is the angle between a and b, then the dot product, $a.b$ is defined as:

$$a.b = |a||b|\cos\theta$$

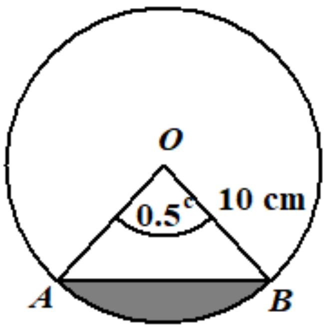
If $\theta = 90°$, then $\cos 90^0 = 0$ then the dot product, $a.b = 0$
Let $a = \overrightarrow{OX}$ and $b = \overrightarrow{OY}$
$\overrightarrow{OX} = 4\mathbf{i} + \mathbf{j}$ and $\overrightarrow{OY} = \mathbf{i} - 4\mathbf{j}$

$$\begin{aligned} \overrightarrow{OX}.\overrightarrow{OY} &= (4 \times 1) + (1 \times -4) \\ &= 4 + (-4) \\ &= 0 \end{aligned}$$

Hence, $\overrightarrow{OX}$ is perpendicular to $\overrightarrow{OY}$.

Q.E.D.

(c) The diagram below, **not drawn to scale**, shows a chord AB which subtends an angle of 0.5^c (0.5 radians) at the centre, O, of a circle of radius 10 cm. Given that the area of triangle $AOB = \frac{1}{2}r^2 \sin\theta$, calculate the area of the shaded region.



SOLUTION:

Data: Diagram showing a chord AB which subtends an angle of 0.5^c (0.5 radians) at the centre, O, of a circle of radius 10 cm. The area of triangle $AOB = \frac{1}{2}r^2 \sin\theta$.

Required to calculate: The area of the shaded region.

Calculation:

Area of sector $AOB = \frac{1}{2}r^2\theta$

$$\begin{aligned} &= \frac{1}{2}(10)^2 \times 0.5 \text{ cm}^2 \\ &= 50 \times 0.5 \text{ cm}^2 \\ &= 25 \text{ cm}^2 \end{aligned}$$

Area of triangle $AOB = \frac{1}{2}r^2 \sin\theta$

$= \frac{1}{2}(10)(10)\sin(0.5)$

$= 23.9712$

Area of the shaded region $= 25 - 23.9712$ cm^2

$= 1.0288$ (correct to 4 d.p)

(d) **(i)** Show that $\cos 2\theta = 2\cos^2\theta - 1$.

SOLUTION:

Required to show: $\cos 2\theta = 2\cos^2\theta - 1$

Proof:

$\cos 2\theta = \cos(\theta + \theta)$

Recall the compound angle formula:

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

Hence, $\cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$

$= \cos^2\theta - \sin^2\theta$

Recall: $\cos^2\theta + \sin^2\theta = 1 \Rightarrow -\sin^2\theta = \cos^2\theta - 1$

$\therefore \cos 2\theta = \cos^2 + (\cos^2\theta - 1)$

$= 2\cos^2\theta - 1$

Q.E.D.

(ii) Hence, solve the equation $\cos 2\theta + \cos\theta + 1 = 0$ for $0 < \theta < 2\pi$.

SOLUTION:

Required to solve: $\cos 2\theta + \cos\theta + 1 = 0$ for $0 < \theta < 2\pi$

Solution:

$\cos 2\theta + \cos\theta + 1 = 0$

$\therefore 2\cos^2\theta - 1 + \cos\theta + 1 = 0$

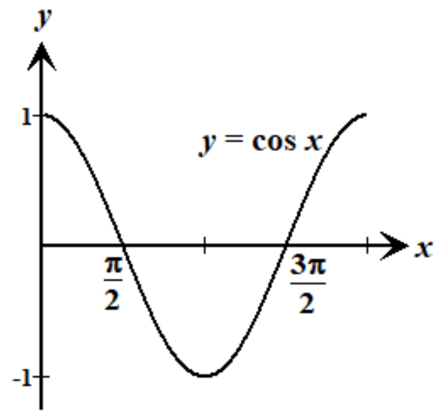
$2\cos^2\theta + \cos\theta = 0$

$\cos\theta(2\cos\theta + 1) = 0$

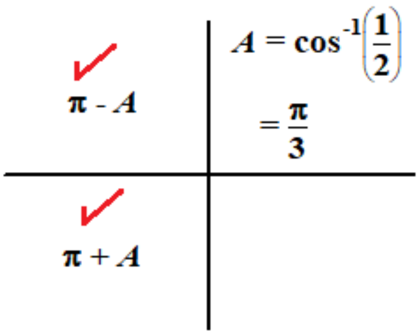
$\cos\theta = 0$ or $-\frac{1}{2}$

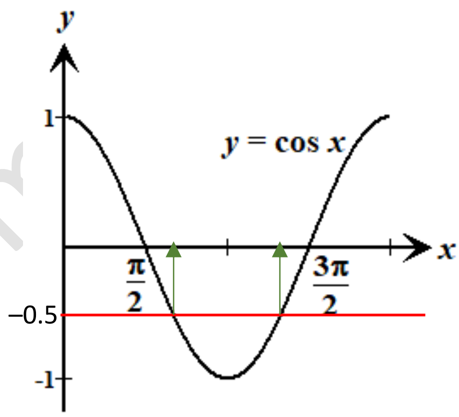
When $\cos\theta = 0$

$\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ for $0 < \theta < 2\pi$



When $\cos\theta = -\frac{1}{2}$, θ lies in the second and third quadrants and the basic acute angle is $cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.





$$\therefore \theta = \pi - \frac{\pi}{3},\ \pi + \frac{\pi}{3}$$

$$= \frac{2\pi}{3},\ \frac{4\pi}{3} \text{ for } 0 < \theta < 2\pi$$

Hence, $\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$ for the given range.

SECTION III

INTRODUCTORY CALCULUS

ALL working must be clearly shown.

4. (a) Given that $f(x)=x(5-x)^2$, determine $f''(x)$.

SOLUTION:

Data: $f(x)=x(5-x)^2$

Required to determine: $f''(x)$

Solution:

$$f(x)=x(5-x)^2$$
$$=x(25-10x+x^2)$$
$$=25x-10x^2+x^3$$
$$\therefore f'(x)=25(1)-10(2x^{2-1})+3x^{3-1}$$
$$=25-20x+3x^2$$
$$f''(x)=0-20(1)+3(2x^{2-1})$$
$$=-20+6x$$
$$=6x-20$$

(b) Differentiate EACH of the following expressions with respect to x, simplifying your answer where possible.

(i) $2\sin 3x+\cos x$

SOLUTION:

Required to differentiate: $2\sin 3x+\cos x$

Solution:

If $y=\sin 3x$

Let $t=3x$

$\therefore\ y=\sin t$

Using the chain rule

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$
$$=\cos t\times 3$$
$$=3\cos 3x$$

$$\therefore \frac{d}{dx}(2\sin 3x + \cos x) = 2(3\cos 3x) + (-\sin x)$$
$$= 6\cos 3x - \sin x$$

(ii) $(1+2x)^3(x+2)$

SOLUTION:

Required to differentiate: $(1+2x)^3(x+2)$

Solution:
Let $y = (1+2x)^3(x+2)$
y is of the form uv, where
$u = (1+2x)^3$

Le t $t = 1+2x$
$$u = t^3$$
$$\frac{du}{dx} = \frac{du}{dt} \times \frac{dt}{dx} \text{ (chain rule)}$$
$$= 3t^2 \times 2$$
$$= 6t^2$$
$$\Rightarrow \frac{du}{dx} = 6(1+2x)^2$$

$v = x+2$

$$\Rightarrow \frac{dv}{dx} = 1$$

Using the product rule:
$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= (x+2)\left(6(1+2x)^2\right) + (1+2x)^3(1)$$
$$= 6(x+2)(1+2x)^2 + (1+2x)^3$$
$$= (1+2x)^2\{6(x+2) + (1+2x)\}$$
$$= (1+2x)^2\{6x+12+1+2x\}$$
$$= (1+2x)^2(8x+13)$$

(c) **(i)** Determine the stationary points on the curve $y = x^3 - 4x^2 + 4x$.

SOLUTION:

Data: $y = x^3 - 4x^2 + 4x$

Required to determine: The stationary points on the curve.

Solution:

At a stationary point, $\dfrac{dy}{dx}=0$

$$\frac{dy}{dx}=3x^{3-1}-4\left(2x^{2-1}\right)+4(1)$$
$$=3x^2-8x+4$$

Let $\dfrac{dy}{dx}=0$

$$3x^2-8x+4=0$$
$$(3x-2)(x-2)=0$$
$$\therefore x=\frac{2}{3}\text{ and }2$$

When $x=\dfrac{2}{3}$

$$y=\left(\frac{2}{3}\right)^3-4\left(\frac{2}{3}\right)^2+4\left(\frac{2}{3}\right)$$
$$=\frac{8}{27}-\frac{16}{9}+\frac{8}{3}$$
$$=\frac{32}{27}$$
$$=1\frac{5}{27}$$

When $x=2$

$$y=(2)^3-4(2)^2+4(2)$$
$$=8-16+8$$
$$=0$$

Hence, the stationary points are $\left(\dfrac{2}{3},1\dfrac{5}{27}\right)$ and $(2, 0)$.

(ii) Providing details, determine the nature of EACH stationary point in (c) (i).

SOLUTION:

Required to determine: The nature of the stationary points of the curve y.

Solution:

$$\frac{d^2y}{dx^2}=3\left(2x^{2-1}\right)-8(1)+0$$
$$=6x-8$$

When $x = \frac{2}{3}$ $\qquad$ $\frac{d^2y}{dx^2} = 6\left(\frac{2}{3}\right) - 8$

$$= -4$$
$$= -\text{ve}$$

$\therefore \left(\frac{2}{3}, 1\frac{5}{27}\right)$ is a maximum point.

When $x = 2$ $\qquad$ $\frac{d^2y}{dx^2} = 6(2) - 8$

$$= 4$$
$$= +\text{ve}$$

$\therefore (2, 0)$ is a minimum point.

5. (a) Determine $\int (4\cos\theta - 6\sin\theta)\, d\theta$.

SOLUTION:

Required to determine: $\int (4\cos\theta - 6\sin\theta)\, d\theta$

Solution:

$\int (4\cos\theta - 6\sin\theta)\, d\theta = 4(\sin\theta) - 6(-\cos\theta) + C,$
(where C is the constant of integration)

$$= 4\sin\theta + 6\cos\theta + C$$

(b) Evaluate $\int_1^2 (3-x)^2\, dx$.

SOLUTION:

Required to evaluate: $\int_1^2 (3-x)^2\, dx$

Solution:

$$\int_1^2 (3-x)^2\ dx = \int_1^2 \left(9-6x+x^2\right)\ dx$$

$$= \left[9x - \frac{6x^2}{2} + \frac{x^3}{3} + C\right]_1^2, \text{ where } C \text{ is the constant of integration}$$

$$= \left[9x - 3x^2 + \frac{x^3}{3} + C\right]_1^2$$

$$= \left\{9(2) - 3(2)^2 + \frac{(2)^3}{3}\right\} - \left\{9(1) - 3(1)^2 + \frac{(1)^3}{3}\right\}$$

(Note that, C, the constant of integration cancels off in a definite integral and may not even be mentioned in the evaluation. However, it is good practice to present it after an integration.)

$$= \left(18 - 12 + 2\frac{2}{3}\right) - \left(9 - 3 + \frac{1}{3}\right)$$

$$= 8\frac{2}{3} - 6\frac{1}{3}$$

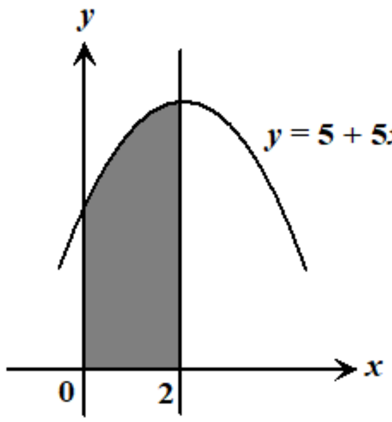
$$= 2\frac{1}{3} \text{ units}$$

(c) Determine the area of the region bounded by the curve $y = 5 + 5x - x^2$, the x – axis, the y – axis and the line $x = 2$.

SOLUTION:

Required to determine: The area of the region bounded by the curve $y = 5 + 5x - x^2$, the x – axis, the y – axis and the line $x = 2$.

Solution:



The shaded region is a sketch of the area required.

Area $= \int_0^2 \left(5 + 5x - x^2\right)\ dx$

(Note that, C, the constant of integration cancels off in a definite integral and may not even be mentioned in the evaluation. However, it is good practice to present it after an integration.)

$$
\begin{aligned}
&= \left[5x + \frac{5x^2}{2} - \frac{x^3}{3} + C\right]_0^2 \\
&= \left(5(2) + \frac{5(2)^2}{2} - \frac{(2)^3}{3}\right) - \left(5(0) + \frac{5(0)^2}{2} - \frac{(0)^3}{3}\right) \\
&= 10 + 10 - 2\frac{2}{3} \\
&= 17\frac{1}{3} \text{ square units}
\end{aligned}
$$

(d) A particle moves in a straight line so that t seconds after passing through a fixed point, O, its acceleration, a, is given by $a = (3t-1)$ ms^{-2}. When $t = 2$, the particle has a velocity, v, of 4 ms^{-1}, and a displacement of 6 m from O. Determine the velocity when $t = 4$.

SOLUTION:

Data: A particle moves in a straight line so that t seconds after passing through a fixed point, O, its acceleration, a, is given by $a = (3t-1)$ ms^{-2}. When $t = 2$, the particle has a velocity, v, of 4 ms^{-1}, and a displacement of 6 m from O.

Required to determine: The velocity when $t = 4$

Solution:

$$
\begin{aligned}
a &= 3t - 1 \\
v &= \int a \, dt \\
v &= \int (3t-1) \, dt \\
v &= \frac{3t^2}{2} - t + C, \text{ where } C \text{ is the constant of integration}
\end{aligned}
$$

When $t = 2, v = 4$

Hence, $4 = \dfrac{3(2)^2}{2} - 2 + C$

$$
\begin{aligned}
4 &= 6 - 2 + C \\
C &= 0 \\
\therefore v &= \frac{3t^2}{2} - t
\end{aligned}
$$

When $t = 4$ $$v = \frac{3(4)^2}{2} - 4$$
$$= 20 \text{ ms}^{-1}$$

NOTE: The displacement of the particle being 6 m from O when $t = 2$ is not used in the question and therefore misleading data.

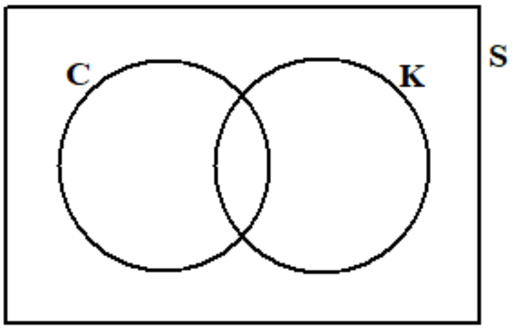
SECTION IV

PROBABILITY AND STATISTICS

ALL working must be clearly shown.

6. (a) At a school canteen, 80% of the students (S) purchase chips (C) and 55% purchase chicken (K). Of the students who purchase chicken, 11% do not purchase chips.

(i) Complete the following Venn diagram to illustrate this information.



SOLUTION:

Data: At a school canteen, 80% of the students (S) purchase chips (C) and 55% purchase chicken (K). Of the students who purchase chicken, 11% do not purchase chips.

Required to complete: The Venn diagram given to illustrate the information given.

Solution:

Percentage of students who purchase chicken = 55%

Since 11% of those who purchase chicken do not purchase chips, the percentage of students who purchases both chips and chicken
$= 55\% - 11\%$
$= 44\%$

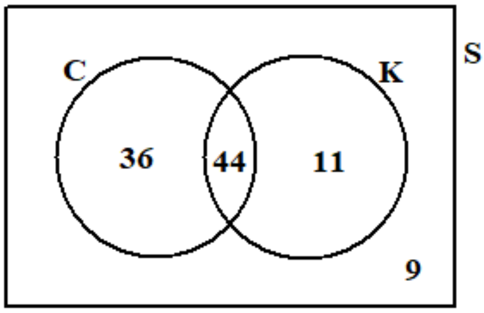
The percentage of students who purchase chicken only $= 11\%$

Percentage of students who purchase chips = 80%

Percentage of students who purchase chips only = 80% –44%
= 36%

Percentage of students who do not purchase either chicken or chips
$= 100 - (36 + 44 + 11)$
$= 9$

The completed Venn diagram looks like:



(ii) Determine the probability that a student chosen at random purchases ONLY chicken or ONLY chips.

SOLUTION:

Required to determine: The probability that a student chosen at random purchases ONLY chicken or ONLY chips.

Solution:
Percentage of students who purchase only chicken or only chips
$= 36 + 11$
$= 47\%$

P (Student purchases only chicken or only chips)

$$= \frac{\text{Percentage of students who purchase only chicken or only chips}}{\text{Percentage of students in the whole class}}$$

$$= \frac{47}{100}$$

$= 47\%$

(b) The probabilities of the occurrence of two events, A and B, are given by $P(A)=\frac{1}{4}, P(B)=\frac{3}{5}$ and $P(A\cup B)=\frac{7}{10}$. Determine

(i) $P(A\cap B)$

Data: $P(A)=\frac{1}{4}, P(B)=\frac{3}{5}$ and $P(A\cup B)=\frac{7}{10}$, where A and B are two events.

Required to determine: $P(A\cap B)$

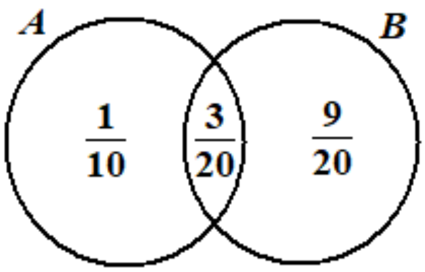
Solution: We recall De Morgan's Law:

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$\therefore \frac{7}{10}=\frac{1}{4}+\frac{3}{5}-P(A\cap B)$$
$$\frac{7}{10}=\frac{17}{20}-P(A\cap B)$$
$$P(A\cap B)=\frac{3}{20}$$

$$P(A)=\frac{1}{4}$$
$$P(A\text{ only})=\frac{1}{4}-\frac{3}{20}$$
$$=\frac{1}{10}$$

$$P(B)=\frac{3}{5}$$
$$P(B\text{ only})=\frac{3}{5}-\frac{3}{20}$$
$$=\frac{9}{20}$$

This is illustrated in the Venn diagram



(ii) $P(A|B)$

SOLUTION:

Required to determine: $P(A|B)$

Solution:

This is a conditional probability and requires the probability of A occurring, given that B has occurred.

By definition:

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$
$$=\frac{\frac{3}{20}}{\frac{3}{5}}$$
$$=\frac{3}{20}\times\frac{5}{3}$$
$$=\frac{1}{4}$$

(c) State, with a reason, whether Events A and B are independent.

SOLUTION:

Required to state: Whether events A and B are independent.

Solution:

If A and B are independent events, then $P(A\cap B)=P(A)\times P(B)$, This rule is sometimes referred to as the product law of probability.

$$P(A\cap B)=\frac{3}{20}$$

$$P(A)\times P(B)=\frac{1}{4}\times\frac{3}{5}$$
$$=\frac{3}{20}$$

So, $P(A\cap B)=P(A)\times P(B)$

Hence, A and B are independent events since they satisfy the law.

(d) The following table shows the marks obtained by 27 students in a Mathematics test.

30	35	39	42	45	45	52	59	61
61	65	69	70	71	75	77	79	81
83	83	85	87	89	90	90	95	98

(i) Construct a stem-and-leaf diagram to display this data.

SOLUTION:

Data: Table showing the marks obtained by 27 students in a Mathematics test.

Required to construct: A stem-and-leaf diagram for the data given.

Solution:

Stem	Leaf					
3	0	5	9			
4	2	5	5			
5	2	9				
6	1	1	5	9		
7	0	1	5	7	9	
8	1	3	3	5	7	9
9	0	0	5	8		

KEY: Stem 10's Leaf 1's
E.g. $3|9 = 39$

(ii) State ONE advantage of using a stem-and-leaf diagram to display data.

SOLUTION:

Required to state: One advantage of using a stem-and-leaf diagram to display data.

Solution:

The data is arranged compactly and the stem is not repeated for multiple data values. The value of each data point can be easily recovered or read off from the plot. Also, the shape of the distribution can be easily seen.

(iii) State the range of values of the marks obtained by the students.

SOLUTION:

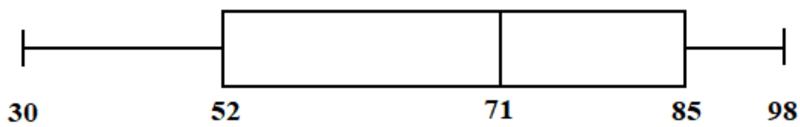
Required to state: The range of values of the marks obtained by the students.

Solution:

Highest mark $= 98$
Lowest mark $= 30$

$\therefore$ Range $= 98 - 30$
$= 68$

(iv) The data are displayed in the following box-and-whisker plot.



State TWO distinct observations about the data as seen in the box-and-whisker plot.

SOLUTION:

Data: Box-and-whisker plot displaying the data given.

Required to state: Two distinct observations about the data as seen in the box-and-whisker plot.

Solution:

- The median is located closer to the upper quartile than it is to the lower quartile. This suggests that the data is slightly skewed to the left or slightly negatively skewed.

- Fifty percent of the students scored between 30 and 71, and the remaining fifty percent scored between 71 and 98. This indicates that there is more variability among the lower scores than among the higher scores.