FAS-PASS Maths

CSEC ADD MATHS 2021

SECTION I

Answer all questions.

ALL working must be clearly shown.

1. (a) (i) Determine the remainder when $f(x)=ax^3+7x^2-7x-3$ is divided by $x-1$.

SOLUTION:
Data: $f(x)=ax^3+7x^2-7x-3$
Required to determine: The remainder when $f(x)$ is divided by $x-1$.
Solution:
If $f(x)$ is any polynomial and $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$.

Hence, the remainder when $f(x)=ax^3+7x^2-7x-3$ is divided by $x-1$ is
$$\begin{aligned} f(1) &= a(1)^3+7(1)^2-7(1)-3 \\ &= a+7-7-3 \\ &= a-3 \end{aligned}$$

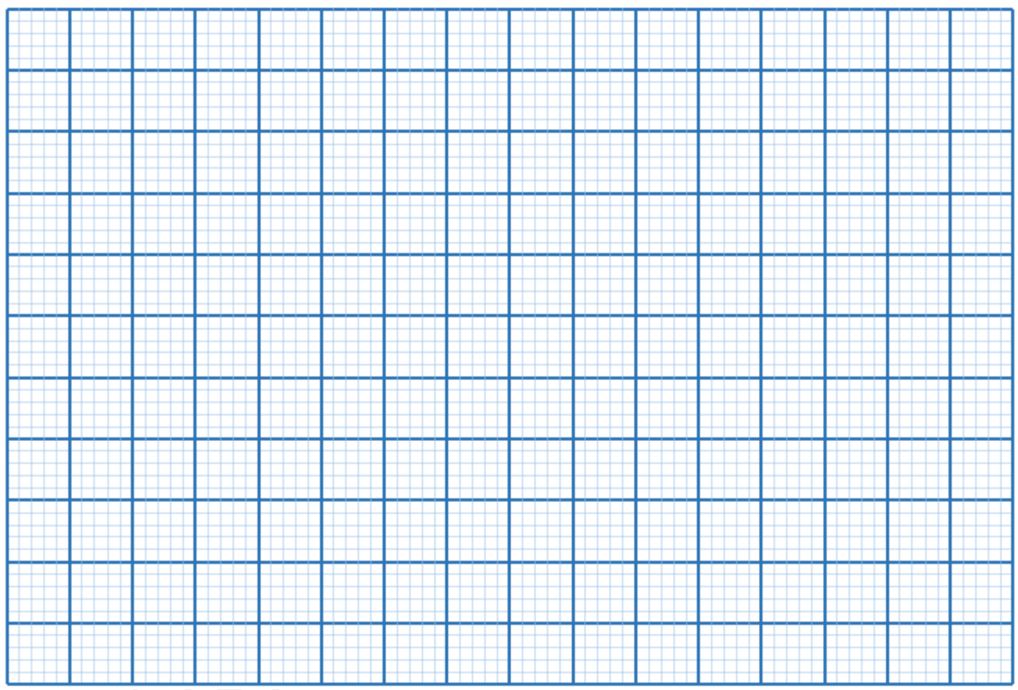
(ii) If the remainder when $f(x)$ is divided by $(x+3)$ is equal to the remainder determined in (a) (i), find the value of *a*.

SOLUTION:
Data: The remainder when $f(x)$ is divided by $(x+3)$ is equal to the remainder determined in (a) (i) above.
Required to find: The value of *a*.
Solution:
When $f(x)$ is divided by $x+3$, the remainder is $f(-3)$.
$$\begin{aligned} f(-3) &= a(-3)^3+7(-3)^2-7(-3)-3 \\ &= -27a+63+21-3 \\ &= -27a+81 \end{aligned}$$

$f(-3)=f(1)$ (data)
$\therefore -27a+81=a-3$
$81+3=a+27a$

$$28a = 84$$
$$a = \frac{84}{28}$$
$$a = 3$$

(b) Consider the function $g(x) = x^2 + (m+4)x + 4m = 0$ which has real and equal roots. Use the discriminant of the given equation to determine the values of *m*. You may use the grid provided to assist you.



SOLUTION:
Data: $g(x) = x^2 + (m+4)x + 4m = 0$ has real and equal roots.
Required to determine: The values of *m* using the discriminant of $g(x)$.
Solution:
$x^2 + (m+4)x + 4m = 0$ has real and equal roots.
Recall: if $ax^2 + bx + c = 0$, then real and equal roots will occur when $b^2 = 4ac$
In the given equation, $a = 1$, $b = m + 4$ and $c = 4m$.
Hence, real and equal roots in the given equation will occur when

$$(m+4)^2 = 4(1)(4m)$$
$$m^2 + 8m + 16 = 16m$$
$$m^2 - 8m + 16 = 0$$
$$(m-4)(m-4) = 0$$
$$m = 4$$

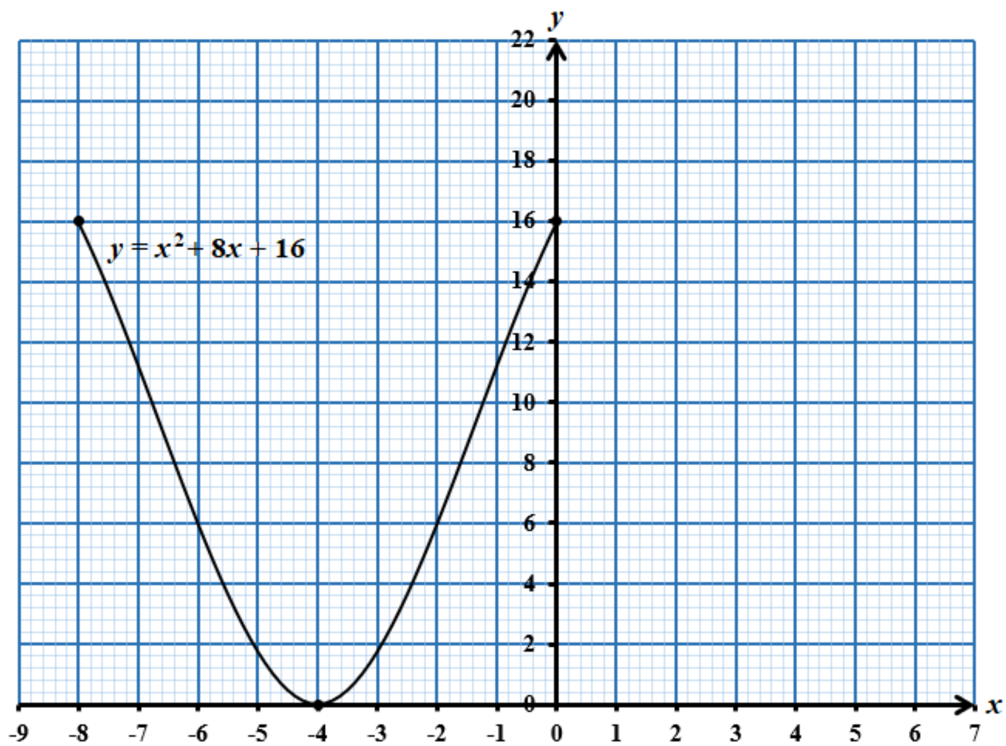
The question asked for the values of *m*. This signals to the candidate that *m* has more than one value. Since *m* has one value, this instruction is misleading. 'Solve for *m*', would have been more appropriate.

The grid is useless to solve for *m* or even to assist in the solving of *m* as was suggested. This is confusing and misleading.
However, we have decided to use the grid to sketch the graph of the function using the value of *m* obtained. Please note that this was NOT asked in the question.

The function cuts the *y*-axis at (0, 16) and has a minimum point at (0, -4). The *x*-axis is a tangent to the curve at the minimum point. The curve is symmetrical about the line with equation, $x = -4$, the axis of symmetry.

$$g(x) = x^2 + 8x + 16 = 0$$
$$= (x+4)^2$$

A sketch of $g(x)$ will look like:



(c) Let $h(x) = 2x^2 + 8x - 10$.

(i) Express $h(x)$ in the form $a(x+b)^2 + c$.

SOLUTION:

Data: $h(x)=2x^2+8x-10$

Required to express: $h(x)$ in the form $a(x+b)^2+c$.

Solution:

$2x^2+8x-10$ is first expressed as

$2(x^2+4x)-10$

$\frac{1}{2}$ coefficient of x is $\frac{1}{2}(+4)=+2$

So, $2x^2+8x-10$ can be written as

$2(x+2)^2+?$

$\downarrow$

$$
\begin{aligned}
2(x+2)(x+2) &= 2(x^2+4x+4) \\
&= 2x^2+8x+8 \\
&\qquad\quad \underline{-18} = ? \\
&\qquad\quad \underline{-10}
\end{aligned}
$$

Hence, $2x^2+8x-10=2(x+2)^2-18$

Alternative Method 2:

$a(x+b)^2+c=a(x^2+2bx+b^2)+c$

$a(x+b)^2+c=ax^2+2abx+ab^2+c$

$2x^2+8x-10=ax^2+2abx+ab^2+c$

Equating coefficients of x^2:

$a=2$

Equating coefficients of x:

$$
\begin{aligned}
2ab &= 8 \\
ab &= 4 \\
(2)b &= 4 \\
b &= \frac{4}{2} \\
b &= 2
\end{aligned}
$$

Equating constants:

$$ab^2 + c = -10$$
$$(2)(2)^2 + c = -10$$
$$8 + c = -10$$
$$c = -10 - 8$$
$$c = -18$$

Hence, $2x^2 + 8x - 10 = 2(x+2)^2 - 18$

Alternative Method 3:

$$h(x) = 2x^2 + 8x + 10$$
$$= 2(x^2 + 4x) - 10$$
$$= 2[(x+2)^2 - 4] - 10$$
$$= 2(x+2)^2 - 8 - 10$$
$$= 2(x+2)^2 - 18$$

(ii) State the minimum value of $h(x)$.

SOLUTION:
Required to state: The minimum value of $h(x)$.
Solution:

$h(x) = 2(x+2)^2 - 18$, where $(x+2)^2 \geq 0\ \forall x$

$$\therefore h(x)_{\text{min}} = 2(0) - 18$$
$$= -18$$

Alternative Method 1:

The axis of symmetry of the function is $x = \dfrac{-(18)}{2(2)}$
$= -2$

$x = -2$ is the x-coordinate of the minimum point of the function and which is the minimum value of the function, $h(x)$

$$h(-2) = 2(-2)^2 + 8(-2) - 10$$
$$= -18$$

Alternative Method 2:

$$h(x) = 2x^2 + 8x - 10$$
$$h'(x) = 2(2x) + 8$$

At the stationary value the first derivative = 0

$$h'(x) = 0$$
$$4x + 8 = 0$$
$$4x = -8$$
$$x = -2$$

$$h(-2) = 2(-2)^2 + 8(-2) - 10$$
$$= -18$$

$$h''(x) = 4$$
$$> 0$$

$\therefore$ Minimum value of $h(x) = -18$ at $x = -2$.

(iii) Determine the value of x for which $h(x)$ is a minimum.

SOLUTION:
Required to determine: The value of x for which $h(x)$ is a minimum.
Solution:
$h(x)_{\min}$ occurs when $2(x+2)^2 = 0$

$$(x+2)^2 = 0$$
$$x + 2 = 0$$
$$x = -2$$

2. (a) Given that $\log_2\left(6+\sqrt{12}\right) - \log_2\left(3+\sqrt{a}\right) = \log 10$, find the value of a.

SOLUTION:
Data: $\log_2\left(6+\sqrt{12}\right) - \log_2\left(3+\sqrt{a}\right) = \log 10$
Required to find: The value of a.
Solution:
Assume $\log 10 = \log_{10} 10 = 1$.

$$\log_2\left(6+\sqrt{12}\right)-\log_2\left(3+\sqrt{a}\right)=\log_{10}10$$

$$\log_2\left(\frac{6+\sqrt{12}}{3+\sqrt{a}}\right)=1$$

$$\frac{6+\sqrt{12}}{3+\sqrt{a}}=2^1$$

$$\frac{6+\sqrt{12}}{3+\sqrt{a}}=2$$

$$6+\sqrt{12}=2\left(3+\sqrt{a}\right)$$

$$6+\sqrt{12}=6+2\sqrt{a}$$

Hence,

$$\sqrt{12}=2\sqrt{a}$$

$$\sqrt{4\times 3}=2\sqrt{a}$$

$$2\sqrt{3}=2\sqrt{a}$$

$$a=3$$

(The question consists of three log terms, two of which are written to the base of 2. The base is missing in the third term and is required for solving the question. The candidate's first thought would be that the base is 2 and which makes the arithmetic to be absurdly complicated. Assuming the base to be 10 is not an expectation of the candidate. Also, if base 10 was the base to be used, the term may well have been stated as lg 10.)

(b) Determine the set of values of x for which $\frac{2-x}{4x-9}<0$.

SOLUTION:

Required to determine: The set of values of x for which $\frac{2-x}{4x-9}<0$.

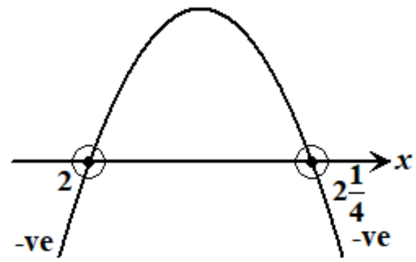
Solution:

$$\frac{2-x}{4x-9}<0$$

$\times\left(4x-9\right)^2$ to maintain the same inequality

$$\left(2-x\right)\left(4x-9\right)<0$$

The coefficient of x is negative. The graph of $y=(2-x)(4x-9)$ has a maximum point and cuts the x-axis at 2 and $2\frac{1}{4}$.



From the diagram, we read off the values of x that satisfy the inequality.

$$\left\{x : x > 2\frac{1}{4} \cup x : x < 2\right\}$$

(c) Alice deposited \$4000 into her new savings account at Bank of Fortune, which pays interest at 8% per annum. The Bank's compounded interest is represented by the geometric progression $A = P\left(1+\frac{R}{100}\right)^T$, where A is the amount of money accumulated after T years, R, the percentage rate of interest per annum and T, a positive integer, the time in years.

Determine the number of years it would take Alice's money to at least triple.

SOLUTION:
Data: Alice deposited \$4000 into her new savings account at Bank of Fortune, which pays interest at 8% per annum. The bank's compounded interest is represented by the geometric progression $A = P\left(1+\frac{R}{100}\right)^T$, where A is the amount of money accumulated after T years, R, the percentage rate of interest per annum and T, a positive integer, the time in years.
Required to determine: The number of years it would take Alice's money to at least triple.
Solution:
If the amount triples, then the amount, $A = \$4\,000 \times 3$
$= \$12\,000$

$P = 4\,000 \qquad R = 8 \qquad T = \text{unknown}$

Hence, $12\,000 = 4\,000\left(1+\frac{8}{100}\right)^T$

$$3 = (1.08)^T$$

Take lg:

$$\lg 3 = T \lg 1.08$$

$$T = \frac{\lg 3}{\lg 1.08}$$

$$T = 14.3$$

So, in 14.3 years the amount is tripled. However, T is a positive integer. Hence, the next integer after 14.3 is 15, so $T=15$.

Alternative Method:
In the geometric progression, the SECOND term is when the first interest is earned and 4000 × 1.08. Hence,

$$12\,000=4\,000(1.08)^n$$
$$n\lg 1.08=\lg 3$$
$$n=\frac{\lg 3}{\lg 1.08}$$
$$n=14.3,\ n\in\mathbb{Z}^+$$
$$\therefore n=15$$

SECTION II

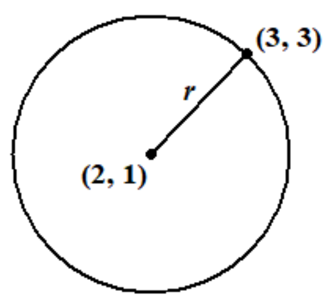
Answer ALL questions.

ALL working must be clearly shown.

3. (a) The coordinates for the center of a circle is $(2, 1)$ and the coordinates for a point on its circumference is $(3, 3)$.

(i) Determine the equation of the circle in the form $x^2+y^2+ax+by+c=0$.

SOLUTION:
Data: A circle has center $(2, 1)$ and a point on its circumference with coordinates $(3, 3)$.
Required To Determine: The equation of the circle in the form $x^2+y^2+ax+by+c=0$.
Solution:



Length of radius $=\sqrt{(3-2)^2+(3-1)^2}$

$= \sqrt{1+4}$

$= \sqrt{5}$

Th equation of the circle is $(x-2)^2 + (y-1)^2 = (\sqrt{5})^2$

$$x^2 - 4x + 4 + y^2 - 2y + 1 = 5$$

$x^2 + y^2 - 4x - 2y = 0$ is of the form

$x^2 + y^2 + (-4)x + (-2)y + 0 = 0$, where $a = -4$, $b = -2$ and $c = 0$.

Alternative Method

Let $C(-g, -f)$ be the center of the circle.

Since $C(2,1)$, $-g = 2$ and $-f = 1$ or $g = -2$ $and f = -1$

General form of the equation of a circle, center $(-g, -f)$ is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Where $c = g^2 + f^2 - r^2$

$$c = (-2)^2 + (-1)^2 - 5$$

$$c = 4 + 1 - 5 = 0$$

Substituting for g, f and c in the general equation, the equation of the circle is:

$$x^2 + y^2 - 4x - 2y + 0 = 0$$

$$x^2 + y^2 - 4x - 2y = 0$$

(ii) The circle intersects the x and y – axes at three points. Determine the coordinates of the three points of intersection.

SOLUTION:

Data: The circle intersects the x and y – axes at three points.

Required to determine: The coordinates of the three points of intersection.

Solution:

$x^2 + y^2 - 4x - 2y = 0$

When $x = 0$:

$$y^2 - 2y = 0$$

$$y(y-2) = 0$$

$$y = 0 \text{ or } y = 2$$

$\therefore$ The circle cuts the y – axis at $(0, 0)$ and $(0, 2)$.

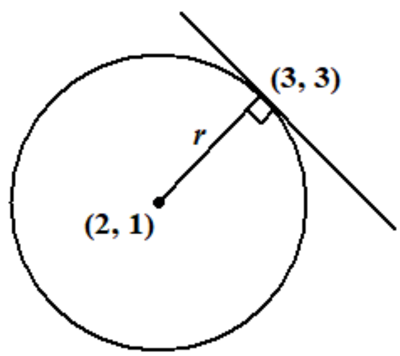
When $y=0$: $$x^2-4x=0$$ $$x(x-4)=0$$ $$x=0 \text{ or } x=4$$

$\therefore$ The circle cuts the $x-$axis at $(0,\ 0)$ and $(4,\ 0)$.

The three points of intersection with the axes are $(0,\ 0)$, $(4,\ 0)$ and $(0,\ 2)$
.

(iii) Determine the equation of the tangent to the circle at the point $(3,\ 3)$.

SOLUTION:
Required to determine: The equation of the tangent to the circle at the point $(3,\ 3)$.
Solution:



Gradient of the radius $=\dfrac{3-1}{3-2}$
$=2$

The angle made by the tangent to a circle and a radius at the point of contact is a right angle.
Hence, the gradient of the tangent $=-\dfrac{1}{2}$.
(The product of the gradients of perpendicular lines $=-1$)

The equation of the tangent is
$$\frac{y-3}{x-3}=\frac{-1}{2}$$
$$2(y-3)=-(x-3)$$
$$2y-6=-x+3$$
$$2y=-x+9$$

(b) The position vectors of two points, P and Q, relative to a fixed origin, O, are given by $OP = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $OQ = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Determine the unit vector in the direction of $\boldsymbol{PQ}$, giving your answer in simplest surd form.

SOLUTION:

Data: $OP = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $OQ = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ represent the position vectors of two points, P and Q, relative to a fixed origin, O.

Required to determine: The unit vector in the direction of PQ, giving your answer in simplest surd form.

Solution:

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$
$$= -\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} -6 \\ 4 \end{pmatrix}$$

Any vector in the direction of $PQ = \alpha\begin{pmatrix} -6 \\ 4 \end{pmatrix}$, where α is a scalar.

$$= \begin{pmatrix} -6\alpha \\ 4\alpha \end{pmatrix}$$

The vector is a unit vector, hence $\begin{vmatrix} -6\alpha \\ 4\alpha \end{vmatrix} = 1$

$$\sqrt{(-6\alpha)^2 + (4\alpha)^2} = 1$$
$$\sqrt{36\alpha^2 + 16\alpha^2} = 1$$
$$\sqrt{52\alpha^2} = 1$$
$$\alpha = \frac{1}{\sqrt{52}}$$
$$= \frac{1}{\sqrt{4 \times 13}}$$
$$= \frac{1}{2\sqrt{13}}$$

Hence, the unit vector in the direction PQ is $\frac{1}{2\sqrt{13}}\begin{pmatrix} -6 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix}$.

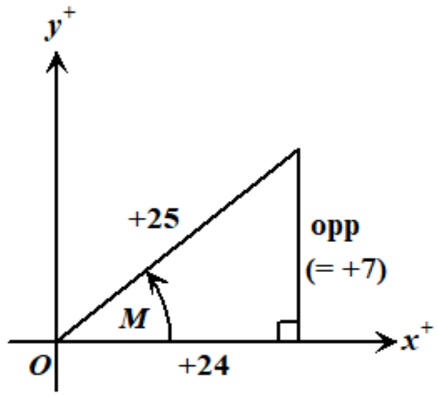
(c) Given that $\cos M = \frac{24}{25}$ and that angle M is acute, determine the value of $\tan 2M$.

SOLUTION:

Data: $\cos M = \frac{24}{25}$ and angle M is acute.

Required to determine: The value of $\tan 2M$.

Solution:



Using Pythagoras' theorem:

$$\begin{aligned}\text{opp} &= \sqrt{(25)^2-(24)^2}\\ &= \sqrt{625-576}\\ &= \sqrt{49}\\ &= +7\end{aligned}$$

$$\tan M = \frac{7}{24}$$

$$\begin{aligned}\tan 2M &= \frac{2\tan M}{1-\tan^2 M}\\ &= \frac{2\left(\frac{7}{24}\right)}{1-\left(\frac{7}{24}\right)^2}\\ &= \frac{\frac{14}{24}}{\frac{527}{576}}\\ &= \frac{14}{\cancel{24}} \times \frac{\cancel{576}^{24}}{527}\\ &= \frac{336}{527}\end{aligned}$$

SECTION III

Answer ALL questions.

ALL working must be clearly shown.

4. (a) (i) Differentiate $\sin x + \cos 4x$ with respect to x.

SOLUTION:
Required to differentiate: $\sin x + \cos 4x$ with respect to x.
Solution:
Let $y = \sin x + \cos 4x$

$$\frac{dy}{dx} = \cos x + (-\sin 4x) \times \frac{d}{dx}(4x)$$
$$= \cos x - 4\sin 4x$$

(ii) Differentiate $\frac{2x^3+2}{2x+1}$ with respect to x.

SOLUTION:
Required to differentiate: $\frac{2x^3+2}{2x+1}$ with respect to x.
Solution:
Let $y = \frac{2x^3+2}{2x+1}$ is of the form $\frac{u}{v}$, where

$u = 2x^3 + 2$ $\quad\quad$ $\frac{du}{dx} = 3\left(2x^{3-1}\right) + 0$
$= 6x^2$

$v = 2x + 1$ $\quad\quad$ $\frac{dv}{dx} = 2(1) + 0$
$= 2$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad \text{(Quotient law)}$$
$$= \frac{(2x+1)(6x^2) - (2x^3+2)(2)}{(2x+1)^2}$$
$$= \frac{12x^3 + 6x^2 - 4x^3 - 4}{(2x+1)^2}$$

$$= \frac{8x^3 + 6x^2 - 4}{(2x+1)^2}$$

(b) Use the principles of differentiation to compute the stationary value of the function $y = x^2 - 4x + 2$.

SOLUTION:
Required to compute: The stationary value of the function $y = x^2 - 4x + 2$ using the principles of differentiation.
Solution:

At a stationary point, $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = 2x^{2-1} - 4(1) + 0$$
$$= 2x - 4$$

Let $2x - 4 = 0$
$x = 2$

The stationary value occurs at $x = 2$.

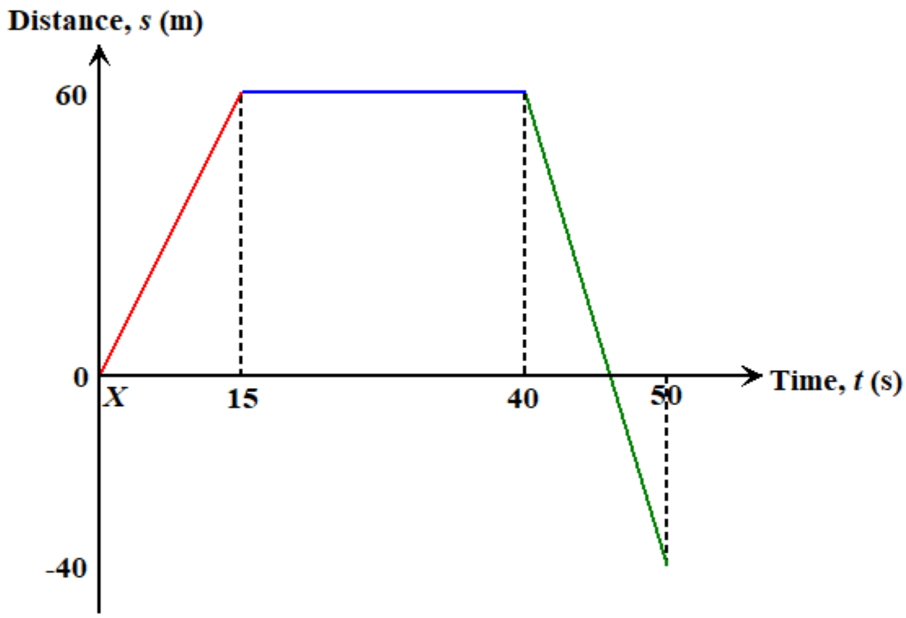
When $x = 2$: $y = (2)^2 - 4(2) + 2$
$= 4 - 8 + 2$
$= -2$

Hence, the stationary value of y is -2 and this occurs at $x = 2$.

(c) A motorist starts from a point X and travels 60 m north to a point Y at a constant speed of 4 ms^{-1}. He stays at Y for 25 seconds and then travels at a constant speed of 10 ms^{-1} for 100 m due south to a point, Z. Calculate

(i) the average speed of the whole journey

SOLUTION:
Data: A motorist starts from a point X and travels 60 m north to a point Y at a constant speed of 4 ms^{-1}. He stays at Y for 25 seconds and then travels at a constant speed of 10 ms^{-1} for 100 m due south to a point, Z.
Required to calculate: The average speed of the whole journey.
Calculation:



Phase 1: 60 m at 4 ms^{-1} (red)

$$\text{Time taken } = \frac{60}{4}$$
$$= 15 \text{ s}$$

Phase 2: At rest for 25 seconds (blue)
The vehicle is at rest so the distance covered remains the same 60 m for the entire 25 s.

Phase 3: 100 m at 10 ms^{-1} (green)

$$\text{Time taken } = \frac{100}{10}$$
$$= 10 \text{ s}$$

$$\text{Average speed of the whole journey } = \frac{\text{Total distance covered}}{\text{Total time taken}}$$
$$= \frac{60+100}{50}$$
$$= 3\frac{1}{5} \text{ ms}^{-1}$$

(ii) the average velocity of the whole journey.

SOLUTION:
Required to calculate: The average velocity of the whole journey.
Calculation:

Average velocity of the whole journey $= \frac{60-100}{50}$ ms^{-1}

$$= -\frac{4}{5} \text{ ms}^{-1}$$

5. (a) Determine the following integrals, giving each answer in its simplest form.

(i) $\int 2x^2 + 3x \, dx$

SOLUTION:
Required to determine: $\int 2x^2 + 3x \, dx$
Solution:

(When the integrand is composed of more than 1 term, they are to be written within brackets)

$$\int \left(2x^2 + 3x\right) dx = \frac{2x^{2+1}}{2+1} + \frac{3x^{1+1}}{1+1} + C$$

$$= \frac{2x^3}{3} + \frac{3x^2}{2} + C, \text{ where } C = \text{a constant}$$

(ii) $\int 2\sin 3x \, dx$

SOLUTION:
Required to determine: $\int 2\sin 3x \, dx$
Solution:
$\int 2\sin 3x \, dx = 2\int \sin 3x \, dx$

Let $t = 3x$

$$\frac{dt}{dx} = 3$$

$$\int 2\sin 3x \, dx = 2\int \sin t \, \frac{dt}{3}$$

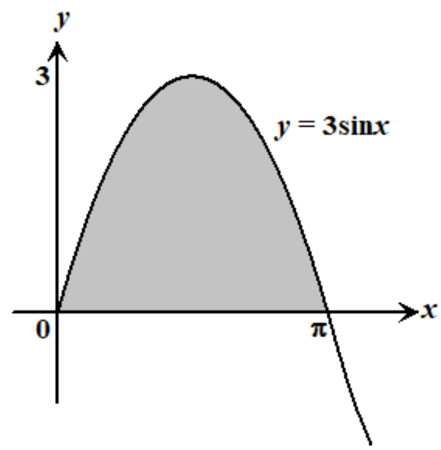
$$= \frac{2}{3}\int \sin t \, dt$$

$$= \frac{2}{3}\left(-\cos t\right) + C, \text{ where } C = \text{a constant}$$

$$= -\frac{2}{3}\cos 3x + C$$

(b) Using an integration method, calculate the area of the region in the first quadrant under the graph $y = 3\sin x$.

SOLUTION:
Required to calculate: The area under the graph $y = 3\sin x$ in the first quadrant using an integration method.
Calculation:



Area of the shaded region $= \int_0^{\pi} 3\sin x \, dx$

$$= 3\int_0^{\pi} \sin x \, dx$$

$$= 3[-\cos x]_0^{\pi}$$

$$= 3\{[-\cos(\pi)] - [-\cos(0)]\}$$

$$= 3\{1+1\}$$

$$= 3(2)$$

$= 6$ square units

(c) A particle starting from rest travels in a straight line with an acceleration, a, given by $a = t^2$, where t is the time in seconds.

(i) Determine the velocity, v, of the particle in terms of time, t.

SOLUTION:
Data: A particle starting from rest travels in a straight line with an acceleration, a, given by $a = t^2$, where t is the time in seconds.
Required to determine: The velocity, v, of the particle in terms of time, t.
Solution:
Let the velocity at time, t, be v.

$v = \int a \ dt$

$v = \int t^2 \ dt$

$v = \frac{t^3}{3} + C$, where $C =$ a constant

Since the particle started from rest,

$v = 0$ at $t = 0$

$0 = \frac{(0)^3}{3} + C$

$C = 0$

Hence, $v = \frac{t^3}{3}$ units s^{-1}.

(ii) Calculate the displacement, *s*, of the particle in the interval of time $t = 0$ to $t = 2$.

SOLUTION:

Required to calculate: The displacement, *s*, of the particle in the interval of time $t = 0$ to $t = 2$.

Calculation:

Let *s* be the displacement from *O* at time, *t*.

$s = \int v \ dt$

$s = \int \frac{t^3}{3} \ dt$

$s = \frac{t^4}{4(3)} + K$, where $K =$ a constant

$s = \frac{t^4}{12} + K$

$s = 0$ at $t = 0$

$0 = \frac{(0)^4}{12} + K$

$\therefore K = 0$

$s = \frac{t^4}{12}$

When $t = 0$ $\quad s = \frac{(0)^4}{12}$

$s = 0$

When $t = 2$ $\qquad s = \dfrac{(2)^4}{12}$

$$s = 1\frac{1}{3}$$

$\therefore$ The displacement is $1\frac{1}{3}$ units.

SECTION IV

Answer ALL questions.

ALL working must be clearly shown.

6. (a) Two fair tetrahedral dice with faces numbered 1, 2, 3, 4 are rolled. The numbers obtained on the turned-down face of each dice are noted.

Create a sample space table listing ALL possible outcomes for the two dice.

SOLUTION:
Data: Two fair tetrahedral dice with faces numbered 1, 2, 3, 4 are rolled. The numbers obtained on the turned-down face of each dice are noted.
Required to create: A sample space table listing ALL possible outcomes for the two dice.
Solution:

In the table a, b = score on die 1, score on die 2

	DIE 1			
DIE 2	1, 1	1, 2	1, 3	1, 4
	2, 1	2, 2	2, 3	2, 4
	3, 1	3, 2	3, 3	3, 4
	4, 1	4, 2	4, 3	4, 4

(b) Using your sample space table created in (a), or otherwise, determine the probability of obtaining a 4

(i) on **both** dice

SOLUTION:
Required to determine: The probability of obtaining a 4 on both dice using the sample space table in (a).

Solution:

P(obtaining a 4 on both dice) $= P(4 \text{ and } 4)$

$$= \frac{1}{4} \times \frac{1}{4}$$

$$= \frac{1}{16}$$

OR There are 16 possible outcomes and 1 outcome for 4, 4. So, the probability of 4 and 4 on both dice = 1/16.

(ii) on **at least one** dice

SOLUTION:

Required to determine: The probability of obtaining a 4 on at least one die.

Solution:

P(obtaining a 4 on at least one die)

$$= P(4 \text{ and } 4') \text{ or } P(4' \text{ and } 4) \text{ or } P(4 \text{ and } 4)$$

$$= \left(\frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{3}{4} \times \frac{1}{4}\right) + \frac{1}{16}$$

$$= \frac{3}{16} + \frac{3}{16} + \frac{1}{16}$$

$$= \frac{7}{16}$$

(iii) on **exactly one** die.

SOLUTION:

Required to determine: The probability of obtaining a 4 on exactly one die.

Solution:

P(obtaining a 4 on exactly one die) $= P(4 \text{ and } 4') \text{ or } P(4' \text{ and } 4)$

$$= \left(\frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{3}{4} \times \frac{1}{4}\right)$$

$$= \frac{6}{16}$$

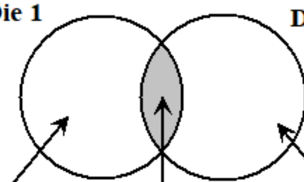
$$= \frac{3}{8}$$

(iv) Show that obtaining a 4 on **both** dice are independent events.

SOLUTION:

Required to show: Obtaining a 4 on both dice are independent events.

Proof:



If two events, A and B are independent then $P(A \cap B) = P(A) \times P(B)$

$$P(4 \text{ and } 4) = \frac{1}{16}$$

$$P(4) \times P(4) = \frac{1}{4} \times \frac{1}{4}$$
$$= \frac{1}{16}$$

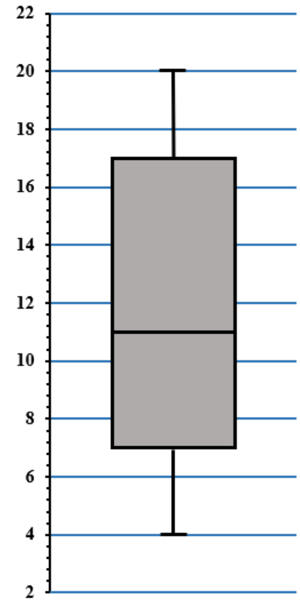
This obeys the law of independent events.

(v) Determine the probability of obtaining a 4 on **both** dice, given that a 4 was obtained on **at least one** die.

SOLUTION:
Required to determine: The probability of obtaining a 4 on **both** dice, given that a 4 was obtained on **at least one** die.
Solution:
Let $A = P(4 \text{ on both dice})$ and $B = P(4 \text{ on at least one dice})$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$
$$= \frac{\frac{1}{16}}{\frac{7}{16}}$$
$$= \frac{1}{7}$$

(c) The scores of a class of 30 students on a Mathematics test were used to draw the box plot below. (The total score possible is 20 marks.)



Using the box plot, determine the following:

(i) The median score

SOLUTION:
Data: Box plot showing the distribution of the scores of a class of 30 students on a Mathematics test.
Required to determine: The median score
Solution:
The median score is 11 (read off).

(ii) The range of the scores

SOLUTION:
Required to determine: The range of the scores.
Solution:
Highest score $= 20$
Lowest score $= 4$

$$\text{Range } = 20 - 4$$
$$= 16$$

(iii) The semi-interquartile range of the scores

SOLUTION:
Required to determine: The semi-interquartile range of the scores
Solution:
Lower quartile, $Q_1 = 7$(read off)
Upper quartile, $Q_3 = 17$(read off)

Interquartile range, $IQR = 17 - 7$
$= 10$

Semi-interquartile range ½ of $IQR = \frac{1}{2}(10)$
$= 5$

(iv) Comment on the shape of the distribution of the scores.

SOLUTION:
Required to comment: On the shape of the distribution of the scores.
Solution:
For the distribution, $Q_3 - Q_2 = 17 - 11 = 6$
And $Q_2 - Q_1 = 11 - 7 = 4$
Hence, $Q_3 - Q_2 > Q_2 - Q_1$
The distribution is non-symmetrical or positively or skewed to the right, with the median closer to the lower quartile than the upper quartile.

(v) A student wants to determine the mean score for the data set.

State ONE reason why it would be impossible to determine the mean score from the box plot.

SOLUTION:
Required to state: One reason why it would be impossible to determine the mean score from the box plot.
Solution:
The individual scores are not given or displayed on a box and whisker plot and which is necessary to obtain the mean.

(vi) State what additional piece of information would be needed to determine the mean score.

SOLUTION:
Required to state: The additional piece of information needed in order to determine the mean score.
Solution:
To obtain the mean, we would need the scores and their corresponding frequencies, since mean, $\bar{x} = \frac{\sum fx}{\sum f}$.

(vii) Given that the sum of the 30 scores for the class is 354 and the sum of the squares of the scores is 4994, determine the standard deviation for the data set.

SOLUTION:
Data: The sum of the 30 scores for the class is 354 and the sum of the squares of the scores is 4994.
Required to determine: The standard deviation for the data set.
Solution:

$\sum x = 354$ and $n = 30$

Mean, $\bar{x} = \frac{354}{30}$
$= 11.8$

$$\begin{aligned}
\text{Standard deviation, } &= \sqrt{\frac{\sum (x-\bar{x})^2}{n}} \\
&= \sqrt{\frac{\sum \left(x^2 - 2x\bar{x} + \bar{x}^2\right)}{n}} \\
&= \sqrt{\frac{\sum x^2 - \sum 2x\bar{x} + \sum \bar{x}^2}{n}} \\
&= \sqrt{\frac{\sum x^2}{n} - \frac{2\bar{x}\sum x}{n} + \frac{n\bar{x}^2}{n}} \\
&= \sqrt{\frac{\sum x^2}{n} - 2\bar{x}\,\bar{x} + \bar{x}^2} \\
&= \sqrt{\frac{\sum x^2}{n} - 2\bar{x}^2 + \bar{x}^2} \\
&= \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} \\
&= \sqrt{\frac{4994}{30} - (11.8)^2} \\
&= 5.218 \\
&\approx 5.22 \text{ (correct to 2 decimal places)}
\end{aligned}$$

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.

By Fayad W. Ali and Shereen A. Khan.