FAS-PASS
Maths

CSEC ADD MATHS 2019

SECTION I

Answer BOTH questions.

ALL working must be clearly shown.

1. (a) The function f is such that $f(x)=2x^3+7x^2+3x$.

(i) Determine all the linear factors of $f(x)$.

SOLUTION:
Data: $f(x)=2x^3+7x^2+3x$
Required to determine: All the linear factors of $f(x)$.
Solution:

$$f(x)=2x^3+7x^2+3x$$
$$=x(2x^2+7x+3)$$
$$=x(2x+1)(x+3)$$

So the linear factors of $f(x)$ are x, $2x+1$ and $x+3$.

(ii) Compute the roots of the function $f(x)$.
(A function does NOT have roots. An equation may have roots or solutions. So, we let $f(x)=0$.)

SOLUTION:
Required to find: The roots of $f(x)=0$.
Solution:

$$f(x)=2x^3+7x^2+3x$$
$$=x(2x+1)(x+3)$$

If $f(x)=0$ then $x(2x+1)(x+3)=0$

and the roots will then be $x=0$ or $-\frac{1}{2}$ or -3

(b) Two functions are such that $g(x)=x^2-x$ and $h(x)=2x-3$.

(i) Determine $gh(x)$.

SOLUTION:

Data: $g(x)=x^2-x$ and $h(x)=2x-3$

Required to determine: $gh(x)$

Solution:

$$g(x)=x^2-x$$

$$\therefore gh(x)=\left[h(x)\right]^2-h(x)$$

$$=(2x-3)^2-(2x-3)$$

$$=4x^2-6x-6x+9-2x+3$$

$$gh(x)=4x^2-14x+12$$

(ii) Given that $hg(x)=2x^2-2x-3$, show that the values of *x*, for which $hg(x)=0$, can be expressed as $\frac{1\pm\sqrt{7}}{2}$.

SOLUTION:

Data: $hg(x)=2x^2-2x-3$

Required to show: The solutions of $hg(x)=0$ are $\frac{1\pm\sqrt{7}}{2}$.

Solution:

When $hg(x)=0$

$$2x^2-2x-3=0$$

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(2)(-3)}}{2(2)}$$

$$=\frac{2\pm\sqrt{4+24}}{4}$$

$$=\frac{2\pm\sqrt{28}}{4}$$

$$=\frac{2\pm2\sqrt{7}}{4}$$

$$=\frac{2\left(1\pm\sqrt{7}\right)}{2(2)}$$

$$=\frac{1\pm\sqrt{7}}{2}$$

Q.E.D.

(c) Solve $3x\log 2+\log 8^{x}=2$.

SOLUTION:
Data: $3x\log 2+\log 8^{x}=2$
Required to find: x
Solution:

$3x\log2 + \log8^{x} = 2$
$3x\log 2 + \log (2^{3})^{x} = 2$
$3x\log 2 + \log 2^{3x} = 2$
$3x\log 2 + 3x\log 2 = 2$
$(3x + 3x) \log2 = 2$
$(6x) \log2 = 2$
$6x = \dfrac{2}{\log2}$
$x = \dfrac{2}{6\log2} = \dfrac{1}{3\log2}$

A value of x is only possible if the base of the terms in logs is given.

For instance, if the base is 10, then $x = \dfrac{1}{3\log_{10}2} = \dfrac{1}{0.903} = 1.11$	For instance, if the base is 2, then $x = \dfrac{1}{3\log_{2}2} = \dfrac{1}{3}$

2. (a) (i) Express $f(x)=-2x^{2}-7x-6$ in the form $a(x+h)^{2}+k$.

SOLUTION:
Data: $f(x)=-2x^{2}-7x-6$
Required to express: $f(x)$ in the form $a(x+h)^{2}+k$.
Solution:

$$-2x^2 - 7x - 6$$
$$= -2\left(x^2 + \frac{7}{2}x\right) - 6$$
$$= -2\left[\left(x + \frac{7}{4}\right)^2 - \frac{49}{16}\right] - 6$$
$$= -2\left(x + \frac{7}{4}\right)^2 + \frac{49}{8} - 6$$
$$= -2\left(x + \frac{7}{4}\right)^2 + \frac{1}{8}$$

So, $-2x^2 - 7x - 6 = -2\left(x + \frac{7}{4}\right)^2 + \frac{1}{8}$ is of the form $a\left(x + h\right)^2 + k$, where $a = -2$, $h = \frac{7}{4}$ and $k = \frac{1}{8}$.

Alternative Method:

$$\begin{aligned} a(x+h)^2 + k &= a(x+h)(x+h) + k \\ &= a(x^2 + 2hx + h^2) + k \\ &= ax^2 + 2ahx + ah^2 + k \end{aligned}$$

So $-2x^2 - 7x - 6 \equiv ax^2 + 2ahx + \left(ah^2 + k\right)$

Equating coefficients:

$a = -2$, $\quad 2ah = -7$,

$$2\left(-2\right)h = -7$$
$$h = \frac{7}{4}$$

$$ah^2 + k = -6$$
$$-2\left(\frac{7}{4}\right)^2 + k = -6$$
$$-2\left(\frac{49}{16}\right) + k = -6$$
$$-6\frac{1}{8} + k = -6$$
$$k = \frac{1}{8}$$

So, $-2x^2 - 7x - 6 \equiv -2\left(x + \frac{7}{4}\right)^2 + \frac{1}{8}$

(ii) State the maximum value of $f(x)$.

SOLUTION:
Required to state: The maximum value of $f(x)$.
Solution:

$$f(x) = -2x^2 - 7x - 6$$
$$= -2\underbrace{\left(x+\frac{7}{4}\right)^2}_{\geq 0\,\forall x} + \frac{1}{8}$$

$\therefore$ The maximum value of $f(x) = -2(0) + \frac{1}{8}$

The maximum value of $f(x)$ is $\frac{1}{8}$.

(iii) State the value of x for which $f(x)$ is a maximum.

SOLUTION:
Required to state: The value of x for which $f(x)$ is a maximum
Solution:

The maximum value of $f(x)$ occurs when $-2\left(x+\frac{7}{4}\right)^2 = 0$

i.e when $x = -\frac{7}{4}$

(iv) Use your answer in (a) (i) to determine all values of x when $f(x) = 0$.

SOLUTION:
Required to determine: The values of x when $f(x) = 0$.
Solution:

$$f(x) = -2\left(x + \frac{7}{4}\right)^2 + \frac{1}{8}$$

$$-2\left(x + \frac{7}{4}\right)^2 + \frac{1}{8} = 0$$

$$-2\left(x + \frac{7}{4}\right)^2 = -\frac{1}{8}$$

$$\left(x + \frac{7}{4}\right)^2 = \frac{1}{16}$$

$$x + \frac{7}{4} = \pm\frac{1}{4}$$

$$x = -\frac{7}{4} \pm \frac{1}{4}$$

$$x = \frac{-7 \pm 1}{4}$$

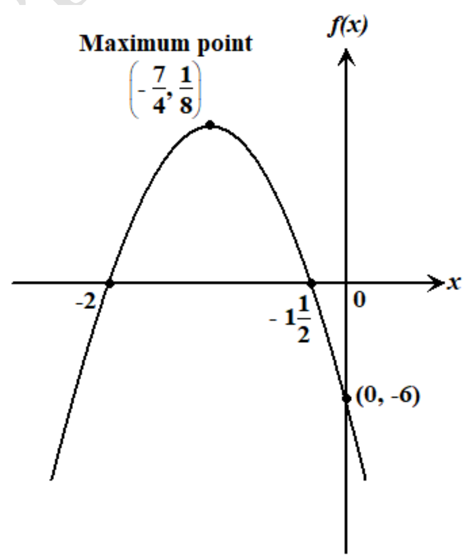
$$x = \frac{-7+1}{4} \text{ or } \frac{-7-1}{4}$$

$$= -\frac{6}{4} \text{ or } -\frac{8}{4}$$

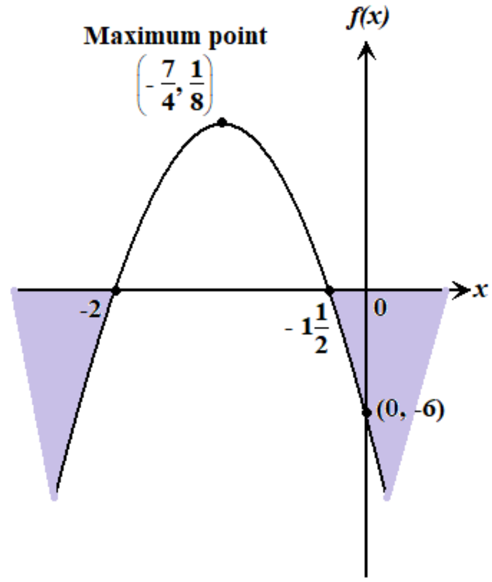
$$= -1\frac{1}{2} \text{ or } -2$$

(v) Sketch the function $f(x)$ and show your solution set to (a) (iv) when $f(x) < 0$.

SOLUTION:
Required to sketch: The function $f(x)$ and write the solution of $f(x) < 0$.
Solution:
$f(0) = -6$



For $f(x) < 0$:



The solution set for $f(x) < 0$ is $\left\{x : x < -2 \cup x > -1\frac{1}{2}\right\}$.

(b) A geometric series can be represented by $\frac{y}{x} + \frac{y^2}{x^3} + \frac{y^3}{x^5} + \ldots$

Prove that $S_\infty = xy\left(x^2 - y\right)^{-1}$.

SOLUTION:

Data: $\frac{y}{x} + \frac{y^2}{x^3} + \frac{y^3}{x^5} + \ldots$ is a geometric series.

Required to prove: $S_\infty = xy\left(x^2 - y\right)^{-1}$

Proof:

For the geometric series $\frac{y}{x} + \frac{y^2}{x^3} + \frac{y^3}{x^5} + \ldots$

$$\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{y^2}{x^3} \div \frac{y}{x}$$
$$= \frac{y^2}{x^3} \times \frac{x}{y}$$
$$= \frac{y}{x^2}$$

The series is a geometric progression with first term, $a = \frac{y}{x}$ and with a common ratio of $\frac{y}{x^2}$.

$$S_\infty = \frac{a}{1-r} \quad ,|r|<1$$

$$= \frac{\frac{y}{x}}{1-\frac{y}{x^2}}$$

$$= \frac{\frac{y}{x}}{\frac{x^2-y}{x^2}}$$

$$= \frac{y}{x} \times \frac{x^2}{x^2-y}$$

$$= \frac{xy}{x^2-y}$$

$$= xy\left(x^2-y\right)^{-1}$$

Q.E.D.

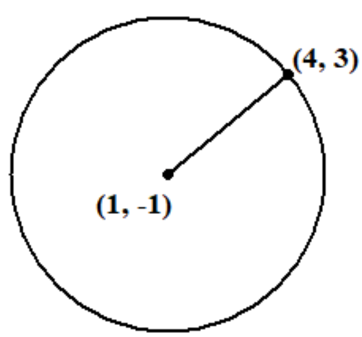
SECTION II

Answer BOTH questions.
ALL working must be clearly shown.

3. (a) A circle with center $(1, -1)$ passes through the point $(4, 3)$.

(i) Calculate the radius of the circle.

SOLUTION:
Data: A circle has center $(1, -1)$ and passes through $(4, 3)$.
Required to calculate: The radius of the circle
Calculation:

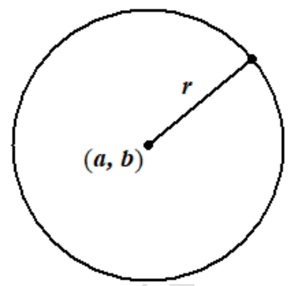


Length of the radius $= \sqrt{(4-1)^2 + (3-(-1))^2}$

$$= \sqrt{(3)^2 + (4)^2}$$
$$= \sqrt{25}$$
$$= 5 \text{ units}$$

(ii) Write the equation of the circle in the form $x^2 + y^2 + 2fx + 2gy + c = 0$.

SOLUTION:
Required to write: The equation of the circle in the form $x^2 + y^2 + 2fx + 2gy + c = 0$.
Solution: Recall for

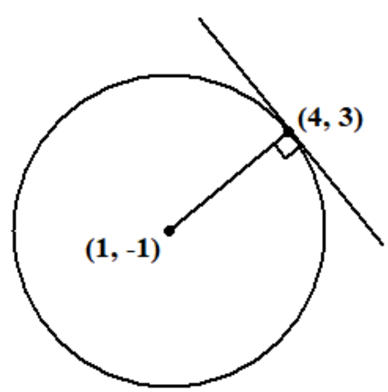


The equation is $(x-a)^2 + (y-b)^2 = r^2$
So, the equation of the given circle is

$$(x-1)^2 + (y-(-1))^2 = (5)^2$$
$$x^2 - 2x + 1 + y^2 + 2y + 1 = 25$$
$x^2 + y^2 - 2x + 2y - 23 = 0$ and which is of the form
$x^2 + y^2 + 2gx + 2fy + c = 0$, where $g = -1$, $f = 1$ and $c = -23$.

(iii) Determine the equation of the tangent to the circle at the point $(4, 3)$.

SOLUTION:
Required to determine: The equation of the tangent to the circle at the point $(4, 3)$.
Solution:



The angle made by a tangent to a circle and a radius at the point of contact is a right angle.

Gradient of the radius:

$$=\frac{3-(-1)}{4-1}=\frac{3+1}{3}=\frac{4}{3}$$

$\therefore$ The gradient of the tangent $=-\frac{3}{4}$ since the product of the gradients of perpendicular lines is -1.

The equation of the tangent to the circle at $(4,\ 3)$ is

$$\frac{y-3}{x-4}=-\frac{3}{4}$$
$$4(y-3)=-3(x-4)$$
$$4y-12=-3x+12$$
$$4y=-3x+24$$

(b) Two vectors **p** and **q** are such that $\mathbf{p}=8\mathbf{i}+2\mathbf{j}$ and $\mathbf{q}=\mathbf{i}-4\mathbf{j}$.

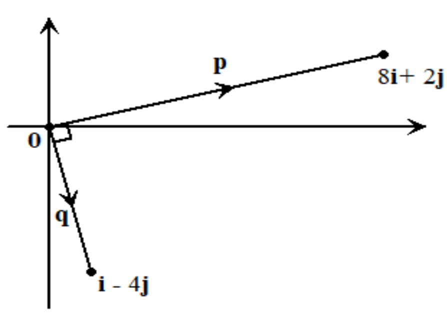
(i) Calculate $\mathbf{p}\,.\mathbf{q}$.

Data: $\mathbf{p}=8\mathbf{i}+2\mathbf{j}$ and $\mathbf{q}=\mathbf{i}-4\mathbf{j}$, where **p** and **q** are two vectors.
Required to calculate: $\mathbf{p}\,.\mathbf{q}$
Calculation:

$$\mathbf{p}\,.\,\mathbf{q}=(8\times 1)+(2\times -4)$$
$$=8-8$$
$$=0$$

(ii) State the angle between the two vectors **p** and **q**.

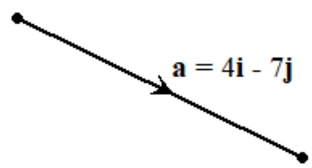
SOLUTION:
Required to state: The angle between vectors **p** and **q**
Solution:



Recall if $\mathbf{a}.\mathbf{b} = 0$ then **a** is perpendicular to **b**.
Since **p.q** = 0 then the angle between **p** and **q** is 90°.

(c) The position vector $\mathbf{a} = 4\mathbf{i} - 7\mathbf{j}$. Find the unit vector in the direction of **a**.

SOLUTION:
Data: $\mathbf{a} = 4\mathbf{i} - 7\mathbf{j}$ is a position vector.
Required to find: The unit vector in the direction of **a**.
Solution:



Any vector in the direction of **a** will be of the form $\alpha(4\mathbf{i} - \mathbf{j})$ where α is a scalar.
A unit vector has a magnitude of 1

So $|\alpha(4i - 7j)| = 1$

$$|4\alpha i - 7\alpha j| = 1$$

$$\sqrt{(4\alpha)^2 + (-7\alpha)^2} = 1$$

$$\sqrt{65\alpha^2} = 1$$

$$\sqrt{65}\alpha = 1$$

$$\alpha = \frac{1}{\sqrt{65}}$$

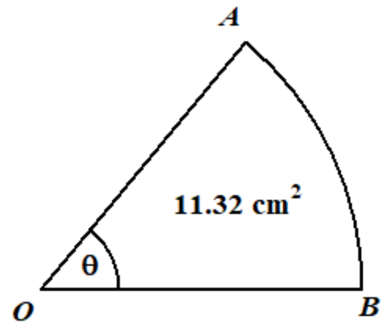
So, the unit vector in the direction of **a** is

$$\frac{1}{\sqrt{65}}(4i - 7j) = \frac{4}{\sqrt{65}}i - \frac{7}{\sqrt{65}}j$$

4. (a) A compass is used to draw a sector of radius 6 cm and area 11.32 cm^2.

(i) Determine the angle of the sector in radians.

SOLUTION:
Data: A sector of radius 6 cm and area 11.32 cm^2 is drawn using a compass.
Required to determine: The angle of the sector in radians
Solution:



Let the sector be AOB and the angle of the sector be θ radians.

Recall: $A=\frac{1}{2}r^2\theta$ ($A=$area, $r=$radius and $\theta=$angle in radians)

So $11.32=\frac{1}{2}(6)^2\times\theta$

$$\theta=\frac{11.32\times 2}{36}\text{ radians}$$
$$=0.628\underline{\underline{8}}\text{ radians}$$
$$\approx 0.629\text{ radians}$$

(ii) Calculate the perimeter of the sector.

SOLUTION:
Required to calculate: The perimeter of the sector
Calculation:
Perimeter of the sector $AOB=(6+\text{arc length } AB+6)$ cm

$$=6+(6\times 0.6288)+6$$
$$=12+6(0.6288)$$
$$=15.7728\text{ cm}$$
$$=15.773\text{ cm to 3 decimal places}$$

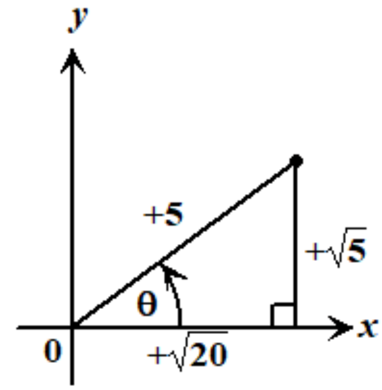
(b) A right-angled triangle XYZ has an angle, θ, where $\sin\theta=\frac{\sqrt{5}}{5}$. Without evaluating θ, calculate the exact value (in surd form if applicable) of

(i) $\cos\theta$

SOLUTION:

Data: Right-angled triangle XYZ has an angle θ such that $\sin\theta=\frac{\sqrt{5}}{5}$.

Required to calculate: $\cos\theta$ in exact form
Calculation:
Assume that θ is acute.



$$\text{adj} = \sqrt{(5)^2 - \left(\sqrt{5}\right)^2} \quad \text{Pythagoras' Theorem}$$
$$= +\sqrt{25-5}$$
$$= +\sqrt{20}$$

$$\therefore \cos\theta = \frac{\sqrt{20}}{5}$$
$$= \frac{\sqrt{4}\sqrt{5}}{5}$$
$$= \frac{2\sqrt{5}}{5} \text{ or } \frac{2}{\sqrt{5}} \text{ in surd form}$$

(ii) $\sin 2\theta$

SOLUTION:
Required to calculate: $\sin 2\theta$
Calculation:
Since θ is acute, then
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\frac{\sqrt{5}}{5} \times \frac{2}{\sqrt{5}}$$
$$= \frac{4}{5}$$

(c) Show that $\tan^2\theta + 1 = \dfrac{1}{\cos^2\theta}$.

SOLUTION:

Required to show: $\tan^2\theta+1=\dfrac{1}{\cos^2\theta}$

Proof:
Consider the lefthand side:

Recall: $\dfrac{\sin\theta}{\cos\theta}=\tan\theta$

$$\begin{aligned}\tan^2\theta+1&=\frac{\sin^2\theta}{\cos^2\theta}+\frac{1}{1}\\&=\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}\qquad\text{Recall}: \sin^2\theta+\cos^2\theta=1\\&=\frac{1}{\cos^2\theta}\\&=\text{R.H.S.}\end{aligned}$$

Q.E.D.

SECTION III

Answer BOTH questions.

ALL working must be clearly shown.

5. (a) The stationary points of a curve are given by $\left(5,\ 11\dfrac{2}{3}\right)$ and $(3,\ 15)$.

(i) Derive an expression for $\dfrac{dy}{dx}$.

SOLUTION:

Data: $\left(5,\ 11\dfrac{2}{3}\right)$ and $(3,\ 15)$ are two stationary points on a curve.

Required to find: $\dfrac{dy}{dx}$

Solution:

$(x-5)$ and $(x-3)$ are factors of $\dfrac{dy}{dx}=0$.

So, $\dfrac{dy}{dx}=a(x-5)(x-3)$, where a is a constant

$=ax^2-8ax+15a$

(ii) Determine the nature of the stationary points

SOLUTION:
Required to determine: The nature of the stationary points.
Solution:

$$\frac{d^2y}{dx^2} = 2ax - 8a$$

When $x = 5$ $\quad \frac{d^2y}{dx^2} = 2a$

> 0 (assuming a is a positive constant)

Hence $\left(5, 11\frac{2}{3}\right)$ is a minimum point.

When $x = 3$ $\quad \frac{d^2y}{dx^2} = -2a$

< 0 (assuming a is a positive constant)

Hence $(3, 15)$ is a maximum point

(iii) Determine the equation of the curve

SOLUTION:
Required to determine: The equation of the curve.
Solution:
Equation of the curve is

$$y = \int \left(ax^2 - 8ax + 15a\right) dx$$

$$y = \frac{ax^3}{3} - 4ax^2 + 15ax + C, \text{ where } C \text{ is a constant}$$

$(3, 15)$ lies on the curve.

$$\therefore 15 = 9a - 36a + 45a + C$$

$$18a + C = 15 \quad \ldots❶$$

$\left(5, 11\frac{2}{3}\right)$ lies on the curve.

$$11\frac{2}{3} = 41\frac{2}{3}a - 100a + 75a + C$$

$$11\frac{2}{3} = 16\frac{2}{3}a + C \quad \ldots❷$$

❶ – ❷

$$3\frac{1}{3}=1\frac{1}{3}a$$

$$\therefore a=2\frac{1}{2}$$

Substituting in ❶ or ❷ will give
$C=-30$

So, if we substitute a = 2 ½ , then $\frac{dy}{dx}=\frac{5}{2}(x-5)(x-3)$

$=\frac{5x^2}{2}-20x+\frac{75}{2}$ and the equation of the curve is

$$y=\frac{5}{6}x^3-10x^2+37\frac{1}{2}x-30$$

(b) Differentiate $\sqrt[3]{(2x+3)^2}$ with respect to *x*, giving your answer in its simplest form.

SOLUTION:

Required to differentiation: $\sqrt[3]{(2x+3)^2}$ with respect to x.

Solution:

Let $y=\sqrt[3]{(2x+3)^2}$

$$\therefore y=(2x+3)^{\frac{2}{3}}$$

Let $t=2x+3$ $\qquad \frac{dt}{dx}=2$

So $y=t^{\frac{2}{3}}$ $\qquad \frac{dy}{dt}=\frac{2}{3}t^{\frac{2}{3}-1}$

$$=\frac{2}{3}t^{-\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \qquad \text{(Chain rule)}$$

$$\frac{dy}{dx} = \frac{2}{3}t^{-\frac{1}{3}} \times 2$$
$$= \frac{4}{3}t^{-\frac{1}{3}}$$
$$= \frac{4}{3t^{\frac{1}{3}}}$$
$$= \frac{4}{3\sqrt[3]{t}}$$

Re-substituting for t we get,

$$\frac{dy}{dx} = \frac{4}{3\sqrt[3]{2x+3}}$$

6. (a) Integrate $3\cos x + 2\sin x$.

SOLUTION:
Required to find: $\int (3\cos x + 2\sin x)\ dx$
Solution:

$$\int (3\cos x + 2\sin x)\ dx = 3\int \cos x\ dx + 2\int \sin x\ dx$$
$$= 3(\sin x) + 2(-\cos x) + C, \text{ where } C \text{ is a constant}$$
$$= 3\sin x - 2\cos x + C$$

(b) Evaluate $\int_1^4 \frac{2\sqrt{x}}{x}\ dx$.

SOLUTION:

Required to evaluate: $\int_1^4 \frac{2\sqrt{x}}{x}\ dx$

Solution:

$$\int_1^4 \frac{2\sqrt{x}}{x}\,dx = \int_1^4 2x^{\frac{1}{2}-1}\,dx$$

$$= \int_1^4 2x^{-\frac{1}{2}}\,dx$$

$$= \left[\frac{2x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C\right]_1^4 \text{, where } C \text{ is a constant}$$

$$= \left[\frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + C\right]_1^4$$

$$= \left[4\sqrt{x} + C\right]_1^4$$

$$= \left(4\sqrt{4}\right) - \left(4\sqrt{1}\right)$$

$$= 8 - 4$$

$$= 4$$

(c) The point $(2, 4)$ lies on the curve whose gradient is given by $\frac{dy}{dx} = -2x + 1$.

Determine:

(i) the equation of the curve

SOLUTION:

Data: $(2, 4)$ is a point on the curve such that $\frac{dy}{dx} = -2x + 1$.

Required to find: The equation of the curve

Solution:

The equation of the curve is

$$y = \int (-2x + 1)\,dx$$

$$y = -\frac{2x^2}{2} + x + C, \text{ where } C \text{ is a constant}$$

$$y = -x^2 + x + C$$

$(2, 4)$ lies on the curve

So $4 = -(2)^2 + 2 + C$

$C = 6$

$\therefore$ The equation of the curve is $y = -x^2 + x + 6$.

(ii) the area under the curve in the finite region in the first quadrant between 0 and 3 on the $x-$axis.

SOLUTION:
Required to find: The area under the curve in the first quadrant between $x=0$ and $x=3$.
Solution:
The area bounded by the curve in the first quadrant between $x=0$ and $x=3$ and the $x-$axis is

$$\int_0^3 \left(-x^2+x+6\right)\,dx = \left[-\frac{x^3}{3}+\frac{x^2}{2}+6x+C\right]_0^3, \text{ where } C \text{ is a constant}$$

$$= \left[-\frac{(3)^3}{3}+\frac{(3)^2}{2}+6(3)\right]-\left[-\frac{(0)^3}{3}+\frac{(0)^2}{2}+6(0)\right]$$

$$= -9+4\frac{1}{2}+18$$

$$= 22\frac{1}{2}-9$$

$$= 13\frac{1}{2} \text{ square units}$$

SECTION IV

Answer only ONE question.
ALL working must be clearly shown.

7. (a) The weights, in kg, of students in a Grade 5 class are displayed in the following stem and leaf diagram

Boys		Girls
9 8	2	8 8 9
9 9 9 7 3	3	2 2 3 5 8 8 8
5 1 1 1 1 0 0	4	0 1 1 2
1	5	

Key:
Boys 8|2 means 28 kg
Girls 2|8 means 28 kg

(i) State the number of students in the class.

SOLUTION:
Data: Stem and leaf diagram showing the weights, in kg, of students in a Grade 5 class.
Required to state: The number of students in a class

Solution:
The number of boys $= 15$
The number of girls $= 14$
Total number of students $= 15 + 14 = 29$

(ii) Construct ONE box-and-whisker plot for the entire Grade 5 class (boys and girls combined).

SOLUTION:
Required to construct: A box-and-whisker plot for the entire Grade 5 class
Solution:

Merging the data for the 29 students in the class

	Boys and Girls
2	8 8 8 9 9
3	2 2 3 3 5 7 8 8 8 9 9 9
4	0 0 0 1 1 1 1 1 1 2 5
5	1

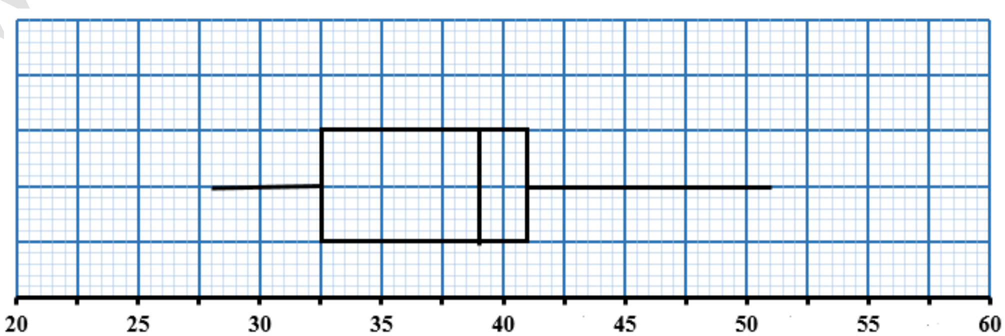
To construct the box-and-whisker plot, we need five statistical indices.
The lowest score = 28
The highest score = 51
The Median $= 15^{\text{th}}$ value $= 39\ (Q_2)$
The lower median, Q_1 is the mean of the 7^{th} and 8^{th} values,

$$Q_1 = \frac{32+33}{2} = 32.5$$

The upper median, Q_3 is the mean of the 22^{nd} and 23^{rd} values,

$$Q_3 = \frac{41+41}{2} = 41$$

The box-and-whisker plot for the entire Grade 5 class is shown below.



(iii) The standard deviation of the weights of the boys is 5.53 kg.

Determine the standard deviation of the weights of the girls. Provide an interpretation of your answer for the girls compared to that given for the boys.

SOLUTION:

Data: The standard deviation of the boys' weights is 5.53 kg

Required to determine: The standard deviation of the weights of the girls and a comparison of the boys' and girls' standard deviations
Solution:
The weights of the girls, in kg, are 28, 28, 29, 32, 33, 35, 38, 38, 38, 40, 41, 41, 42.

Standard deviation $=\sqrt{\dfrac{\sum(x-\bar{x})^2}{n}}$, where x =values, $\bar{x}$ =mean and $n=$ number of values.

$$\bar{x}=\frac{28+28+29+32+32+33+35+38+38+38+40+41+41+42}{14}$$

$$=\frac{495}{14}$$

$$=35.36$$

We now calculate the deviation of each score from the mean, $x_i-\bar{\bar{x}}$.
Then we square these deviations and calculate the sum.

x_i	$x_i-\bar{\bar{x}}$	$(x_i-\bar{\bar{x}})^2$
28	−7.36	54.17
28	−7.36	54.17
29	−6.36	40.45
32	−3.36	11.29
32	−3.36	11.29
33	−2.36	5.57
35	−0.36	0.13
38	2.64	6.97
38	2.64	6.97
38	2.64	6.97
40	4.64	21.53
41	5.64	31.81
41	5.64	31.81
42	6.64	44.09
		327.22

$$\sum(x-\bar{x})^2 = 327.22$$

$$\frac{\sum(x-\bar{x})^2}{n} = \frac{327.22}{14}$$
$$= 23.37$$

$$\text{Standard deviation } = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$
$$= \sqrt{23.37}$$
$$= 4.83$$

The standard deviation of the weights of the girls (4.83) is less than that of the boys (5.53).

This means that the data showing the weights of the girls has a less spread or is of a lesser variability than that for the boys. In the case of the girls, their weights are more clustered around the mean.

(iv) Determine the number of students above the 20th percentile for this class.

SOLUTION:
Required to determine: The number of students above the 20th percentile of the class

Solution:
There are 15 boys and 14 girls, which is a total of 29 students in the class.
Finding the 20th percentile: $\frac{20}{100} \times 29 = 5.8$
We take the nearest whole number which is 6 and called the index

The data set written from the smallest to largest will be
28, 28, 28, 29, 29, 32, 32, 33, …
↑
6th value

The number of students whose score is more than 32 will be 22 since 7 have scores of 32 or less.

(b) A vendor has 15 apples on a tray: 5 red, 6 green and 4 yellow. A customer requests 3 apples but does NOT specify a colour.

Determine the probability that the apples chosen

(i) contain one of EACH colour

SOLUTION:
Data: A vendor has 5 red, 6 green and 4 yellow apples on a tray. A customer requests 3 apples without specifying the colours.
Required to find: The probability that the customer gets one of each colour of apple
Solution:
There are 6 possible ways that this can happen. The customer can get RGY or RYG or GRY or GYR or YRG or YGR

$$\text{P(RGY)} = \text{P}(R \text{ and } G \text{ and } Y) = \frac{5}{15} \times \frac{6}{14} \times \frac{4}{13} = \frac{4}{91}$$

$$\text{P(RYG)} = \text{P}(R \text{ and } Y \text{ and } G) = \frac{5}{15} \times \frac{4}{14} \times \frac{6}{13} = \frac{4}{91}$$

$$\text{P(GRY)} = \text{P}(G \text{ and } R \text{ and } Y) = \frac{6}{15} \times \frac{5}{14} \times \frac{4}{13} = \frac{4}{91}$$

$$\text{P(GYR)} = \text{P}(G \text{ and } Y \text{ and } R) = \frac{6}{15} \times \frac{4}{14} \times \frac{5}{13} = \frac{4}{91}$$

$$\text{P(YRG)} = \text{P}(Y \text{ and } R \text{ and } G) = \frac{4}{15} \times \frac{5}{14} \times \frac{4}{13} = \frac{4}{91}$$

$$\text{P(YGR)} = \text{P}(Y \text{ and } G \text{ and } R) = \frac{4}{15} \times \frac{6}{14} \times \frac{5}{13} = \frac{4}{91}$$

$$\text{P(Customer gets one of each colour)} \frac{4}{91} \times 6 = \frac{24}{91}$$

(ii) are ALL of the same colour.

SOLUTION:
Required to find: The probability that the customer gets all three apples of the same colour
Solution:
P(3 apples drawn at random are the same colour)

$$= \text{P}(RRR) \text{ or } \text{P}(GGG) \text{ or } \text{P}(YYY)$$

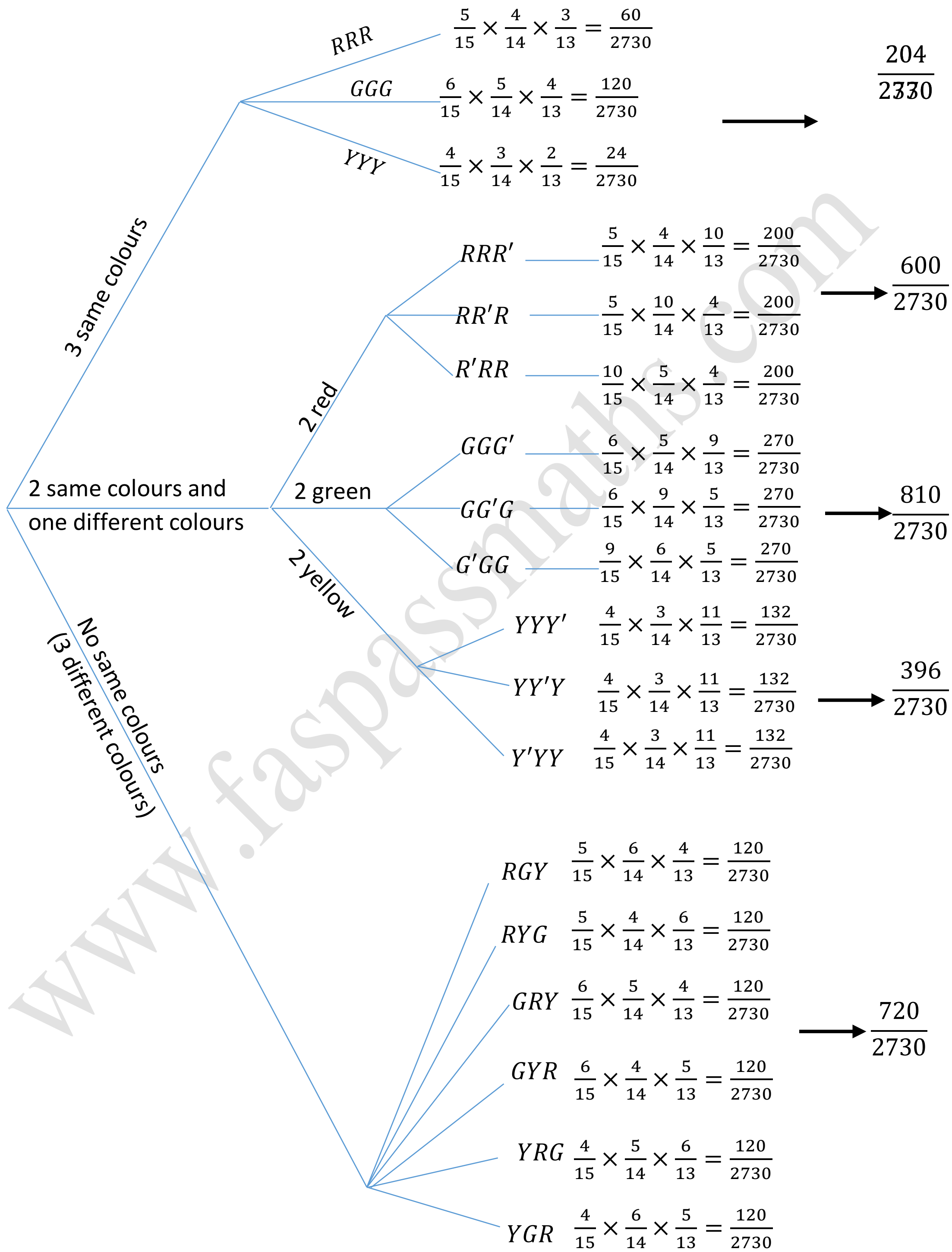
$$= \left(\frac{5}{15} \times \frac{4}{14} \times \frac{3}{13}\right) + \left(\frac{6}{15} \times \frac{5}{14} \times \frac{4}{13}\right) + \left(\frac{4}{15} \times \frac{3}{14} \times \frac{2}{13}\right)$$

$$= \frac{60 + 120 + 24}{15 \times 14 \times 13}$$

$$= \frac{204}{15 \times 14 \times 13}$$

$$= \frac{34}{455}$$

The probability tree diagram illustrates all possible outcomes for these events.

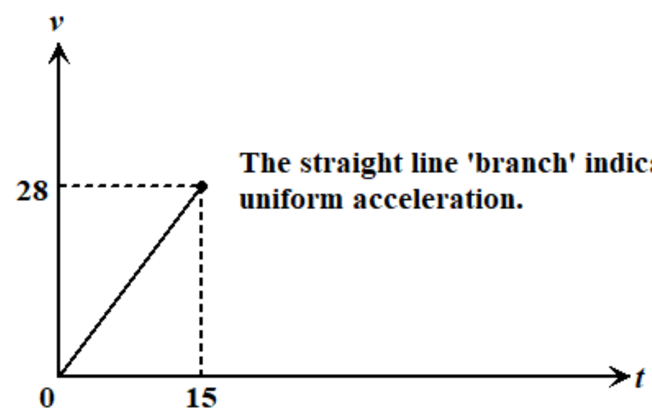


$$\text{Total Probability} = \frac{204+600+810+396+720}{2730} = \frac{2730}{2730} = 1$$

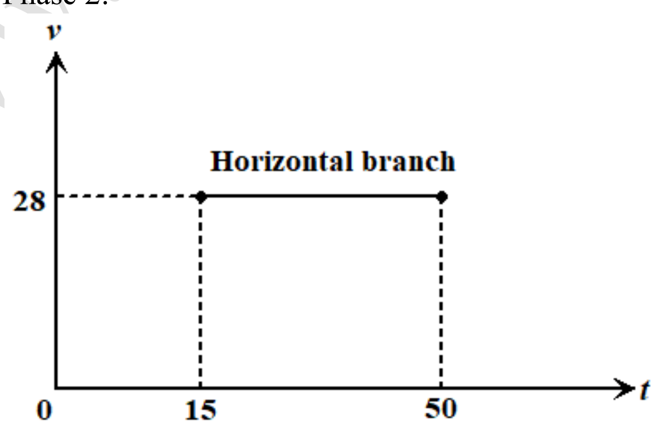
8. (a) A car has stopped at a traffic light. When the light turns green, it accelerates uniformly, to a speed of 28 ms^{-1} in 15 seconds. The car continues to travel at this speed for another 35 seconds, before it has to stop 10 seconds later at another traffic light.

(i) On the grid provided, draw a speed-time graph showing the information above.

SOLUTION:
Data: A car has stopped at a traffic light. When the light turns green, it accelerates uniformly, to a speed of 28 ms^{-1} in 15 seconds. The car continues to travel at this speed for another 35 seconds, before it has to stop 10 seconds later at another traffic light.
Required to draw: A speed-time graph to show the motion of the car
Solution:
Phase 1:

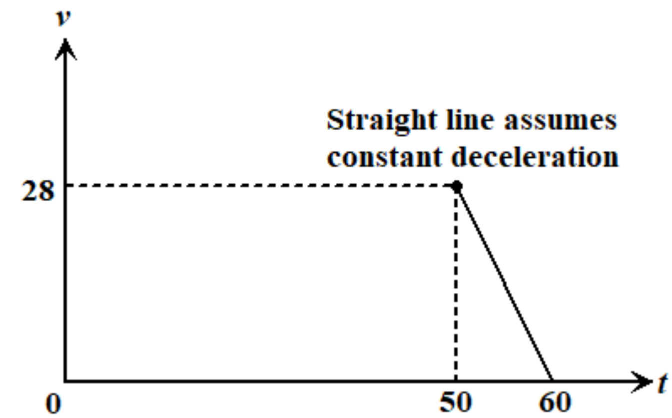


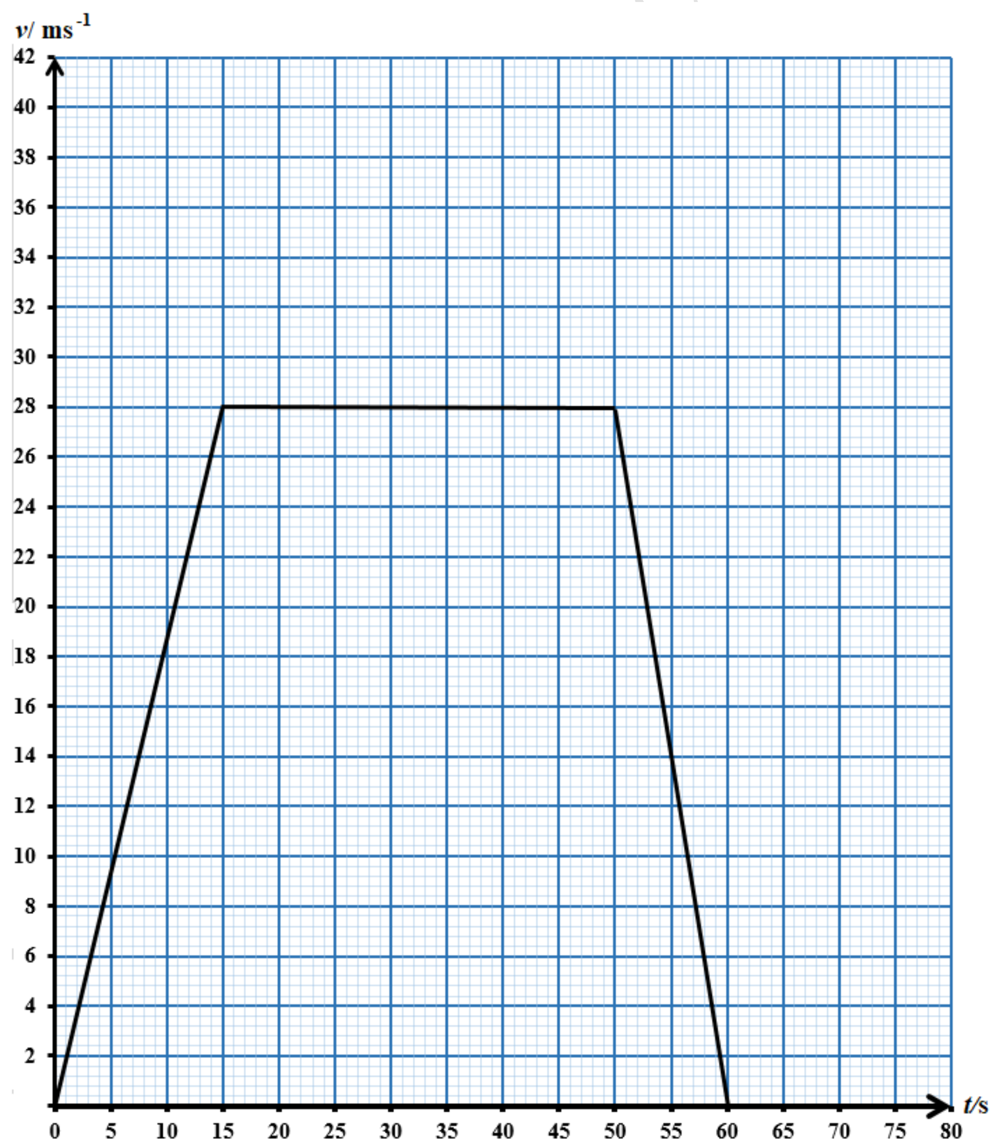
Phase 2:



The horizontal branch (gradient of 0) indicates there is no acceleration and hence constant velocity.

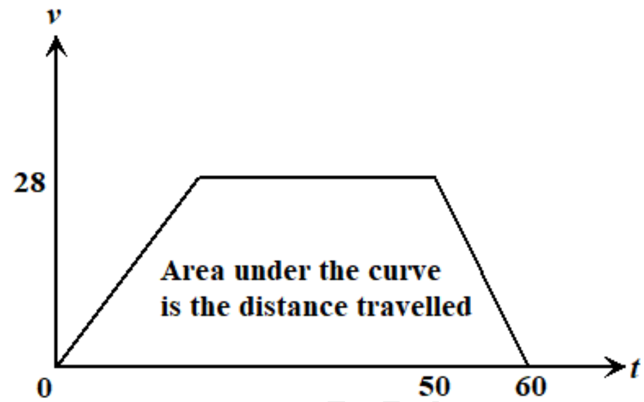
Phase 3:


v
Straight line assumes
constant deceleration
28
0
50
60
t


v/ ms -1
42
40
38
36
34
32
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
5
10
15
20
25
30
35
40
45
50
55
60
65
70
75
80
t/s

(ii) Calculate the distance the car travelled between the two traffic lights.

SOLUTION:
Required to calculate: The distance travelled by the car between the two traffic lights
Calculation:



Distance travelled $= \frac{1}{2}(60+35)\times 28$
$= 1330 \text{ m}$

(iii) Calculate the average speed of the car over this journey, **giving your answer in kmh^{-1}**.

SOLUTION:
Required to calculate: The average speed of the journey in kmh^{-1}
Calculation:

$$\text{Average speed} = \frac{\text{Total distance covered}}{\text{Total time taken}}$$

$$= \frac{\frac{1\,330}{1000}\text{ km}}{\frac{60}{3\,600}\text{ h}}$$

$$= \frac{1.33}{\frac{1}{60}}$$

$$= 79.8 \text{ kmh}^{-1}$$

(b) A particle moves in a straight line such that t seconds after passing a fixed point, O, its acceleration, a, in ms^{-2}, is given by $a = 12t - 17$. Given that its speed at O is 10 ms^{-1}, determine

(i) the values of t for which the particle is stationary

SOLUTION:

Data: A particle moving in a straight line passes a fixed point, O, after t second with acceleration, $a = 12t - 17$. Its speed at O is 10 ms^{-1}.

Required to determine: the values of t for which the particle is stationary

Solution:

Let the velocity at t be v.

$$v = \int (12t - 17)\ dt$$

$v = \frac{12t^2}{2} - 17t + C,$ where C is a constant

$v = 10$ where $t = 0$

$\therefore 10 = 6(0)^2 - 17(0) + C$

$C = 10$

Hence, $v = 6t^2 - 17t + 10$

At a stationary point, $v = 0$

Let $6t^2 - 17t + 10 = 0$

$$(6t - 5)(t - 2) = 0$$

$$\therefore t = \frac{5}{6} \text{ or } 2$$

$\therefore$ the particle is stationarywhen $t = \dfrac{5}{6}$ seconds or 2 seconds

(ii) the distance the particle travels in the fourth second.

SOLUTION:

Required to calculate: The distance the particle travelled in the 4th second

Calculation:

Let the distance from O at time t be s.

$s = \int v\ dt$

$s = \int (6t^2 - 17t + 10)\ dt$

$s = \dfrac{6t^3}{3} - \dfrac{17t^2}{2} + 10t + K$, where K is a constant

$s = 0$ when $t = 0$

$$0=2(0)^3-\frac{17}{2}(0)^2+10(0)+K$$

$$K=0$$

So, $s=2t^3-\frac{17}{2}t^2+10t$

When $t=3$

$$\begin{aligned} s &= 2(3)^3-\frac{17(3)^2}{2}+10(3) \\ &= 54-\frac{17(9)}{2}+30 \\ &= 7.5 \end{aligned}$$

When $t=4$

$$\begin{aligned} s &= 2(4)^3-\frac{17(4)^2}{2}+10(4) \\ &= 128-(17\times 8)+40 \\ &= 32 \end{aligned}$$

So, the distance travelled in the 4th second $=32-7.5$
$=24.5$ m