FAS-PASS Maths

# CSEC ADD MATHS 2017

## SECTION I

**Answer BOTH questions.**

**ALL working must be clearly shown.**

**1. (a)** The function $f$ is defined by

$$f(x)=\frac{2x+p}{x-1},\ x\neq 1 \text{ and } p \text{ is a constant.}$$

**(i)** Determine the inverse of $f(x)$.

**SOLUTION:**

**Data:** $f(x)=\dfrac{2x+p}{x-1},\ x\neq 1$

**Required to find:** $f^{-1}(x)$

**Solution:**

Let $y=f(x)$

$$y=\frac{2x+p}{x-1}$$

$$y(x-1)=1(2x+p)$$

$$xy-y=2x+p$$

$$xy-2x=y+p$$

$$x(y-2)=y+p$$

$$x=\frac{y+p}{y-2}$$

Replace $x$ by $y$:

$$y=\frac{x+p}{x-2}$$

$$f^{-1}(x)=\frac{x+p}{x-2},\ x\neq 2$$

**(ii)** If $f^{-1}(8)=5$, find the value of $p$.

**SOLUTION:**

**Data:** $f^{-1}(8)=5$

**Required to find:** $p$

**Solution:**

If $f^{-1}(8)=5$

Then, $\frac{8+p}{8-2}=5$

$\frac{8+p}{6}=5$

$8+p=30$

$p=30-8$

$=22$

**(b)** Given that the remainder when $f(x)=x^3-x^2-ax+b$ is divided by $x+1$ is 6, and that $x-2$ is factor, determine the values of *a* and *b*.

**SOLUTION:**

**Data:** $f(x)=x^3-x^2-ax+b$ when divided by $x+1$ gives a remainder of 6 and $x-2$ is factor of $f(x)$.

**Required to find:** The value of *a* and of *b*

**Solution:**

Recall: If $f(x)$ is a polynomial and $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$. If $f(a)=0$, then $(x-a)$ is factor of $f(x)$.

Hence, $f(-1)=6$ and $f(2)=0$

When $f(-1)=6$

$(-1)^3-(-1)^2-a(-1)+b=6$

$-1-1+a+b=6$

$a+b=8 \quad \ldots❶$

When $f(2)=0$

$(2)^3-(2)^2-a(2)+b=0$

$8-4-2a+b=0$

$-2a+b=-4$

$2a-b=4 \quad \ldots❷$

Equation ❶ + Equation ❷:

$a+b=8$

$2a-b=4$

$3a=12$

$a=4$

Substitute $a = 4$ into equation ❶:

$4 + b = 8$

$b = 4$

$\therefore a = 4$ and $b = 4$.

**(c)** The values of the variables $P$ and $x$ in Table 1 obtained from an experiment are thought to obey a law of the form $P = Ax^{-k}$.

**TABLE 1**

| $x$ | 1.58 | 2.51 | 3.98 | 6.30 | 10.0 |
|---|---|---|---|---|---|
| $P$ | 121.5 | 110.6 | 106.2 | 99.1 | 93.8 |

**(i)** Use logarithms to reduce the equation to linear form.

**SOLUTION:**
**Data:** Table showing the values of variables $P$ and $x$ related by the equation $P = Ax^{-k}$.
**Required To Reduce:** $P = Ax^{-k}$ to linear form
**Solution:**

$P = Ax^{-k}$

Take lg:

$\lg P = \lg\left(Ax^{-k}\right)$

$\lg P = \lg A + \lg x^{-k}$

$\lg P = \lg A - k \lg x$

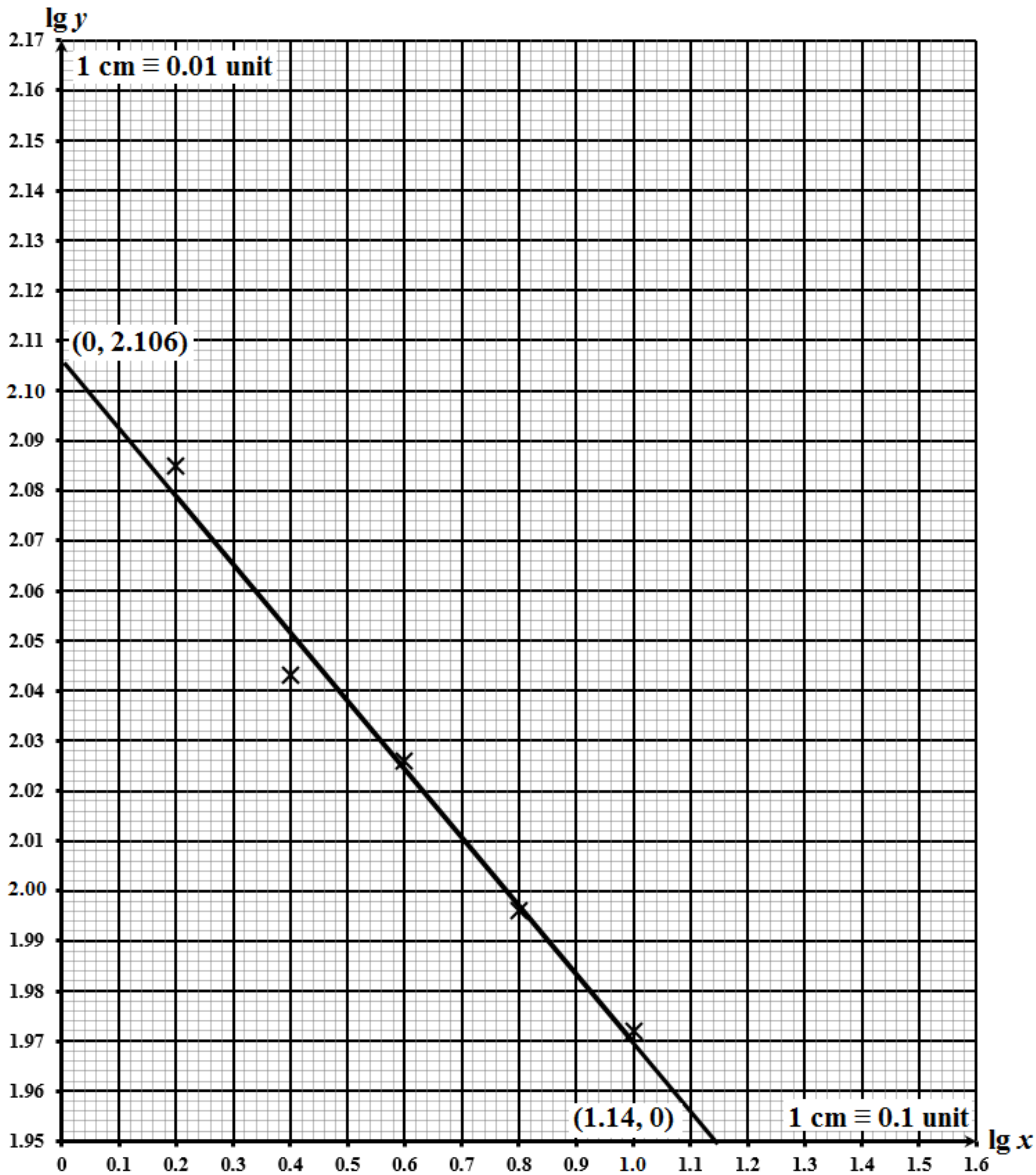
$\lg P = -k \lg x + \lg A$ which is of the form $Y = mX + C$, where $Y = \lg P$ (a variable), $m = -k$ (a constant), $X = \lg x$ (a variable) and $C = \lg A$ (a constant).

**(ii)** Using a suitable scale, plot the best fit line of the equation in (c) (i) on the graph paper provided. Use the space below to show your working.

**SOLUTION:**
**Required to plot:** The best fit line of the equation in (c) (i)
**Solution:**
If $\lg P$ vs $\lg x$ is drawn, a straight line would be obtained with a gradient of $-k$ and intercept on the vertical axis of $\lg A$.

| $\lg x$ | 0.20 | 0.40 | 0.60 | 0.80 | 1.00 |
|---|---|---|---|---|---|
| $\lg P$ | 2.085 | 2.043 | 2.026 | 1.996 | 1.972 |

We plot these points on the graph page provided.



**(iii)** Hence, estimate the constants $A$ and $k$.

**SOLUTION:**
**Required To Estimate:** $A$ and $k$
**Solution:**

$$\text{Gradient} = \frac{2.106 - 1.95}{0 - 1.14}$$
$$= \frac{0.156}{-1.14}$$
$$= -0.1368$$
$$\therefore -k = -0.1368$$
$$k = 0.1368$$
$$k = 0.14 \text{ (correct to 2 decimal places)}$$

The intercept on the vertical axis is 2.106.
Hence, $\lg A = 2.106$

$$A = \text{antilog}(2.106)$$
$$= 127.643$$
$$= 127.64 \text{ (correct to 2 decimal places)}$$

**2. (a)** The quadratic equation $2x^2 + 6x + 7 = 0$ has roots $\alpha$ and $\beta$.
Calculate the value of $\frac{1}{\alpha} + \frac{1}{\beta}$.

**SOLUTION:**
**Data:** $\alpha$ and $\beta$ are the roots of $2x^2 + 6x + 7 = 0$.
**Required to calculate:** $\frac{1}{\alpha} + \frac{1}{\beta}$
**Calculation:**
If $ax^2 + bx + c = 0$
$\div a$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

If the roots are $\alpha$ and $\beta$ then

$$(x - \alpha)(x - \beta) = 0$$
$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Equating coefficients:

$$\left(\alpha+\beta\right)=-\frac{b}{a}$$

$$\alpha\beta=\frac{c}{a}$$

Hence, if $\alpha$ and $\beta$ are the roots of $2x^2+6x+7=0$, then

$$\alpha+\beta=\frac{-(6)}{2}$$
$$=-3$$

and

$$\alpha\beta=\frac{7}{2}$$

$$\frac{1}{\alpha}+\frac{1}{\beta}$$

$$\frac{\beta+\alpha}{\alpha\beta}=\frac{\alpha+\beta}{\alpha\beta}$$

$$=\frac{-3}{\frac{7}{2}}$$

$$=-\frac{6}{7}$$

(b) Determine the range of values of $x$ for which, $\frac{2x+3}{x+1}\geq 0$.

**SOLUTION:**

**Required to find:** The range of values of $x$ for which $\frac{2x+3}{x+1}\geq 0$.

**Solution:**

$$\frac{2x+3}{x+1}\geq 0$$

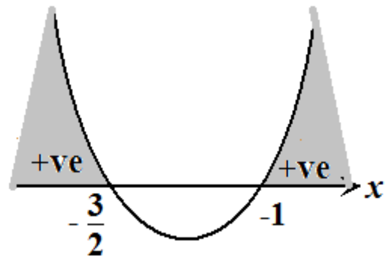
$$\times\left(x+1\right)^2$$

$$\left(2x+3\right)\left(x+1\right)\geq 0$$

Let $y=\left(2x+3\right)\left(x+1\right)$

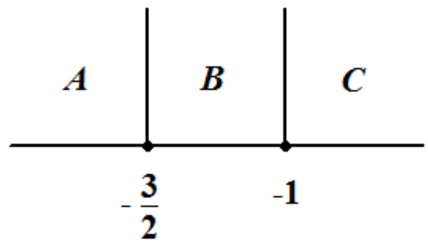
If we let $y=0$ we see that the curve cuts the $x-$axis at $-\frac{3}{2}$ and at $-1$.

The coefficient of $x^2>0$ in the quadratic, therefore the curve is a parabola and has a minimum point.




Hence, $\frac{2x+3}{x+1} \geq 0$ for $\{x : x \geq -1\} \cup \left\{x : x \leq -\frac{3}{2}\right\}$

Alternatively,




Test a point in a region, say $x = 0$ in *C*.

$(2(0)+3)(0+1) \geq 0$

$3 \geq 0$ (True)

Therefore, solutions are Region *C*, not *B* and Region *A*.

That is, $\{x : x \geq -1\} \cup \left\{x : x \leq -\frac{3}{2}\right\}$

**(c)** An accountant is offered a five-year contract with an annual increase. The accountant earned a salary of \$53 982.80 and \$60 598.89 in the third and fifth years respectively. If the increase follows a geometric series, calculate

**(i)** the amount paid in the first year

**SOLUTION:**

**Data:** An accountant is paid \$53 982.80 in the third year and \$60 598.89 in the fifth year of five year contract. The salary increase follows a geometric series.

**Required to calculate:** The amount the account earned in the first year

**Calculation:**

Let the 1st term of a G.P. $=a$, number of terms $=n$ and the common ratio $=r$

$T_n = n^{\text{th}}$ term

$= ar^{n-1}$

Hence,

$T_3 = ar^{3-1}$

$ar^2 = 53982.80$

And,

$T_5 = ar^{5-1}$

$ar^4 = 60598.89$

$$\frac{T_5}{T_3} = \frac{ar^4}{ar^2}$$

$$r^2 = \frac{60598.89}{53982.80}$$

Recall: $ar^2 = 53982.80$

So, $$a = \frac{53982.80}{r^2}$$

$$= \frac{53982.80}{\frac{60598.89}{53982.80}}$$

$$= 48089.04$$

Therefore, the amount paid in the first year is \$48 089.04

**(ii)** the TOTAL salary earned at the end of the contract.

**SOLUTION:**
**Required to calculate:** The total salary earned at the end of the five year contract
**Calculation:**
The amount paid at the end of the five-year contract is the sum of the first 5 terms of the G.P.

$$r = \sqrt{\frac{60598.89}{53982.80}}$$

$= 1.06$ (correct to 2 decimal places)

$$S_n = \frac{a(r^n - 1)}{r - 1},\ |r| > 1$$

$$S_5 = \frac{48\,089.04(1.06^5 - 1)}{1.06 - 1}$$

$$= \frac{48\,089.04(1.338\,23 - 1)}{0.06}$$

$$= \$271085.93$$

**SECTION II**

**Answer BOTH questions.**

**ALL working must be clearly shown.**

**3. (a)** A circle $C$ has an equation $x^2 + y^2 + 4x - 2y - 20 = 0$.

**(i)** Express the equation in the form $(x+f)^2 + (y+g)^2 = r^2$.

**SOLUTION**:
**Data:** Circle $C$ has equation $x^2 + y^2 + 4x - 2y - 20 = 0$.
**Required to express:** The equation of $C$ in the form of $(x+f)^2 + (y+g)^2 = r^2$
**Solution:**

$$x^2 + y^2 + 4x - 2y - 20 = 0$$

$$x^2 + 4x + y^2 - 2y - 20 = 0$$

$$(x+2)^2 - 4 + (y-1)^2 - 1 - 20 = 0$$

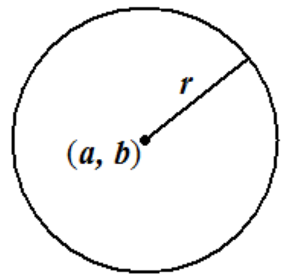
$$(x+2)^2 + (y-1)^2 = 20 + 1 + 4$$

$(x+2)^2 + (y-1)^2 = (5)^2$ is of the form $(x+f)^2 + (y+g)^2 = r^2$, where $f = 2, g = -1$ and $r = 5$.

**(ii)** State the coordinates of the center and the value of the radius of circle $C$.

**SOLUTION:**
**Required to find:** The coordinates of the center and the radius of $C$
**Solution:**
The equation of the circle is $(x+2)^2 + (y-1)^2 = (5)^2$.

Recall: If the equation of a circle is of the form $(x+a)^2+(y+b)^2=r^2$, then the center is $(a, b)$ and the radius is $r$.




$\therefore$ The centre of the circle is $(-(2), 1)=(-2, 1)$ and the radius is $\sqrt{(5)^2}=5$ units.

**(iii)** Determine the points of intersection of circle $C$ and the equation $y=4-x$.

**SOLUTION:**
**Required to find:** The points of intersection of the circle and the line $y=4-x$.
**Solution:**
To determine the points of intersection of $C$ and $y=4-x$, we solve both equations simultaneously.
Substitute $y=4-x$ into the equation of the circle.

$$(x+2)^2+(4-x-1)^2-(5)^2=0$$
$$(x+2)^2+(3-x)^2-25=0$$
$$x^2+4x+4+9-6x+x^2-25=0$$
$$2x^2-2x-12=0$$

$\div 2$

$$x^2-x-6=0$$
$$(x-3)(x+2)=0$$
$$\therefore x=3 \text{ or } -2$$

When $x=3$
$y=4-3$
$=1$

When $x=-2$

$y = 4 - (-2)$
$\quad = 6$

$\therefore$ The points of intersection are $(3, 1)$ and $(-2, 6)$.

**(b)** Given that $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{q} = \mathbf{i} + 5\mathbf{j}$, determine

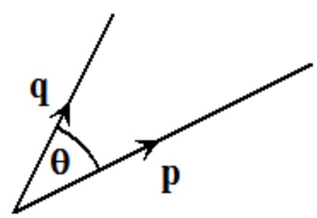
**(i)** the product of the vectors, **p** and **q**
(presumably this means the dot product)

**SOLUTION:**
**Data:** $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{q} = \mathbf{i} + 5\mathbf{j}$
**Required to find:** The product of **p** and **q**
**Solution:**
The product of **p** and **q** is $\mathbf{p}.\mathbf{q}$

$\mathbf{p}.\mathbf{q} = (2 \times 1) + (3 \times 5)$
$\quad = 2 + 15$
$\quad = 17$

**(ii)** the angle between the two vectors

**SOLUTION:**
**Required to find:** The angle between vectors **p** and **q**
**Solution:**
Let the angle between **p** and **q** be $\theta$.



$$p.q = |p||q|\cos\theta$$

$$17 = \sqrt{(2)^2 + (3)^2} \times \sqrt{(1)^2 + (5)^2} \cos\theta$$

$$17 = \sqrt{13}\sqrt{26}\cos$$

$$\cos\theta = \frac{17}{\sqrt{13}\sqrt{26}}$$

$$\theta = \cos^{-1}\left(\frac{17}{\sqrt{13}\sqrt{26}}\right)$$

$$= 22.35^\circ$$

$$= 22.4^\circ \text{ (correct to the nearest } 0.1^\circ)$$

**4. (a)** Figure 1 shows a plot of land, ABCD (**not drawn to scale**). Section ABC is used for building and the remainder for farming. The radius BC is 10 m and angle BCD is a right angle.

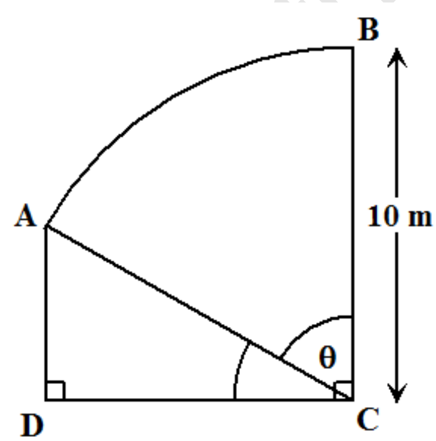


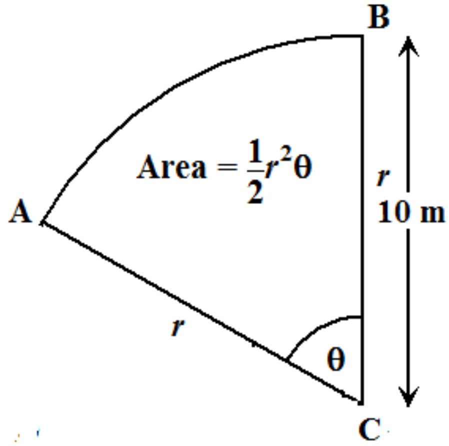

**Figure 1**

**(i)** If the building space is $\frac{50\pi}{3}$ m$^2$, calculate the angle ACB in radians.

**SOLUTION**:
**Data:** Diagram showing a plot of land. Section ABC is used for building and has an area of $\frac{50\pi}{3}$ m$^2$. The remainder is used for farming. BC $10$ m.
**Required to calculate: Angle** ACB in radians
**Calculation:**
(Presumably the region ACB is a sector)

Area of ACB $= \frac{50\pi}{3}$ (data)

Area of a sector $=\frac{1}{2}r^2\theta$, where r is radius and $\theta$ is the angle in radians




Hence,

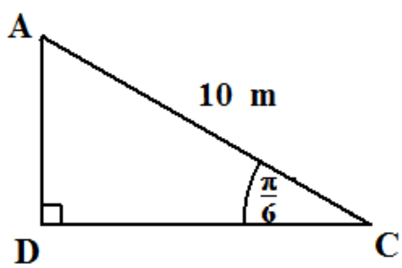
$$\frac{1}{2}(10)^2\theta=\frac{50\pi}{3}$$

$$50\theta=\frac{50\pi}{3}$$

$$\theta=\frac{\pi}{3}\text{ radians}$$

**(ii)** Working in radians, calculate the area used for farming.

**SOLUTION:**
**Required to calculate:** The area of land used for farming
**Calculation:**
Region ACD is used for farming.




$$A\hat{C}D=\frac{\pi}{2}-\frac{\pi}{3}$$

$$=\frac{\pi}{6}$$

$$\frac{DC}{10} = \cos\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{2}$$
$$DC = \frac{10\sqrt{3}}{2}$$
$$= 5\sqrt{3}$$

$$\text{Area of } \Delta ACD = \frac{1}{2}(10)\left(5\sqrt{3}\right)\sin\left(\frac{\pi}{6}\right)$$
$$= \frac{1}{2}\times 10\times 5\sqrt{3}\times\frac{1}{2}$$
$$= \frac{25\sqrt{3}}{2}\ \text{m}^2 \text{ (in exact form)}$$

**(b)** Given that

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\cos\frac{\pi}{3} = \frac{1}{2} \text{ and}$$
$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

show without using a calculate that

$$\frac{\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right)}{\sin\frac{2\pi}{3}} = \frac{\sqrt{2}+\sqrt{6}}{2\sqrt{3}}$$

**SOLUTION:**

**Data:** $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

**Required to show:** $\dfrac{\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right)}{\sin\frac{2\pi}{3}} = \dfrac{\sqrt{2}+\sqrt{6}}{2\sqrt{3}}$

**Proof:**

Using the compound angle formula

$$\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right)=\cos\frac{\pi}{4}\cos\frac{\pi}{3}+\sin\frac{\pi}{4}\cos\frac{\pi}{3}$$
$$=\frac{\sqrt{2}}{2}\times\frac{1}{2}+\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}$$
$$=\frac{\sqrt{2}}{4}+\frac{\sqrt{2}\sqrt{3}}{4}$$

$$\sin\frac{2\pi}{3}\equiv\sin\left(\pi-\frac{\pi}{3}\right)$$
$$=\sin\pi\cos\frac{\pi}{3}-\cos\pi\sin\frac{\pi}{3}$$
$$=\left(0\times\frac{1}{2}\right)-\left(-1\times\frac{\sqrt{3}}{2}\right)$$
$$=\frac{\sqrt{3}}{2}$$

$$\frac{\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right)}{\sin\frac{2\pi}{3}}=\frac{\frac{\sqrt{2}+\sqrt{2}\sqrt{3}}{4}}{\frac{\sqrt{3}}{2}}$$
$$=\frac{\sqrt{2}+\sqrt{6}}{\not{4}_2}\times\frac{\not{2}}{\sqrt{3}}$$
$$=\frac{\sqrt{2}+\sqrt{6}}{2\sqrt{3}}$$

**Q.E.D.**

**(c)** Prove the identity

$$1-\frac{\cos^2\theta}{1+\sin\theta}=\sin\theta$$

**SOLUTION:**

**Required to prove:** $1-\frac{\cos^2\theta}{1+\sin\theta}=\sin\theta$

**Proof:**

Simplify L.H.S.:

$$1-\frac{\cos^2\theta}{1+\sin\theta}$$

Recall: $\sin^2\theta+\cos^2\theta=1$

$$\cos^2\theta = 1 - \sin^2\theta$$

This is a difference of two squares:

$\cos^2\theta = (1-\sin\theta)(1+\sin\theta)$

So,

$$1 - \frac{\cos^2\theta}{1+\sin\theta} = 1 - \frac{(1-\sin\theta)\cancel{(1+\sin\theta)}}{\cancel{1+\sin\theta}}$$

$$= 1 - (1-\sin\theta)$$

$$= 1 - 1 + \sin\theta$$

$$= \sin\theta = RHS$$

**Q.E.D.**

## SECTION III

**Answer BOTH questions.**

**ALL working must be clearly shown.**

**5. (a)** Differentiate the expression $(1+2x)^3(x+3)$ with respect to *x*, simplifying your answer

**SOLUTION:**

**Required to differentiate:** $(1+2x)^3(x+3)$ with respect to *x*.

**Solution:**

Let $y = (1+2x)^3(x+3)$

$y$ is of the form $y = uv$, where

$u = (1+2x)^3$

$v = x + 3 \Rightarrow \frac{dv}{dx} = 1$

Let $t = 1 + 2x \Rightarrow \frac{dt}{dx} = 2$

So, $u = t^3 \Rightarrow \frac{du}{dt} = 3t^2$

Apply the chain rule:

$$\frac{du}{dx} = \frac{du}{dt} \times \frac{dt}{dx}$$

$$= 3t^2 \times 2$$

$$= 6t^2$$

Re: $t = 1 + 2x$

$$\frac{du}{dx} = 6(1+2x)^2$$

Apply the product rule:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= (x+3)6(1+2x)^2 + (1+2x)^3(1)$$
$$= (1+2x)^2\{6(x+3)+(1+2x)\}$$
$$= (1+2x)^2\{6x+18+1+2x\}$$
$$= (1+2x)^2(8x+19)$$

**(b)** The point $P(-2, 0)$ lies on the curve $y = 3x^3 + 2x^2 - 24x$. Determine the equation of the normal to the curve at point *P*.

**SOLUTION:**
**Data:** $P(-2, 0)$ lies on the curve $y = 3x^3 + 2x^2 - 24x$.
**Required To Find:** The equation of the normal to the curve at *P*
**Solution:**
(As a point of interest, the point *P*, (2, 0) does NOT lie on the curve)

$$y = 3x^3 + 2x^2 - 24x$$

Gradient function, $\frac{dy}{dx} = 3\left(3x^{3-1}\right) + 2\left(2x^{2-1}\right) - 24$

$$= 9x^2 + 4x - 24$$

The gradient of the tangent at $P = 9(-2)^2 + 4(-2) - 24$

$$= 36 - 8 - 34$$
$$= 4$$

Hence, the gradient of the normal at $P = -\frac{1}{4}$ (The products of the gradients of perpendicular lines $= -1$)

The equation of the normal at *P* is

$$\frac{y-0}{x-(-2)} = -\frac{1}{4}$$
$$4(y) = -1(x+2)$$
$$4y = -x - 2$$

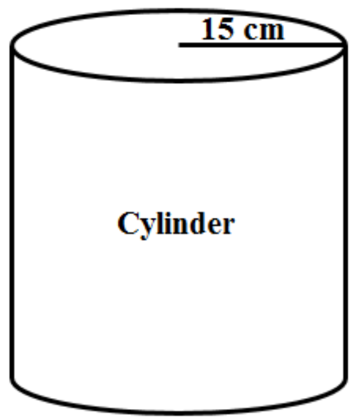
$4y + x + 2 = 0$ or any other equivalent form.

**(c)** Water is poured into a cylindrical container of radius 15 cm. The height of the water increases at a rate of 2 cms$^{-1}$. Given that the formula for the volume of a cylinder is $\pi r^2 h$, determine the rate of increase of the volume of water in the container in terms of $\pi$.

**SOLUTION:**
**Data:** The rate of increase of the height of water in a cylindrical container is 2 cms$^{-1}$. Volume of a cylinder is $\pi r^2 h$.

**Required to find:** The rate of increase of the volume of water in the container

**Solution:**




Let the height of water be $h$ cm and time be $t$ s and $V$ be the volume of the cylinder.

$V = \pi r^2 h$

$V = \pi (15)^2 h$

$V = 225\pi h$

Since the height increases at the rate of 2 cms$^{-1}$, then $\frac{dh}{dt} = +2$ cms$^{-1}$

Required to calculate $\dfrac{dV}{dt}$

By the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$
$$= 225\pi \times 2$$
$$= 450\pi \text{ cm}^3\text{s}^{-1} \text{ (Positive} \Rightarrow \text{increase in the rate of volume)}$$

**6. (a)** Show that $\int_0^{\frac{\pi}{4}} (\sin x + 4\cos x)\, dx = \dfrac{3\sqrt{2}+2}{2}$.

**SOLUTION:**

**Required to show:** $\int_0^{\frac{\pi}{4}} (\sin x + 4\cos x)\, dx = \frac{3\sqrt{2}+2}{2}$

**Proof:**

$\int (\sin x + 4\cos x)\, dx = -\cos x + 4(\sin x) + C$ where $C$ is a constant

$$\therefore \int_0^{\frac{\pi}{4}} (\sin x + 4\cos x)\, dx = [-\cos x + 4\sin x]_0^{\frac{\pi}{4}}$$
$$= \left(-\cos\frac{\pi}{4} + 4\sin\frac{\pi}{4}\right) - (-\cos(0) + 4\sin(0))$$
$$= -\frac{1}{\sqrt{2}} + 4\left(\frac{1}{\sqrt{2}}\right) - (-(1) + 4(0))$$
$$= \frac{3}{\sqrt{2}} - (-1)$$
$$= \frac{3}{\sqrt{2}} + 1$$
$$= \frac{3}{\sqrt{2}} + \frac{1}{1}$$
$$= \frac{3+\sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{3\sqrt{2}+2}{2}$$

**Q.E.D.**

**(b)** Determine the equation of a curve whose gradient function $\frac{dy}{dx} = x + 2$, and which passes through the point $P(2, 3)$.

**SOLUTION:**

**Data:** Curve has gradient function $\frac{dy}{dx} = x + 2$ and passes through $P(2, 3)$.

**Required to find:** The equation of the curve

**Solution:**

$$\frac{dy}{dx} = x + 2$$

$\therefore$ The equation of the curve is $y = \int (x+2)\, dx$

$$y = \int (x+2)\, dx$$

$y = \frac{x^2}{2} + 2x + C$ (where $C$ is the constant of integration)

$P(2, 3)$ lies on the curve.
When $x = 2$, $y = 3$ satisfies the equation

$$3 = \frac{(2)^2}{2} + 2(2) + C$$
$$3 = 2 + 4 + C$$
$$C = -3$$

$\therefore$ The equation of the curve is $y = \frac{x^2}{2} + 2x - 3$.

**(c)** Evaluate $\int_1^2 (4-x)^2 \, dx$.

**SOLUTION:**

**Required to evaluate:** $\int_1^2 (4-x)^2 \, dx$

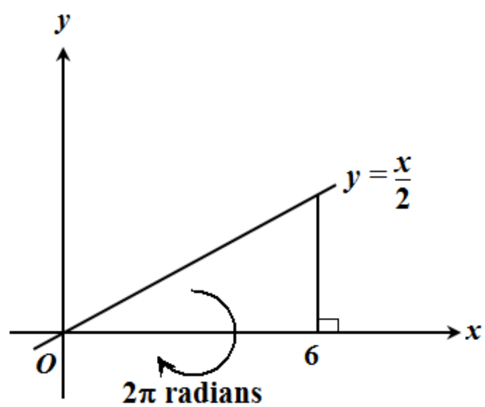
**Solution:**

$$\int_1^2 (4-x)^2 \, dx = \int_1^2 \left(16 - 8x + x^2\right) dx$$
$$= \left[16x - 4x^2 + \frac{x^3}{3}\right]_1^2$$
$$= \left[16(2) - 4(2)^2 + \frac{(2)^3}{3}\right] - \left[16(1) - 4(1)^2 + \frac{(1)^3}{3}\right]$$
$$= \left(32 - 16 + 2\frac{2}{3}\right) - \left(16 - 4 + \frac{1}{3}\right)$$
$$= 18\frac{2}{3} - 12\frac{1}{3}$$
$$= 6\frac{1}{3} \text{ units}^2$$

**(d)** Calculate the volume of the solid formed when the area enclosed by the straight line $y = \frac{x}{2}$ and the $x$ – axis for $x = 0$ to $x = 6$ is rotated through $2\pi$ about the $x$ – axis.

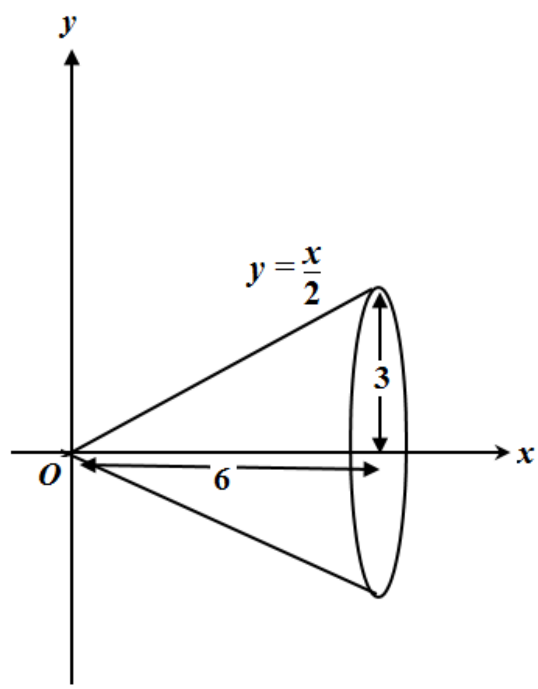
**SOLUTION:**

**Data:** A solid is formed by rotating the area enclosed by the line $y = \frac{x}{2}$ and the $x$ – axis for $x = 0$ to $x = 6$ through $2\pi$ about the $x$ – axis.
**Required to calculate:** The volume of the solid formed
**Calculation:**




When $x = 6$

$$y = \frac{6}{2}$$
$$= 3$$




The solid generated is a cone of radius 3 units and height 6 units.

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \pi \times (3)^2 \times 6$$
$$= 18\pi \text{ units}^3$$

**Alternative Method:**

$$V = \pi \int_{x_1}^{x_2} y^2 \; dx$$
$$= \pi \int_0^6 \left(\frac{x}{2}\right)^2 \; dx$$
$$= \pi \int_0^6 \frac{x^2}{4} \; dx$$
$$= \pi \left[\frac{x^3}{12}\right]_0^6$$
$$= \pi \left\{\frac{(6)^3}{12} - \frac{(0)^3}{12}\right\}$$
$$= 18\pi \text{ units}^3$$

## SECTION IV

**Answer only ONE question.**

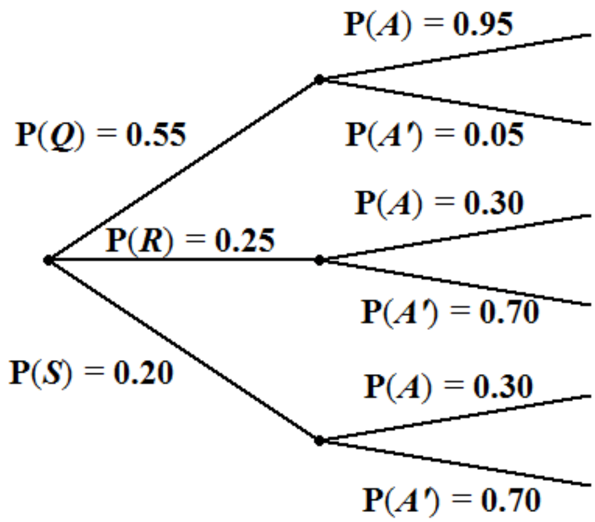
**ALL working must be clearly shown.**

**7. (a)** The probability of a final-year college student receiving a reply for an internship programme from three accounting firms, $Q$, $R$ and $S$, is 0.55, 0.25 and 0.20 respectively. The probability that a student receives a reply from firm $Q$ and is accepted is 0.95. The probability that a student receives a reply from firms $R$ and $S$ and is accepted is 0.30 for each of them.

**(i)** Draw a tree diagram to illustrate the information above.

**SOLUTION:**
**Data:** The probability a student receives a reply for an internship programme from accounting firms $Q$, $R$ and $S$ are 0.55, 0.25 and 0.20 respectively. The probability that a student receives a reply from firm $Q$ and is accepted is 0.95. The probability that a student receives a reply from firms $R$ and $S$ and is accepted is 0.30 for each of them.

**Required to draw:** A tree diagram to illustrate the information given
**Solution:**
Let $A$ represent the event that a student is accepted.




**(ii)** Determine the probability that the student will be accepted for an internship programme.

**SOLUTION:**
**Required to find:** The probability that a student will be accepted for an internship programme
**Solution:**

$$
\begin{aligned}
P(A) &= P(Q \text{ and } A \text{ or } R \text{ and } A \text{ or } S \text{ and } A) \\
&= P(Q \text{ and } A) + P(R \text{ and } A) + P(S \text{ and } A) \\
&= (0.55 \times 0.95) + (0.25 \times 0.3) + (0.2 \times 0.3) \\
&= 0.5225 + 0.075 + 0.06 \\
&= 0.6575
\end{aligned}
$$

**(b)** Table 2 shows the length, in cm, of 20 spindles prepared by a carpenter to build a railing for an existing staircase.

**TABLE 2**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1.5 | 3.2 | 6.1 | 9.4 | 11.0 | 12.6 | 17.0 | 18.5 | 20.2 | 24.4 |
| 25.2 | 25.2 | 28.3 | 28.8 | 29.1 | 30.4 | 32.5 | 34.6 | 38.3 | 38.4 |

Determine

**(i)** the mean length

**SOLUTION:**
**Data:** Table showing the lengths, in cm, of 20 spindles prepared by a carpenter.
**Required to find:** The mean length
**Solution:**

Mean length, $\bar{x} = \frac{\sum x}{n}$, where $x$ is the length of a spindle and $n$ is the number of spindles.

$$\bar{x} = \begin{pmatrix} 1.5+3.2+6.1+9.4+11.0+12.6+17.0+18.5+20.2+24.4 \\ +25.2+25.2+28.3+28.8+29.1+30.4+32.5+34.6+38.3+38.4 \end{pmatrix} \div 20$$
$$= 434.7 \div 20$$
$$= 21.735 \text{ cm}$$

**(ii)** the modal length

**SOLUTION:**
**Required to find:** The modal length
**Solution:**
There is only one length which occurs more than once and which is 25.2. Hence, the modal length is 25.2 cm.

**(iii)** the median length

**SOLUTION:**
**Required to find:** The median length
**Solution:**
When arranged in ascending order of magnitude there will be two middle values as the number of measurements is an even number
$10^{\text{th}}$ length $= 24.4$
$11^{\text{th}}$ length $= 25.2$

$$\therefore \text{Median length } = \frac{24.4+25.5}{2}$$
$$= 24.8 \text{ cm}$$

**(iv)** the interquartile range for the data

**SOLUTION:**
**Required to find:** The interquartile range for the data.
**Solution:**

The 5th and 6th values are 11.0 and 12.6
Hence,
$$Q_1 = \frac{11.0 + 12.6}{2} = 11.8$$
The 15th and 16th values are 29.1 and 30.4
Hence,
$$Q_3 = \frac{29.1 + 30.4}{2} = 29.75$$

The interquartile range (I.Q.R.) $= \text{Upper quartile} - \text{Lower Quartile}$
$$= Q_3 - Q_1$$
$$= 29.75 - 11.8$$
$$= 17.95 \text{ cm}$$

**(c)** A school cafeteria sells 20 chicken patties, 10 lentil patties and 25 saltfish patties daily. On a particular day, the first student ordered 2 patties but did not specify the type. The vendor randomly selects 2 patties.

**(i)** Calculate the probability that the first patty selected was saltfish.

**SOLUTION:**
**Data:** A school cafeteria sells 20 chicken patties, 10 lentil patties and 25 saltfish patties daily. The vendor selects two patties at random to sell to a customer on a particular day.
**Required to find:** The probability that the first patty was saltfish
**Solution:**

Number of chicken patties ($C$) $= 20$
Number of lentil patties ($L$) $= 10$
Number of saltfish patties ($S$) $= \underline{25}$
$\underline{55}$

$P(S) = \frac{20}{55}$ $\quad P(L) = \frac{10}{55}$ $\quad P(C) = \frac{20}{55}$

If the first patty is saltfish, the second can be either saltfish or not saltfish.
$$P(S \text{ and } S) = \frac{25}{55} \times \frac{24}{54}$$
$$P(S \text{ and } S') = \frac{25}{55} \times \frac{30}{54}$$
$$P(S) = \left(\frac{25}{55} \times \frac{24}{54}\right) + \left(\frac{25}{55} \times \frac{30}{54}\right) = \frac{1350}{2970} = \frac{5}{11}$$

**(ii)** Given that the first patty was saltfish, calculate the probability that the second patty was NOT saltfish.

**SOLUTION:**
**Data:** The first patty selected was saltfish.
**Required to find:** The probability that the second patty was not saltfish
**Solution:**
If the first patty is saltfish, number of saltfish patties remaining $= 25 - 1$
$= 24$

Number of patties remaining $= 55 - 1$
$= 54$

Number of patties remaining that were not saltfish $= 54 - 24$
$= 30$

$$P(\text{Second patty is not saltfish}) = \frac{30}{54}$$

$$P(\text{S and S}') = \frac{25}{55} \times \frac{30}{54} = \frac{25}{99}$$

**8. (a)** The displacement, $s$, of a particle from a fixed point $O$, is given by $s = t^3 - \frac{5}{2}t^2 - 2t$ metres at time, $t$ seconds.

**(i)** Determine the velocity of the particle at $t = 3.5$ s, clearly starting the correct unit.

**SOLUTION:**
**Data:** The displacement, $s$, of a particle from a point O is $s = t^3 - \frac{5}{2}t^2 - 2t$ metres at time, $t$ seconds.
**Required to find:** The velocity of the particle at $t = 3.5$.
**Solution:**
Let the velocity at time $t$ be $v$ ms$^{-1}$.

$$v = \frac{ds}{dt}$$
$$= 3t^{3-1} - \frac{5}{2}\left(2t^{2-1}\right) - 2$$
$$= 3t^2 - 5t - 2$$

When $t = 3.5$

$v = 3(3.5)^2 - 5(3.5) - 2$
$= 17.25 \text{ ms}^{-1}$

**(ii)** If the particle is momentarily at rest, find the time, *t*, at this position.

**SOLUTION:**
**Required to find:** The time that the particle is momentarily at rest
**Solution:**
At instantaneous rest, $v = 0$.
Let $3t^2 - 5t - 2 = 0$
$(3t+1)(t-2) = 0$

$$t = 2 \text{ or } -\frac{1}{3}$$

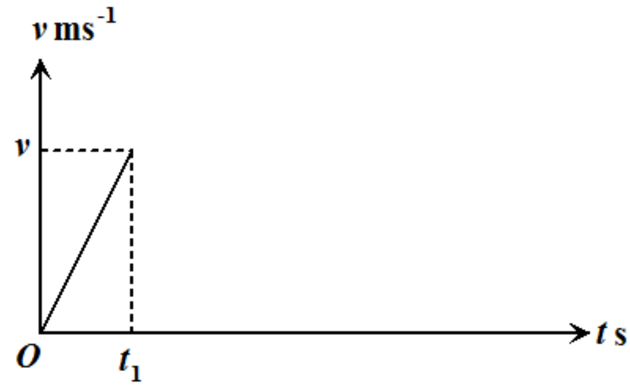
*t* cannot be negative.
So $t = 2$ only at this position of momentary rest
(As a point of interest, the velocity of the vehicle is NOT zero for any value of *t* when *t* lies between 0 and 2 (exclusive) or for *t* greater than 2. The particle is therefore at instantaneous rest at *t* = 2 and NOT momentary rest.)

**(b)** A vehicle accelerates uniformly from rest for 75 m and then travels for another 120 m at its maximum speed. The vehicle later stops at a traffic light. The distance from rest to the traffic light is 240 m and the time for the journey is 15 seconds.

**(i)** In the space below, sketch a velocity – time graph to illustrate the motion of the vehicle.

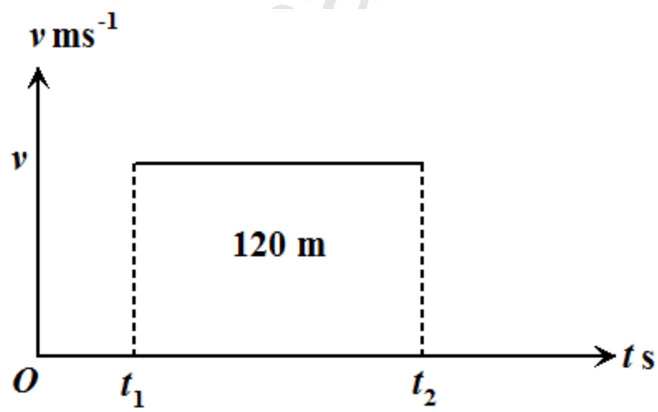
**SOLUTION:**
**Data:** A vehicle accelerates uniformly from rest for 75 m and then travels for another 120 m at its maximum speed. It then stops at a traffic light 240 m away. The time for the journey is 15 seconds.
(As a point of interest-a vehicle cannot travel)

**Required to sketch:** The velocity – time graph to illustrate the journey
**Solution:**
Let us look at the journey of the vehicle in different phases
Phase 1:



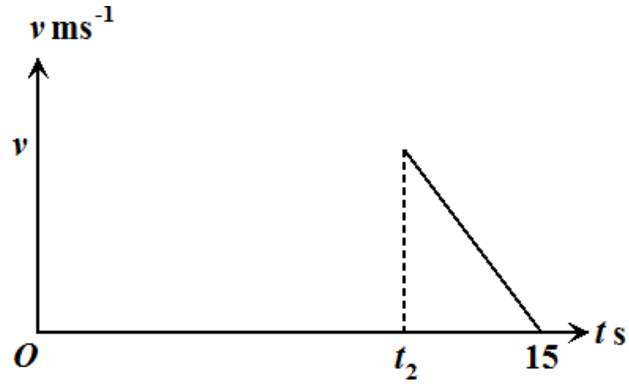

The straight line 'branch' implies uniform acceleration
Let the maximum velocity reached $= v$
Let the time taken $= t_1$
The area under the graph $= 75$
(Assuming that the maximum speed was attained after covering the 75 m distance)

Phase 2:



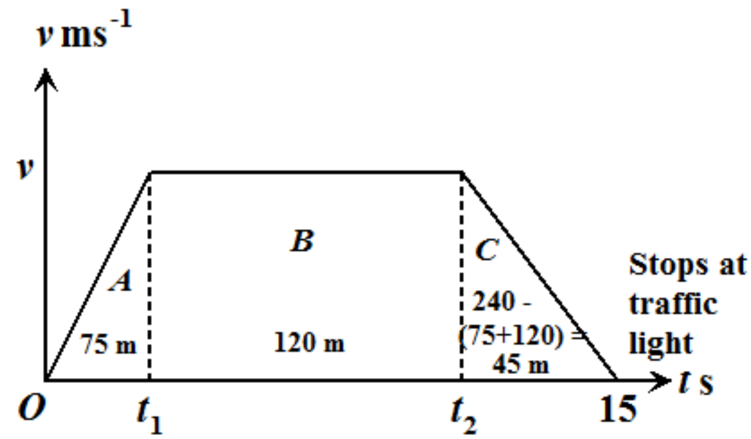

The horizontal branch indicates constant speed. Let the time for this phase be from $t_1$ to $t_2$

Phase 3:




The straight line 'branch' is assuming uniform deceleration as the vehicles proceeds from a constant velocity to stopping at the traffic light. If the deceleration was NOT constant, the branch would be a curve.

The completed velocity – time graph looks like:




**(ii)** Calculate the length of time the vehicle maintains constant speed.

**SOLUTION**:
**Required to calculate:** The length of time the vehicles travels at constant speed
**Calculation:**
Let the regions *A*, *B* and *C* be as shown on the diagram.
Region *A*: Vehicles accelerates for a period of *t*, to reach a maximum speed of *v*.

$$\therefore \frac{vt_1}{2} = 75$$

$$vt_1 = 150 \qquad \ldots❶$$

Region *B*: Constant speed from $t_1$ to $t_2$ of $v$ $\text{ms}^{-1}$.

$\therefore v(t_2 - t_1) = 120$ …❷

$vt_2 - vt_1 = 120$

Region $C$: Decelerates from rest from $t_2$ to 15 s, that is, for $(15 - t_2)$ s.

Hence, $\dfrac{v \times (15 - t_2)}{2} = 45$

$v(15 - t_2) = 90$ …❸

$15v - vt_2 = 90$

Substitute equation ❶ into equation ❷:

$vt_2 - 150 = 120$

$vt_2 = 270$ …❹

Substitute equation ❹ into equation ❸:

$15v - 270 = 90$

$15v = 90 + 270$

$= 360$

$v = \dfrac{360}{15}$

$= 24 \text{ ms}^{-1}$

Substitute $v = 24$ into equation ❶:

$24t_1 = 150$

$t_1 = \dfrac{150}{24}$

$= 6\dfrac{1}{4}$ s

Substitute $v = 24$ into equation ❸:

$24(15 - t_2) = 90$

$15 - t_2 = \dfrac{90}{24}$

$= 3\dfrac{3}{4}$

$t_2 = 15 - 3\dfrac{3}{4}$

$= 11\dfrac{1}{4}$ s

$\therefore$ The length of time that the vehicle maintains a constant speed

$= t_2 - t_1$

$= 11\frac{1}{4} - 6\frac{1}{4}$

$= 5$ s

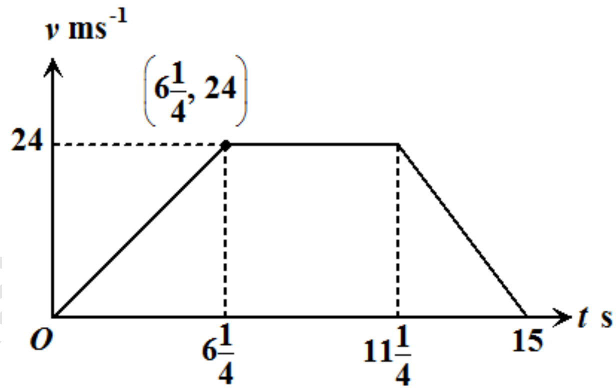
**(iii)** Calculate the maximum velocity attained.

**SOLUTION:**
**Required to calculate:** The maximum velocity attained
**Calculation:**
Maximum velocity $= v$

$= 24$ ms$^{-1}$ (already done)

**(iv)** Determine the acceleration of the vehicle.

**SOLUTION:**
**Required to find:** The acceleration of the vehicle.
**Solution:**




Considering the 'branch' for the first phase of the journey.

Gradient $= \dfrac{24-0}{6\frac{1}{4}-0}$

$= \dfrac{24}{\frac{25}{4}}$

$= \dfrac{96}{25}$

$= 3\dfrac{21}{25}$ ms$^{-2}$

**Alternative Method:**
$v = u + at$ for constant acceleration

When $u = 0,\ v = 24$ and $t = 6\frac{1}{4}$

$$24 = 0 + 6\frac{1}{4}$$

$$t = \frac{24}{6\frac{1}{4}}$$

$$= 3\frac{21}{25}\ \text{ms}^{-2}$$