FAS-PASS Maths

CSEC ADD MATHS 2016

SECTION I

Answer BOTH questions.

ALL working must be clearly shown.

1. (a) The domain for the function $f(x)=2x-5$ is $\{-2, -1, 0, 1\}$.

(i) Determine the range for the function.

SOLUTION:
Data: The domain of the function $f(x)=2x-5$ is $\{-2, -1, 0, 1\}$.
Required To Determine: The range for the function.
Solution:
$f(x)=2x-5$
The domain is $\{-2, -1, 0, 1\}$.

x	$f(x)$
-2	-9
-1	-7
0	-5
1	-3

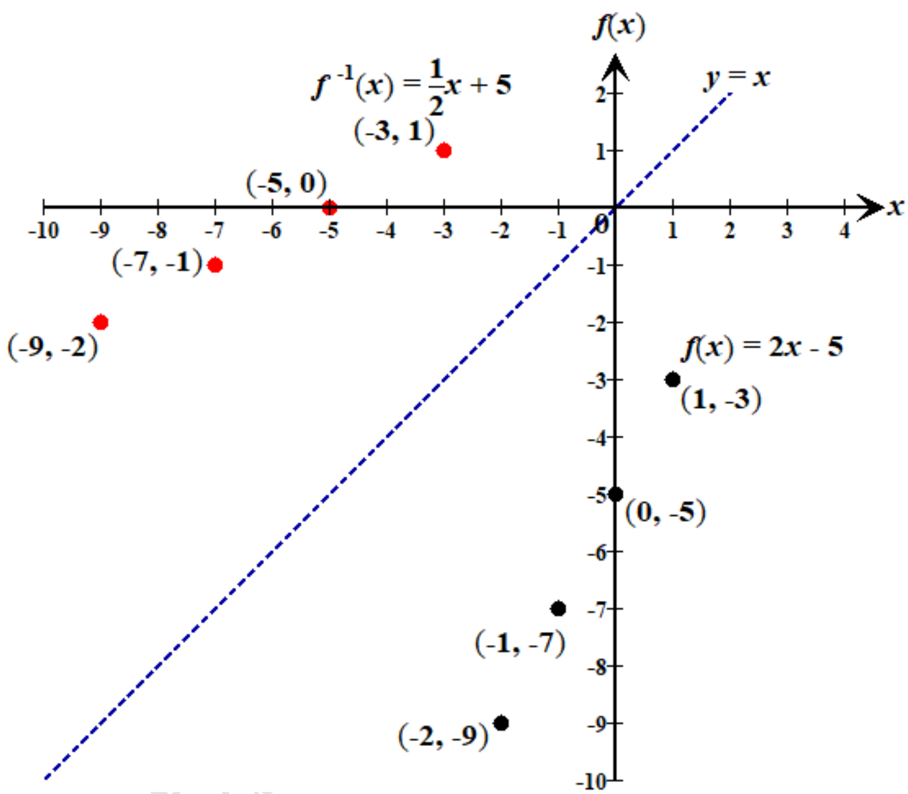
Hence, the range of $f(x)$ is $f(x)=\{-9, -7, -5, -3\}$.

(ii) Find $f^{-1}(x)$.

SOLUTION:
Required To Find: $f^{-1}(x)$
Solution:
$f(x)=2x-5$
Let $y=2x-5$
$2x-5=y$
$2x=y+5$
$x=\dfrac{y+5}{2}$
Replace y by x:
$f^{-1}(x)=\dfrac{x+5}{2}$ or $\dfrac{1}{2}(x+5)$ for $x=\{-2, -1, 0, 1\}$

(iii) Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same axes.

SOLUTION:
Required To Sketch: The graphs of $f(x)$ and $f^{-1}(x)$ on the same axes.
Solution:



(iv) Comment on the relationship between the two graphs.

SOLUTION:
Required To Comment: On the relationship between the graph of $f(x)$ and the graph of $f^{-1}(x)$.
Solution:

$$f(x) \xrightarrow[\text{in the line } y=x]{\text{Reflection}} f^{-1}(x)$$

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(b) Solve the equation $2^{2x+1}+5\left(2^{x}\right)-3=0$.

SOLUTION:
Data: $2^{2x+1}+5\left(2^{x}\right)-3=0$
Required To Solve: For *x*.
Solution:

$$2^{2x+1}+5\left(2^{x}\right)-3=0$$
$$2^{2x}\times 2+5\left(2^{x}\right)-3=0$$
$$2\left(2^{x}\right)^{2}+5\left(2^{x}\right)-3=0$$

Let $t=2^{x}$

$$\therefore 2t^{2}+5t-3=0$$
$$(2t-1)(t+3)=0$$
$$\therefore t=\frac{1}{2} \text{ or } -3$$

Hence,

$$2^{x}=\frac{1}{2}$$
$$2^{x}=2^{-1}$$

Equating indices:

$$x=-1$$

$$2^{x}=-3$$

Taking lg:

$\lg 2^{x}=\lg(-3)$ which has no real values.

$$\therefore 2^{x}\neq -3$$

$$\therefore x=-1 \text{ only.}$$

(c) (i) Given that $T=kp^{\left(\frac{h}{c}\right)}$, make *c* the subject of the formula.

SOLUTION:

Data: $T=kp^{\left(\frac{h}{c}\right)}$
Required To Make: *c* the subject of the formula.
Solution:

$$T=kp^{\left(\frac{h}{c}\right)}$$

Taking lg:

$$\lg T = \lg\left(kp^{\frac{h}{c}}\right)$$

$$\lg T = \lg k + \lg p^{\frac{h}{c}}$$

$$\lg T = \lg k + \left(\frac{h}{c}\right)\lg p$$

$$\left(\frac{h}{c}\right)\lg p = \lg T - \lg k$$

$$\frac{h}{c} = \frac{\lg T - \lg k}{\lg p}$$

$$\frac{c}{h} = \frac{\lg p}{\lg T - \lg k}$$

$$c = \frac{h\lg p}{\lg T - \lg k}$$

$$c = \frac{\lg p^h}{\lg\left(\frac{T}{k}\right)}$$

(ii) Solve the equation $\log(x+1)+\log(x-1)=2\log(x+2)$.

SOLUTION:
Data: $\log(x+1)+\log(x-1)=2\log(x+2)$
Required To Find: x
Solution:

$$\log(x+1)+\log(x-1)=2\log(x+2)$$

$$\log\{(x+1)(x-1)\}=\log(x+2)^2$$

Remove log:

$$(x+1)(x-1)=(x+2)^2$$

$$x^2-1=(x+2)^2$$

$$x^2-1=x^2+4x+4$$

$$-1=4x+4$$

$$-5=4x$$

$$x=\frac{-5}{4}$$

NOTE: However, if $x=-\frac{5}{4}$ then $\log(x-1)=\log\left(-\frac{5}{4}-1\right)$, which is the log of a negative number, which does not exist. Realistically, the question has no real solutions.

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2. (a) (i) Determine the nature of the roots of the quadratic equation $2x^2 + 3x - 9 = 0$.

SOLUTION:
Data: $2x^2 + 3x - 9 = 0$
Required To Determine: The nature of the roots of the quadratic equation.
Solution:
$2x^2 + 3x - 9 = 0$ is of the form $ax^2 + bx + c = 0$, where $a = 2$, $b = 3$ and $c = -9$.

$$b^2 = (3)^2$$
$$= 9$$

$$4ac = 4(2)(-9)$$
$$= -72$$

$b^2 > 4ac$ and so, the roots are real and distinct (different).

(ii) Given that $f(x) = 2x^2 + 3x - 9$, sketch the graph of the quadratic function, clearly indicating the minimum value.

SOLUTION:
Data: $f(x) = 2x^2 + 3x - 9$
Required To Sketch: The graph of $f(x)$.
Solution:
Calculating the x – intercepts of $f(x)$:

$$f(x) = 0$$
$$\therefore 2x^2 + 3x - 9 = 0$$
$$(2x - 3)(x + 3) = 0$$
$$x = 1\frac{1}{2} \text{ or } -3$$

$\therefore f(x)$ cuts the x – axis at $\left(1\frac{1}{2}, 0\right)$ and $(-3, 0)$.

Calculating the coordinates of the minimum turning point of $f(x)$:
$f(x)$ is a quadratic function.

$\therefore$ The axis of symmetry is $x = \frac{-(3)}{2(2)} = -\frac{3}{4}$

The coefficient of $x^2 > 0$ and so $f(x)$ has a minimum turning point.
Hence, the y – coordinate of the minimum turning point is

$$f\left(-\tfrac{3}{4}\right) = 2\left(-\tfrac{3}{4}\right)^2 + 3\left(-\tfrac{3}{4}\right) - 9$$
$$= -10\tfrac{1}{8}$$

$\therefore$ The minimum turning point is $\left(-\frac{3}{4},\ -10\frac{1}{8}\right)$.

OR

$$f(x) = 2x^2 + 3x - 9$$
$$f'(x) = 2(2x) + 3$$
$$= 4x + 3$$

A stationary point occurs when $f'(x) = 0$.

Let $4x + 3 = 0$

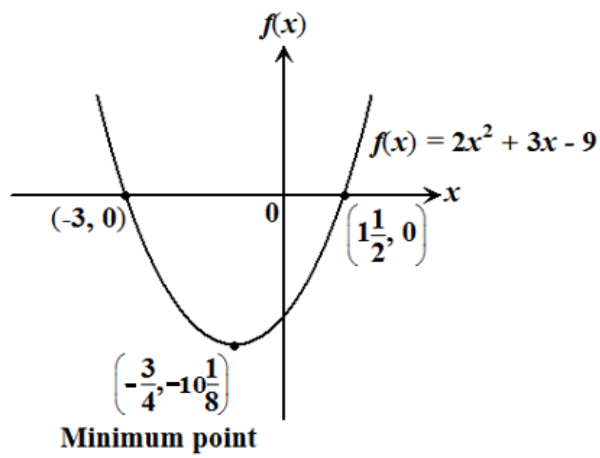
$$x = -\frac{3}{4}$$

$$f\left(-\tfrac{3}{4}\right) = 2\left(-\tfrac{3}{4}\right)^2 + 3\left(-\tfrac{3}{4}\right) - 9$$
$$= -10\tfrac{1}{8}$$

$\therefore$ The stationary point is $\left(-\frac{3}{4},\ -10\frac{1}{8}\right)$.

$f''(x) = 4 > 0$

Hence, $\left(-\frac{3}{4},\ -10\frac{1}{8}\right)$ is a minimum point.



(b) Evaluate $\sum_{1}^{25} 3^{-n}$.

SOLUTION:

Required To Evaluate: $\sum_{1}^{25} 3^{-n}$

Solution:

When $n=1 \Rightarrow 3^{-n} = 3^{-1} = \frac{1}{3}$

When $n=2 \Rightarrow 3^{-n} = 3^{-2} = \frac{1}{3^2}$

When $n=3 \Rightarrow 3^{-n} = 3^{-3} = \frac{1}{3^3}$ and so on.

$$\therefore \sum_{1}^{25} 3^{-n} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^{25}}$$

$$= \frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{1}{3}\right)^2 + \ldots + \frac{1}{3}\left(\frac{1}{3}\right)^{24}$$

This is a geometric progression with first term, $a = \frac{1}{3}$, common ratio, $\frac{1}{3}$ and number of terms $n = 25 - 1 + 1 = 25$.

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r} \quad \text{for } |r| < 1$$

$$\therefore S_{25} = \frac{\frac{1}{3}\left(1 - \left(\frac{1}{3}\right)^{25}\right)}{1 - \frac{1}{3}}$$

$$= \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^{25}\right)$$

(c) A man invested \$$x$ in a company in January 2010, on which he earns quarterly dividends. At the end of the second, third and fourth quarter in 2011, he earned \$100, \$115 and \$130 respectively. Calculate the total dividends on his investment by the end of 2016.

SOLUTION:
Data: A man invested \$$x$ in a company in January 2010 and earns quarterly dividends. At the end of the second, third and fourth quarter in 2011, he earned \$100, \$115 and \$130 respectively.

Required To Calculate: The total dividends on his investment by the end of 2016.
Calculation:

Year	2^{nd} quarter	3^{rd} quarter	4^{th} quarter	…
2011	\$100	\$115	\$130	…

This is in an arithmetic progression with common difference,
$d = \$115 - \$100 = \$15$.
From January 2010 to the end of 2016, there are 7 × 4 = 28 quarters.
The investor earns from the first quarter of 2010 to the last quarter of 2016, for which there are 28 quarters earning dividends.
In the AP of 28 terms, 100 is therefore the sixth term.

$$T_n = a + (n-1)d$$
$$100 = a + (6-1) \times 15$$
$$a = 25$$

Hence, in the AP, $a = 25$, $d = 15$ and $n = 28$

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$S_{28} = \frac{28}{2}[2 \times 25 + (28-1)15]$$

$$S_{28} = 14(50 + 405) = \$6370$$

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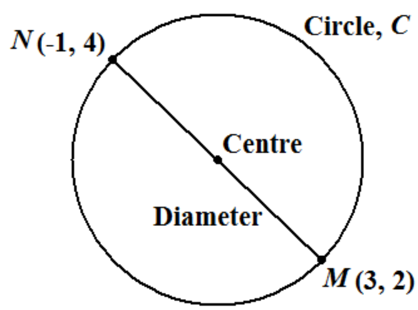
SECTION II

Answer BOTH questions.

ALL working must be clearly shown.

3. (a) (i) The points $M\ (3, 2)$ and $N\ (-1, 4)$ are the ends of a diameter of circle *C*. Determine the equation of circle *C*.

SOLUTION:
Data: The points $M\ (3, 2)$ and $N\ (-1, 4)$ are the ends of a diameter of circle *C*.



Required To Determine: The equation of the circle, *C*.
Solution:

The coordinates of the centre will be $\left(\frac{-1+3}{2}, \frac{4+2}{2}\right) = (1, 3)$.

The length of the diameter $= \sqrt{(3-(-1))^2 + (2-4)^2}$

$= \sqrt{(4)^2 + (-2)^2}$

$= \sqrt{20}$

$= 2\sqrt{5}$ units

$\therefore$ The length of the radius is $\frac{2\sqrt{5}}{2} = \sqrt{5}$ units.

The equation of the circle is $(x-1)^2 + (y-3)^2 = (\sqrt{5})^2$ or

$x^2 - 2x + 1 + y^2 - 6y + 9 - 5 = 0$ which reduces to

$x^2 + y^2 - 2x - 6y + 5 = 0$.

(ii) Find the equation of the tangent to the circle C at the point $P\left(-1,\ 6\right)$.

SOLUTION:
Required To Find: The equation of the tangent to C at the point $P\left(-1,\ 6\right)$.
Solution:
Let the centre of the circle be X.

$\therefore$ The gradient of the radius $XN = \dfrac{3-4}{1-(-1)} = -\dfrac{1}{2}$

$\therefore$ The gradient of the tangent at $N = \dfrac{-1}{-\frac{1}{2}} = 2$

NOTE: The point $(-1,\ 4)$ lies on the circle. Hence, it is impossible that $(-1,\ 6)$ lies on the circle and a tangent could never be drawn at the point $(-1,\ 6)$.

Alternative Method:
Centre of circle is $(-g, -f) = (1,\ 3) \Rightarrow g = -1$ and $f = -3$
The point at which the tangent is drawn is $(x,\ y) = (-1,\ 4)$.

$$\therefore \text{The gradient of the tangent } = \frac{-(x+g)}{y+f} = \frac{-(-1+(-1))}{4+(-3)} = \frac{2}{1} = 2$$

Using P as $(-1,\ 4)$

$$\frac{y-4}{x-(-1)} = 2$$

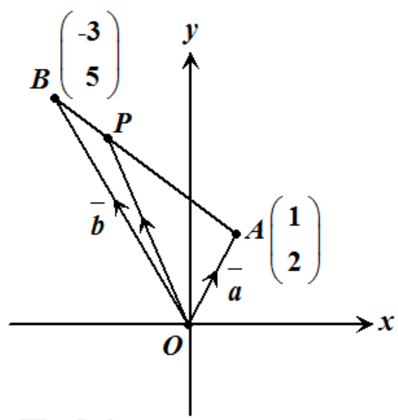
Equation of the tangent at $(-1,\ 4)$ is

$$y-4 = 2(x+1)$$
$$y-4 = 2x+2$$
$$y = 2x+6$$

(b) The position vector of two points A and B, relative to a fixed origin, O, are $\bar{a}$ and $\bar{b}$ respectively, where $\bar{a}=\begin{pmatrix}1\\2\end{pmatrix}$ and $\bar{b}=\begin{pmatrix}-3\\5\end{pmatrix}$. P lies on $\overrightarrow{AB}$ such that $\overrightarrow{PB}=\frac{1}{4}\overrightarrow{AB}$. Find the coordinates of $\overrightarrow{OP}$.

SOLUTION:

Data: The position vectors of two points A and B are $\bar{a}=\begin{pmatrix}1\\2\end{pmatrix}$ and $\bar{b}=\begin{pmatrix}-3\\5\end{pmatrix}$ respectively. P lies on $\overrightarrow{AB}$ such that $\overrightarrow{PB}=\frac{1}{4}\overrightarrow{AB}$.



Required To Find: The coordinates of $\overrightarrow{OP}$.
Solution:

$$\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}$$
$$=-\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}-3\\5\end{pmatrix}$$
$$=\begin{pmatrix}-4\\3\end{pmatrix}$$

$$\overrightarrow{PB}=\frac{1}{4}\begin{pmatrix}-4\\3\end{pmatrix}$$
$$=\begin{pmatrix}-1\\\frac{3}{4}\end{pmatrix}$$

$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$

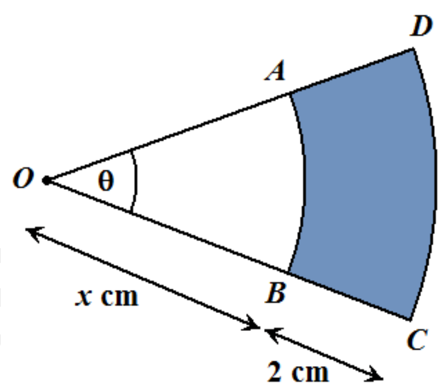
$$= \begin{pmatrix} -3 \\ 5 \end{pmatrix} + -\begin{pmatrix} -1 \\ \frac{3}{4} \end{pmatrix}$$

$$= \begin{pmatrix} -3+1 \\ 5-\frac{3}{4} \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 4\frac{1}{4} \end{pmatrix}$$

$\therefore$ The coordinates of $P = \left(-2,\ 4\frac{1}{4}\right)$.

4. (a) The following diagram (not drawn to scale) shows two sectors, *AOB* and *DOC*. *OB* and *OC* are x cm and $(x+2)$cm respectively and angle $AOB = \theta$.



If $\theta = \frac{2\pi}{9}$ radians, calculate the area of the shaded region in terms of x.

SOLUTION:
Data:

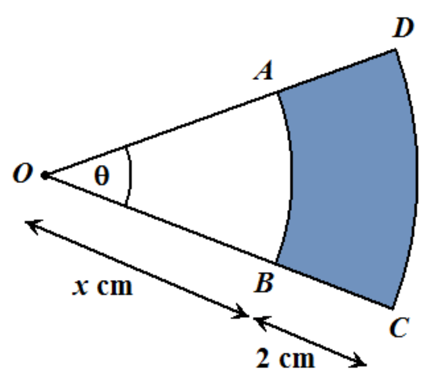


Diagram showing two sectors AOB and DOC with OB and OC are x cm a $(x+2)$ cm respectively and $AOB = \theta = \frac{2\pi}{9}$ radians.

Required To Calculate: The area of the shaded region, in terms of x.

Calculation:

The area of the sector $AOB = \frac{1}{2}(x)^2\left(\frac{2\pi}{9}\right)$ cm^2

$$= \frac{2\pi x^2}{2(9)} \text{ cm}^2$$

$$= \frac{\pi x^2}{9} \text{ cm}^2$$

The area of the sector $DOC = \frac{1}{2}(x+2)^2\left(\frac{2\pi}{9}\right)$ cm^2

$$= \frac{\pi(x+2)^2}{9} \text{ cm}^2$$

Hence, the area of the shaded region $= \frac{\pi(x+2)^2}{9} - \frac{\pi x^2}{9}$

$$= \frac{\pi}{9}\left((x+2)^2 - x^2\right) \text{ cm}^2$$

$$= \frac{\pi}{9}(4x+4) \text{ cm}^2$$

(b) Given that $\cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 45° = \frac{\sqrt{2}}{2}$, without the use of a calculator, evaluate $\cos 105°$, in surd form, giving your answer in the simplest terms.

SOLUTION:

Data: $\cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 45° = \frac{\sqrt{2}}{2}$.

Required To Calculate: $\cos 105°$, in surd form, without the use of a calculator.

Calculation:

$$\cos 30° = \frac{\sqrt{3}}{2}$$

$$\begin{aligned}\therefore \cos 60° &= \cos\left[2(30°)\right] \\ &= 2\cos^2 30° - 1 \\ &= 2\left(\frac{\sqrt{3}}{2}\right)^2 - 1 \\ &= \frac{3}{2} - 1 \\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\cos 105° &= \cos(60° + 45°) \\ &= \cos 60° \cos 45° - \sin 60° \sin 45° \\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{2} - \sqrt{3}\sqrt{2}}{4} \\ &= \frac{\sqrt{2}\left(1-\sqrt{3}\right)}{4} \\ &= \frac{\sqrt{2}\left(1-\sqrt{3}\right)}{2\sqrt{2}\sqrt{2}} \\ &= \frac{1-\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

Note, we may rationalize the surd to obtain:

$$\frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$

(c) Prove that the identity $\dfrac{\sin(\theta+\alpha)}{\cos\theta\cos\alpha} \equiv \tan\theta + \tan\alpha$.

SOLUTION:

Required To Prove: $\dfrac{\sin(\theta+\alpha)}{\cos\theta\cos\alpha} \equiv \tan\theta + \tan\alpha$

Proof:

Consider the left hand side:

$$\begin{aligned}\frac{\sin(\theta+\alpha)}{\cos\theta\cos\alpha} &= \frac{\sin\theta\cos\alpha + \cos\theta\sin\alpha}{\cos\theta\cos\alpha} \\ &= \frac{\sin\theta\cos\alpha}{\cos\theta\cos\alpha} + \frac{\cos\theta\sin\alpha}{\cos\theta\cos\alpha} \\ &= \frac{\sin\theta}{\cos\theta} + \frac{\sin\alpha}{\cos\alpha} \\ &= \tan\theta + \tan\alpha\end{aligned}$$

SECTION III
Answer BOTH questions.
ALL working must be clearly shown.

5. (a) Find $\frac{dy}{dx}$ given that $y=\sqrt{5x^2-4}$, simplifying your answer.

SOLUTION:
Data: $y=\sqrt{5x^2-4}$
Required To Find: $\frac{dy}{dx}$
Solution:

$y=\sqrt{5x^2-4}$

Let $t=5x^2-4 \Rightarrow \frac{dt}{dx}=2(5x)=10x$

So $y=\sqrt{t}=t^{\frac{1}{2}} \Rightarrow \frac{dy}{dt}=\frac{1}{2}t^{\frac{1}{2}-1}=\frac{1}{2}t^{-\frac{1}{2}}=\frac{1}{2\sqrt{t}}$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$
$$=\frac{1}{2\sqrt{t}}\times 10x$$
$$=\frac{5x}{\sqrt{t}}$$

Recall: $t=5x^2-4$

$$\frac{dy}{dx}=\frac{5x}{\sqrt{5x^2-4}}$$

Alternative Method:

$y=\sqrt{5x^2-4}$
$y=5x^2-4$

$2y\frac{dy}{dx}=10x$ [Differentiating implicitly wrt x]

$$\frac{dy}{dx}=\frac{10x}{2y}$$

$$\frac{dy}{dx}=\frac{5x}{y}$$

$$\frac{dy}{dx}=\frac{5x}{\sqrt{5x^2-4}}$$

(b) The point $P(1, 8)$ lies on the curve with equation $y = 2x(x+1)^2$. Determine the equation of the normal to the curve at the point P.

SOLUTION:
Data: $P(1, 8)$ lies on the curve $y = 2x(x+1)^2$.
Required To Find: The equation of the normal to the curve at *P*.
Solution:

$$y = 2x(x+1)^2$$
$$y = 2x(x^2+2x+1)$$
$$y = 2x^3+4x^2+2x$$

Gradient function, $\frac{dy}{dx} = 2(3x^2)+4(2x)+2$
$= 6x^2+8x+2$

$\therefore$ The gradient of the tangent at $P = 6(1)^2+8(1)+2$
$= 6+8+2$
$= 16$

$\therefore$ The gradient of the normal at $P = -\frac{1}{16}$
(The product of the gradients of perpendicular lines $= -1$)

The equation of the normal at *P* is

$$\frac{y-8}{x-1} = -\frac{1}{16}$$
$$16(y-8) = -1(x-1)$$
$$16y-128 = -x+1$$
$$x+16y-129 = 0$$

(c) Obtain the equation for EACH of the two tangents drawn to the curve $y = x^2$ at the points where $y = 16$.

SOLUTION:
Data: Equation of a curve is $y = x^2$.
Required To Find: The equation of the two tangents at $y = 16$.

Solution:
When $y = 16$

$16 = x^2$

$x = \pm\sqrt{16}$

$x = \pm 4$

$\therefore$ The two points where the tangent exist at $y = 16$ are $(4, 16)$ and $(-4, 16)$.

Gradient function, $\dfrac{dy}{dx} = 2x$

Gradient at $x = 4$ is $\dfrac{dy}{dx} = 2(4)$

$= 8$

$\therefore$ The equation of the tangent at $x = 4$ is

$$\frac{y-16}{x-4} = 8$$

$$y - 16 = x - 32$$

$$y = 8x - 16$$

Gradient at $x = -4$ is $\dfrac{dy}{dx} = 2(-4)$

$= -8$

$\therefore$ The equation of the tangent at $x = -4$ is

$$\frac{y-16}{x-(-4)} = 8$$

$$y - 16 = -8(x + 4)$$

$$y - 16 = -8x - 32$$

$$y = -8x - 16$$

6. (a) (i) Find $\int (3\cos\theta - 5\sin\theta)\, d\theta$

SOLUTION:

Required To Find: $\int (3\cos\theta - 5\sin\theta)\, d\theta$

Solution:

$$\int (3\cos\theta - 5\sin\theta)\, d\theta = 3\int \cos\theta\, d\theta - 5\int \sin\theta\, d\theta$$

$$= 3(\sin\theta) - 5(-\cos\theta) + C \quad (C \text{ is a constant})$$

$$= 3\sin\theta + 5\cos\theta + C$$

(ii) Evaluate $\int_1^3 \left(\dfrac{2}{x^2} - 3 + 2x^3\right) dx$.

SOLUTION:

Required To Evaluate: $\int_1^3 \left(\frac{2}{x^2} - 3 + 2x^3 \right) dx$

Solution:

$$\int_1^3 \left(\frac{2}{x^2} - 3 + 2x^3 \right) dx = \int_1^3 \left(2x^{-2} - 3 + 2x^3 \right) dx$$

$$= \left[\frac{2x^{-2+1}}{-2+1} - 3x + \frac{2x^{3+1}}{3+1} + C \right]_1^3$$

(where C is a constant)

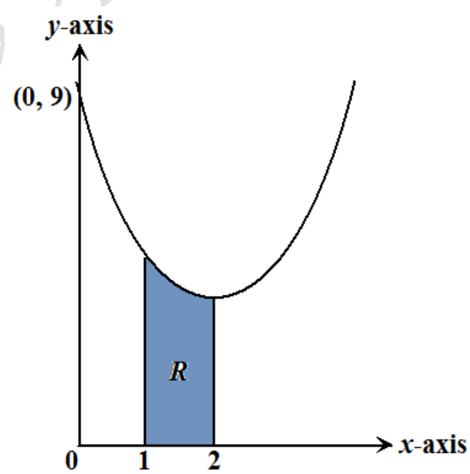
$$= \left[-\frac{2}{x} - 3x + \frac{x^4}{2} + C \right]_1^3$$

$$= \left[-\frac{2}{3} - 3(3) + \frac{(3)^4}{2} \right] - \left[-\frac{2}{1} - 3(1) + \frac{(1)^4}{2} \right]$$

$$= \left[-\frac{2}{3} - 9 + \frac{81}{2} \right] - \left[-2 - 3 + \frac{1}{2} \right]$$

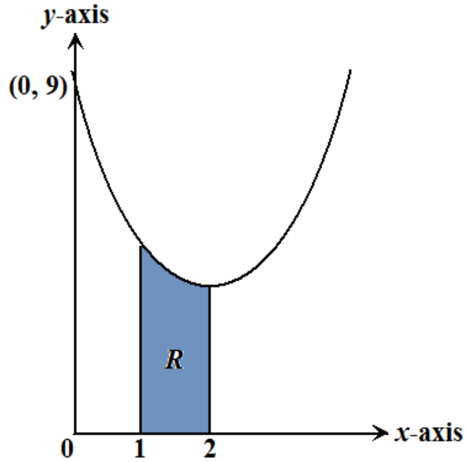
$$= 35\frac{1}{3}$$

(b) The following figure shows the finite region R bounded by the lines $x = 1$, $x = 2$ and the arc of the curve $y = (x-2)^2 + 5$.



Calculate the area of the region R.

SOLUTION:
Data:



The equation of the curve is $y=(x-2)^2+5$.
Required To Calculate: The area of *R*.
Calculation:

The area of $R=\int_{x_1}^{x_2} y\ dx$

$$=\int_1^2 \left[(x-2)^2+5\right] dx$$

$$=\int_1^2 \left(x^2-4x+4+5\right) dx$$

$$=\int_1^2 \left(x^2-4x+9\right) dx$$

$$=\left[\frac{x^3}{3}-2x^2+9x+C\right]_1^2 \text{ (where } C \text{ is a constant)}$$

$$=\left[\frac{(2)^3}{3}-2(2)^2+9(2)\right]-\left[\frac{(1)^3}{3}-2(1)^2+9(1)\right]$$

$$=2\frac{2}{3}-8+18-\frac{1}{3}+2-9$$

$$=5\frac{1}{3} \text{ square units}$$

(c) The point $P(1, 2)$ lies on the curve which has a gradient function given by $\frac{dy}{dx} = 3x^2 - 6x$. Find the equation of the curve.

SOLUTION:

Data: $P(1, 2)$ lies on the curve with gradient function $\frac{dy}{dx} = 3x^2 - 6x$.

Required To Find: The equation of the curve.
Solution:

The equation of a curve is $y = \int \frac{dy}{dx}\, dx$

$$y = \int (3x^2 - 6x)\, dx$$

$$y = \frac{3x^3}{3} - \frac{6x^2}{2} + C \text{ (where } C \text{ is a constant)}$$

$$y = x^3 - 3x^2 + C$$

$P(1, 2)$ lies on the curve.
$2 = (1)^3 - 3(1)^2 + C$
$2 = 1 - 3 + C$
$C = 4$

$\therefore$The equation of the curve is $y = x^3 - 3x^2 + 4$.

SECTION IV
Answer only ONE question.
ALL working must be clearly shown.

7. (a) Use the data set provided below to answer the questions which follow.

15	16	18	18	20	21	22	22
22	25	28	30	30	32	35	40
41	52	54	59	60	65	68	75

(i) Construct a stem-and-leaf diagram to represent the data given.

SOLUTION:
Data:

15	16	18	18	20	21	22	22
22	25	28	30	30	32	35	40
41	52	54	59	60	65	68	75

Required To Construct: A stem-and-leaf diagram for the data.
Solution:

Stem	**Leaf**						
1	5	6	8	8			
2	0	1	2	2	2	5	8
3	0	0	2	5			
4	0	1					
5	2	4	9				
6	0	5	8				
7	5						

Key
6|3 = 63
Leaf unit = 1
Stem unit = 10

(ii) State an advantage of using the stem-and-leaf diagram to represent the given data.

SOLUTION:
Required To State: An advantage of using the stem-and-leaf diagram to represent data.
Solution:

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In the stem-and-leaf diagram the data is arranged compactly and the stem is not repeated for multiple data values.

(iii) Determine the mode.

SOLUTION:
Required To Determine: The mode of the data set.
Solution:
The mode is 22 since this score occurs more than any other score.

(iv) Determine the median.

SOLUTION:
Required To Determine: The median of the data set.
Solution:
There are 24 values.
12^{th} value is 30.
13^{th} value is 30.

$\therefore$ The median is $\frac{30+30}{2}=30$.

(v) Determine the interquartile range.

SOLUTION:
Required To Determine: The interquartile range of the data set.
Solution:
The lower Quartile, Q_1 is between the 6^{th} and 7^{th} values.

The 6^{th} value is 21.

The 7^{th} value is 22.

$Q_1=\frac{21+22}{2}=21.5$

The upper Quartile, Q_3 is between the 18^{th} and 19^{th} values.

The 18^{th} value is 52.

The 19^{th} value is 54

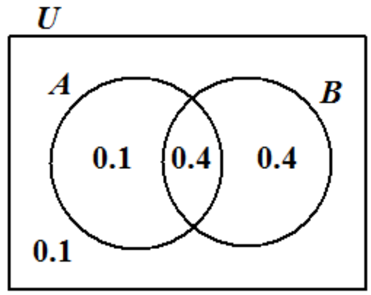
$Q_3=\frac{52+54}{2}=53$

$\therefore$ The interquartile range $=Q_3-Q_1=53-21.5=31.5$

(b) Two events, A and B, are such that $P(A)=0.5$, $P(B)=0.8$ and $P(A\cup B)=0.9$.

(i) Determine $P(A\cap B)$.

SOLUTION:
Data: $P(A)=0.5$, $P(B)=0.8$ and $P(A\cup B)=0.9$
Required To Calculate: $P(A\cap B)$
Calculation:



$P(A\cup B)=P(A)+P(B)-P(A\cap B)$ (De Morgan's Law)

$P(A\cap B)=P(A)+P(B)-P(A\cup B)$
$=0.5+0.8-0.9$
$=0.4$ (as illustrated)

(ii) Determine $P(A|B)$.

SOLUTION:
Required To Find: $P(A|B)$
Solution:

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$
$$=\frac{0.4}{0.8}$$
$$=0.5$$

(iii) State, giving a reason, whether or not A and B are independent events.

SOLUTION:
Required To State: Whether or not A and B are independent events.
Solution:
Independent events have no effect on subsequent events.

If A and B are independent events then

$$P(A \cap B) = P(A) \times P(B)$$

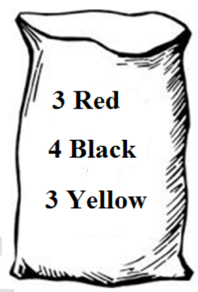
That is, $0.4 = 0.5 \times 0.8$

$0.4 = 0.4$ which is true.

$\therefore A$ and B are independent events.

(c) A bag contains 3 red balls, 4 black balls and 3 yellow balls. Three balls are drawn at random **with replacement** from the bag. Find the probability that the balls drawn are all of the same colour.

SOLUTION:
Data: A bag contains 3 red balls, 4 black balls and 3 yellow balls. Three balls are drawn at random from the bag, with replacement.
Required To Calculate: The probability that all three balls drawn are of the same colour.
Calculation:



Let R be the event a red ball is chosen.
Let B be the event a black ball is chosen.
Let Y be the event a yellow ball is chosen.

Total number of balls $= 3 + 4 + 3$

$= 10$

$P(\text{Balls are of the same colour})$

$= P(R \,\&\, R \,\&\, R)$ or $P(B \,\&\, B \,\&\, B)$ or $P(Y \,\&\, Y \,\&\, Y)$

$$= \left(\frac{3}{10} \times \frac{3}{10} \times \frac{3}{10}\right) + \left(\frac{4}{10} \times \frac{4}{10} \times \frac{4}{10}\right) + \left(\frac{3}{10} \times \frac{3}{10} \times \frac{3}{10}\right)$$

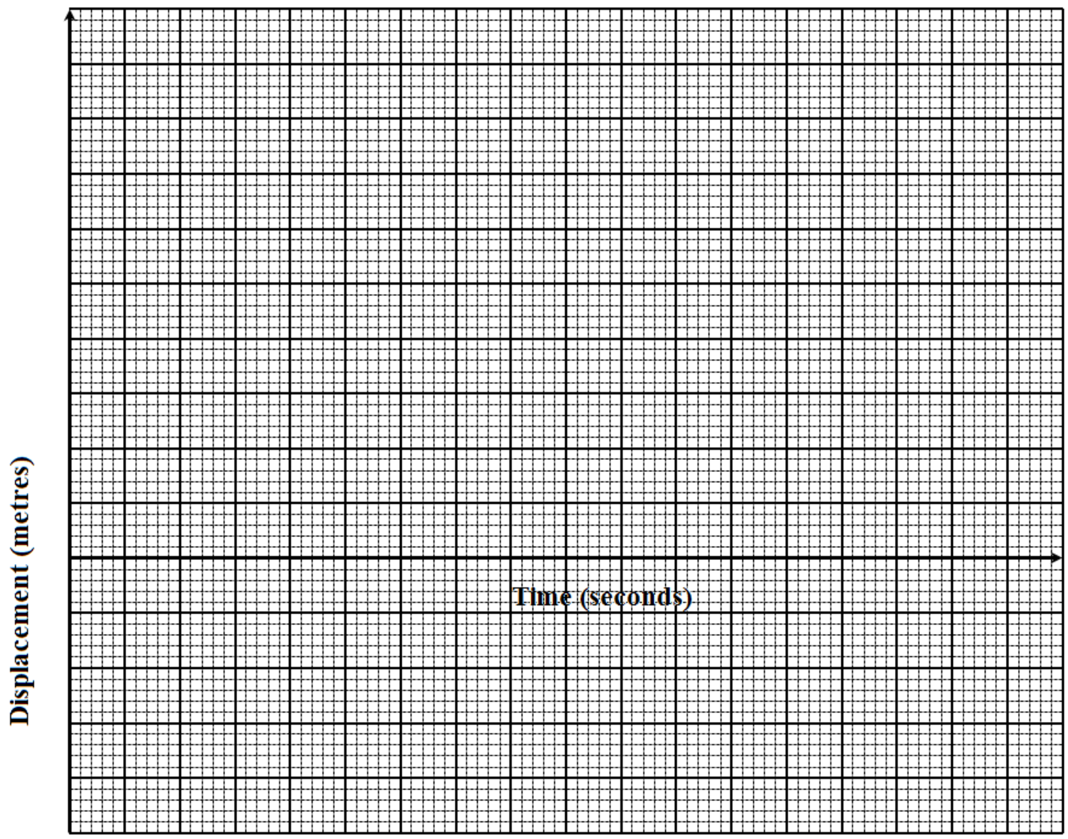
$$= \frac{27}{1000} + \frac{64}{1000} + \frac{27}{1000}$$

$$= \frac{118}{1000}$$

$$= \frac{59}{500}$$

8. (a) A motorist starts from a point, *X*, and travels 100 m due North to a point *Y*, at a constant speed of 5 ms^{-1}. He stays at *Y* for 40 seconds and then travels at a constant speed of 10 ms^{-1} for 200 m due South to a point *Z*.

(i) On the following grid, draw a displacement-time graph to display this information.



SOLUTION:
Data: A motorist starts from a point, *X*, and travels 100 m due North to a point *Y*, at a constant speed of 5 ms^{-1}. He stays at *Y* for 40 seconds and then travels at a constant speed of 10 ms^{-1} for 200 m due South to a point *Z*.
Required To Draw: A displacement-time graph to display this information.
Solution:

Time taken to travel from *X* to *Y* $= \dfrac{\text{Total distance}}{\text{Speed}}$

$$= \frac{100 \text{ m}}{5 \text{ ms}^{-1}}$$
$$= 20 \text{ s}$$

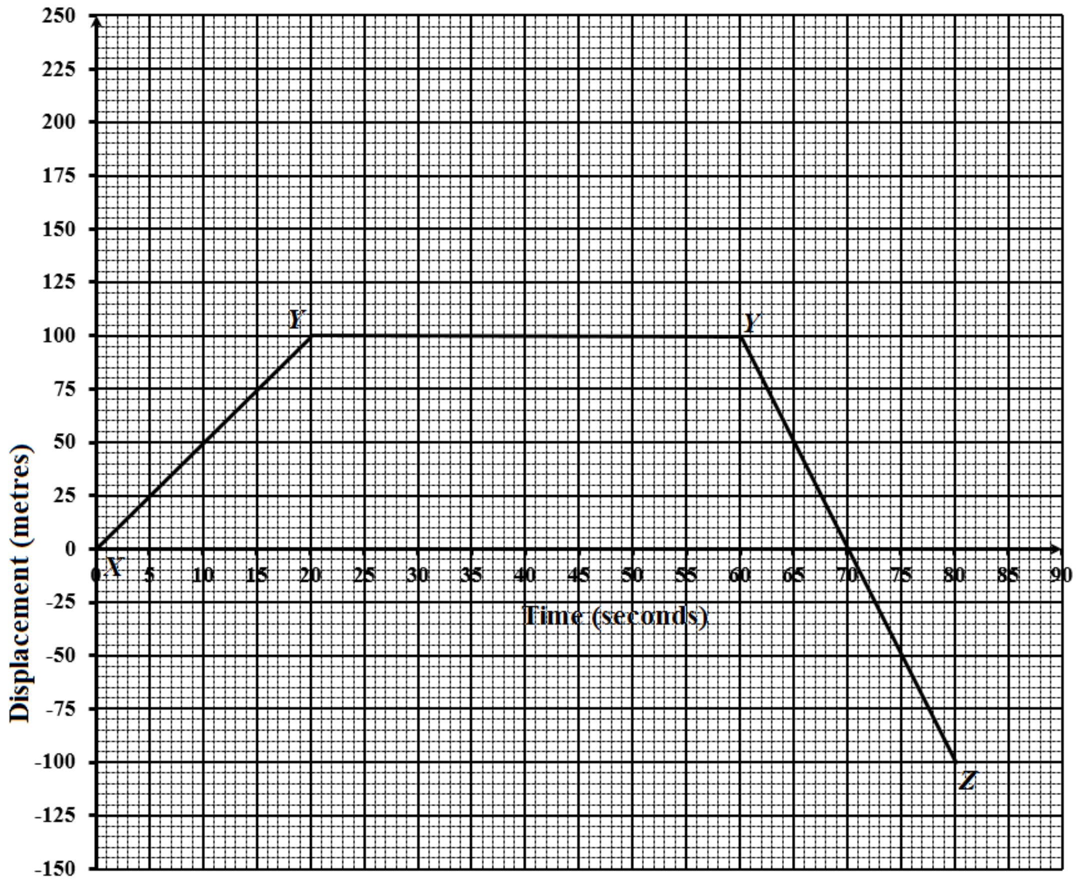
(Straight line indicates constant speed)

Stopping at Y for 40 seconds is indicated by a horizontal branch.

Time taken to travel from Y to $Z = \frac{\text{Total distance}}{\text{Speed}}$

$$= \frac{200 \text{ m}}{10 \text{ ms}^{-1}}$$
$$= 20 \text{ s}$$

(Straight line indicates constant speed)



(ii) Calculate the average speed for the whole journey.

SOLUTION:
Required To Calculate: The average speed for the whole journey.
Calculation:

$$\text{Average speed} = \frac{\text{Total distance covered}}{\text{Total time taken}}$$
$$= \frac{100+200}{20+40+20}$$
$$= \frac{300 \text{ m}}{80 \text{ s}}$$
$$= 3\frac{3}{4} \text{ ms}^{-1}$$

(iii) Calculate the average velocity of the whole journey.

SOLUTION:
Required To Calculate: The average velocity of the whole journey.
Calculation:
Displacement from X to Y $=100$ m
Displacement from Y to Z $=-200$ m

$$\text{Average velocity} = \frac{\text{Total displacement}}{\text{Total time taken}}$$
$$= \frac{100+(-200)}{20+40+20} \text{ ms}^{-1}$$
$$= \frac{-100}{80} \text{ ms}^{-1}$$
$$= -\frac{5}{4} \text{ ms}^{-1}$$

(b) A particle starting from rest, travels in a straight line with an acceleration, a, given by $a = \cos t$ where t is the time in seconds.

(i) Find the velocity of the particle in terms of t.

SOLUTION:
Data: The acceleration, a, of a particle is $a = \cos t$ ms^{-2}.
Required To Find: The velocity of the particle in terms of t.

Solution:
Let the velocity at time t be v.

$$v = \int a \; dt$$
$$v = \int \cos t \; dt$$
$$v = \sin t + C \text{ (where } C \text{ is a constant)}$$

When $t = 0$, $v = 0$ (from data)

$\therefore 0 = \sin(0) + C$

$C = 0$

$\therefore$ The velocity of the particle, $v = \sin t$ ms^{-1}.

(ii) Calculate the displacement of the particle in the interval of time $t = \pi$ to $t = 2\pi$.

SOLUTION:
Required To Calculate: The displacement of the particle during $t = \pi$ to $t = 2\pi$.
Calculation:
Let the displacement at time *t* be *s*.

$s = \int v \, dt$

$s = \int \sin t \, dt$

$s = -\cos t + K$ (where K is a constant)

$s = 0$ when $t = 0$

$\therefore 0 = -\cos(0) + k$

$0 = -1 + K$

$K = 1$

$s = 1 - \cos t$

When $t = \pi \Rightarrow s = 1 - \cos(\pi) = 1 - (-1) = 2$

When $t = 2\pi \Rightarrow s = 1 - \cos(2\pi) = 1 - 1 = 0$

Hence, the displacement from $t = \pi$ to $t = 2\pi$ is $(0 - 2) = -2$ units.