CSEC ADD MATHS 2015

1. (a) **Data:** $f(x)=x^2+5,\ x\geq 1$ and $g(x)=4x-3,\ x\in R$.

Required to Calculate: $gf(2)$

Calculation:

$$f(2)=(2)^2+5$$
$$=9$$

$$\therefore gf(2)=g(9)$$
$$=4(9)-3$$
$$=33$$

OR

$$gf(x)=4\left(x^2+5\right)-3$$
$$\therefore gf(2)=4\left((2)^2+5\right)-3$$
$$=33$$

(Note: The given domain for $f(x)$ and $g(x)$ are of no consequence in the question).

(b) **Data:** $h(x)=\dfrac{3x+5}{x-2},\ x\in R,\ x\neq 2$.

Required to Calculate: $h^{-1}(x)$

Calculation:

$$\text{Let } y=\frac{3x+5}{x-2}$$
$$xy-2y=3x+5$$
$$xy-3x=5+2y$$
$$x(y-3)=2y+5$$
$$x=\frac{2y+5}{y-3}$$

Replace y by x to obtain:

$$h^{-1}(x)=\frac{2x+5}{x-3},\ x\neq 3$$

(c) **Data:** $(x-2)$ is a factor of $k(x)=2x^3-5x^2+x+2$.

Required to Factorise: $k(x)$ completely.

Solution:

If $(x-2)$ is a factor of $k(x)=2x^3-5x^2+x+2$, then $k(x)$ is divisible by $(x-2)$

$$\begin{array}{r}
2x^2-x-1 \\
x-2\overline{)\,2x^3-5x^2+x+2} \\
-\;\underline{2x^3-4x^2} \\
-x^2+\;\;x+2 \\
-\;\underline{-x^2+2x} \\
-x+2 \\
-\;\underline{-x+2} \\
\underline{0}
\end{array}$$

We factorise $2x^2-x-1=(2x+1)(x-1)$

Hence, $k(x)=(x-2)(2x+1)(x-1)$

(d) **(i)** **Data:** $16^{x+2}=\dfrac{1}{4}$

Required to Calculate: x

Calculation:

$$16^{x+2}=\frac{1}{4}$$

$$\left(2^4\right)^{x+2}=\frac{1}{(2)^2}$$

$$2^{4x+8}=2^{-2}$$

Equating indices since the bases are equal, we obtain

$$4x+8=-2$$

$$\therefore 4x=-10$$

$$x=-2\frac{1}{2}$$

(ii) **Data:** $\log_3(x+2)+\log_3(x-1)=\log_3(6x-8)$

Required to Calculate: x

Calculation:

$\log_3(x+2)+\log_3(x-1)=\log_3(6x-8)$
$\therefore \log_3\{(x+2)(x-1)\}=\log_3(6x-8)$ (Product law)

Remove $\log_3$, we obtain

$(x+2)(x-1)=6x-8$

$x^2+x-2=6x-8$

$x^2-5x+6=0$

$(x-2)(x-3)=0$

Hence, $x=2$ or 3

When $x=2$, there are no terms of the equation that result in $\log_3(-\text{ve})$ or $\log_3(0)$.

When $x=3$ there are no terms of the equation that result in $\log_3(-\text{ve})$ or $\log_3(0)$.

Hence, $x=2$ or 3.

2. (a) Data: $f(x)=3x^2-9x+4$

(i) Required to Express: $f(x)$ in the form $a(x+b)^2+c$, where a, b and $c\in R$.

Solution:

$a(x+b)^2+c=a(x+b)(x+b)+c$

$=a(x^2+2bx+b^2)+c$

$=2ax^2+2abx+ab^2+c$

Hence,

$3x^2-9x+4=ax^2+2abx+ab^2+c$

Equating coefficients:

$a=3$

$2ab=-9$

$\therefore 2(3)b=-9$

$b=-1\frac{1}{2}$

$$ab^2 + c = 4$$

$$\therefore 3\left(-1\frac{1}{2}\right)^2 + c = 4$$

$$\frac{27}{4} + c = 4$$

$$c = 4 - \frac{27}{4}$$

$$= 4 - 6\frac{3}{4}$$

$$= -2\frac{3}{4}$$

$\therefore f(x) = 3x^2 - 9x + 4$ can be written as $3\left(x - 1\frac{1}{2}\right)^2 - 2\frac{3}{4}$, where $a = 3 \in R$, $b = -1\frac{1}{2} \in R$ and $c = -2\frac{3}{4} \in R$.

OR

$$3x^2 - 9x + 4 = 3\left(x^2 - 3x\right) + 4$$

One half the coefficient of x is $\frac{1}{2}(-3) = -1\frac{1}{2}$

$$\therefore 3\left(x^2 - 3x\right) + 4$$

$$= 3\left(x - 1\frac{1}{2}\right)^2 + *$$

$$\downarrow$$

$$3\left(x - 1\frac{1}{2}\right)\left(x - 1\frac{1}{2}\right)$$

$$= 3\left(x^2 - 3x + \frac{9}{4}\right)$$

$$= 3x^2 - 9x + 6\frac{3}{4}$$

$$6\frac{3}{4} - 2\frac{3}{4} = 4$$

$$\therefore * = -2\frac{3}{4}$$

If $3x^2 - 9x + 4 = 3\left(x - 1\frac{1}{2}\right)^2 - 2\frac{3}{4}$ and which is of the form $a\left(x+b\right)^2 + c$, where $a = 3 \in R$, $b = -1\frac{1}{2} \in R$ and $c = -2\frac{3}{4} \in R$.

(ii) **Required to State:** The minimum point of $f\left(x\right)$.

Solution:

$$f\left(x\right) = 3x^2 - 9x + 4 \equiv 3\left(x - 1\frac{1}{2}\right)^2 - 2\frac{3}{4}$$

$$\left(x - 1\frac{1}{2}\right)^2 \geq 0 \ \forall x$$

$$\therefore 3\left(x - 1\frac{1}{2}\right)^2 \geq 0 \ \forall x$$

Hence, if the graph of $f\left(x\right)$ is drawn, the minimum value of $f\left(x\right)$ is $0 - 2\frac{3}{4} = -2\frac{3}{4}$. The minimum value occurs when $3\left(x - 1\frac{1}{2}\right)^2 = 0$ that is when $x = 1\frac{1}{2}$. Therefore, the minimum point on the curve of $f\left(x\right)$ is $\left(1\frac{1}{2}, -2\frac{3}{4}\right)$.

(b) **Data:** The equation $3x^2 - 6x - 4 = 0$ has roots α and β.

Required to Calculate: The value of $\frac{1}{\alpha} + \frac{1}{\beta}$.

Calculation:

Recall: If $ax^2 + bx + c = 0$
$\div a$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

If the roots are α and β, then $\left(x-\alpha\right)\left(x-\beta\right) = 0$.

$\therefore x^2 - \left(\alpha+\beta\right)x + \alpha\beta = 0$

Equating coefficients, we get:

$\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

So, in the expression $3x^2 - 6x - 4 = 0$

$$\alpha+\beta=\frac{-(-6)}{3}$$
$$=2$$

$$\alpha\beta=-\frac{4}{3}$$

$$\frac{1}{\alpha}+\frac{1}{\beta}$$
$$\frac{\beta+\alpha}{\alpha\beta}=\frac{\alpha+\beta}{\alpha\beta}$$
$$=\frac{2}{-\frac{4}{3}}$$
$$=-1\frac{1}{2}$$

So, $\frac{1}{\alpha}+\frac{1}{\beta}=-1\frac{1}{2}$.

(c) **Data:** Equation of the curve is $2x^2-y+19=0$ and the equation of the line is $y+11x=4$.

Required to Find: The points of intersection of the line and curve.

Solution:

To determine the points of intersection we solve the two equations simultaneously.

Let $y+11x=4$ $\quad\ldots(1)$

$2x^2-y+19=0$ $\quad\ldots(2)$

Equation (1) + Equation (2)

$$y+11x-4+2x^2-y+19=0$$
$$2x^2+11x+15=0$$
$$(2x+5)(x+3)=0$$
$$x=-\frac{5}{2}\text{ or }-3$$

Substitute $x=-\frac{5}{2}$ in equation (1)

$$y + 11\left(-\frac{5}{2}\right) - 4 = 0$$
$$y = 4 + \frac{55}{2}$$
$$y = 31\frac{1}{2}$$

Substitute $x = -3$ in equation (1)

$$y + 11(-3) - 4 = 0$$
$$y = 37$$

$\therefore$ The points of intersection are $\left(-2\frac{1}{2},\ 31\frac{1}{2}\right)$ and $(-3,\ 37)$.

(d) **Data:** Starting salary of employee \$36 000 and which increases \$2 400 per annum.

Required to Calculate: The salary of the employee in the 9th year.

Calculation:

Year	Salary
1	36 000
2	36 000 + 2 400 = 36 000 + (2-1) 2 400
3	36 000 + 2 400 + 2 400 = 36 000 + (3-1) 2 400
4	$36\ 000 + 2(2\ 400) + 2\ 400$ = 36 000 + (4-1) 2 400
9	$36\ 000 + (9-1)(2\ 400)$

The annual salary is in arithmetic progression, with the first term, $a = \$36\ 000$ and the common difference, $d = \$2\ 400$.

The n^{th} term, $T_n = a + (n-1)d$, where n = number of term.

$$T_9 = \$36\ 000 + (9-1)\$2\ 400$$
$$= \$\left(36\ 000 + 8(\$2\ 400)\right)$$
$$= \$55\ 200$$

3. (a) **Data:** The equation of the circle is $x^2 + y^2 - 12x - 22y + 152 = 0$.

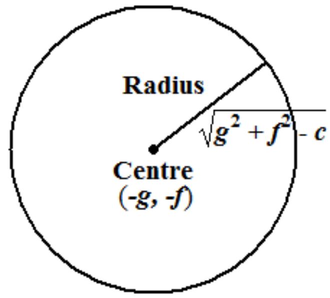
(i) **Required to Determine**: The coordinates of the center.

Solution:

The equation of the circle $x^2+y^2-12x-22y+152=0$ can be written as $x^2+y^2+2(-6)x+2(-11)y+152=0$ and which is of the general form $x^2+y^2+2gx+2fy+c=0$.

In this case, $g=-6,\ f=-11$ and $c=152$.

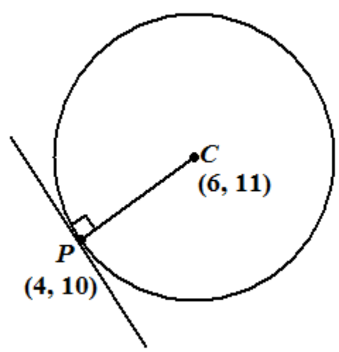
$\therefore$ The center of the circle is $(-(-6), -(-11))=(6, 11)$.



(ii) **Required to Determine:** The length of the radius.
Solution:

The length of the radius $=\sqrt{(-6)^2+(-11)^2-(152)}$
$=\sqrt{36+121-152}$
$=\sqrt{5}$ units

(iii) **Required to Determine:** The equation of the normal to the circle at $(4, 10)$.
Solution:



Let the center of the circle be $(6, 11)$.

Let P be the point $(4, 10)$.

The angle made by the tangent to a circle and the radius at the point of contact is a right angle.
Hence, the normal to the circle at P is the line (radius) CP.

The gradient of $CP = \frac{11-10}{6-4}$

$= \frac{1}{2}$

The equation of the normal to the circle at P is

$$\frac{y-10}{x-4} = \frac{1}{2}$$

$$2y - 20 = x - 4$$

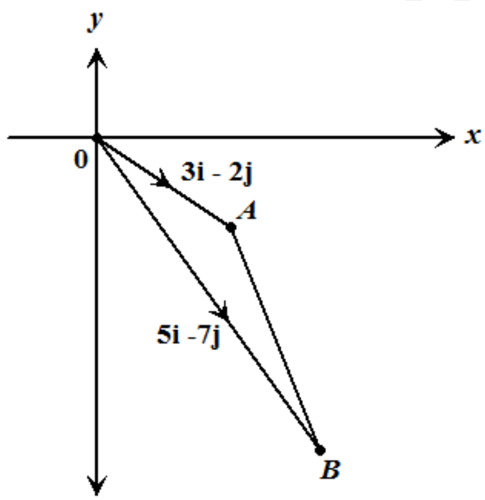
$$2y = x + 16$$

$$y = \frac{1}{2}x + 8$$

(b) **Data:** $OA = 3\mathbf{i} - 2\mathbf{j}$ and $OB = 5\mathbf{i} - 7\mathbf{j}$.

(i) **Required to Find:** The unit vector AB.

Solution:



$$AB = AO + OB$$

$$= -(3\mathbf{i} - 2\mathbf{j}) + 5\mathbf{i} - 7\mathbf{j}$$

$$= 2\mathbf{i} - 5\mathbf{j}$$

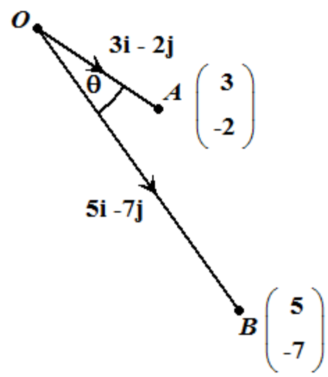
Any vector in the direction of $AB = \alpha(2\mathbf{i} - 5\mathbf{j}) = 2\alpha\mathbf{i} - 5\alpha\mathbf{j}$, where α is a scalar.

Since the vector is a unit vector, then its modulus is 1.

$$\sqrt{(2\alpha)^2+(-5\alpha)^2}=1$$
$$\sqrt{4\alpha^2+25\alpha^2}=1$$
$$\sqrt{29\alpha^2}=1$$
$$\alpha=\frac{1}{\sqrt{29}}$$

$\therefore$ The unit vector in the direction of $AB=\frac{1}{\sqrt{29}}(2\mathbf{i}-5\mathbf{j})$ or $\frac{2}{\sqrt{29}}\mathbf{i}-\frac{5}{\sqrt{29}}\mathbf{j}$

(ii) **Required to Find:** The acute angle AOB (to 1 decimal place).
Solution:

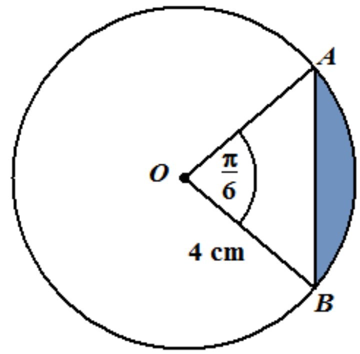


Let $A\hat{O}B=\theta°$
Hence, $OA.OB=|OA||OB|\cos\theta$ (Dot product law)

$$OA.OB=(3\times5)+(-2\times-2)$$
$$=29$$
$$|OA|=\sqrt{(3)^2+(-2)^2}$$
$$=\sqrt{13}$$
$$|OB|=\sqrt{(5)^2+(-7)^2}$$
$$=\sqrt{74}$$

So, $29=\sqrt{13}\sqrt{74}\cos\theta$
$$\cos\theta=\frac{29}{\sqrt{13}\sqrt{74}}$$
$$\theta=\cos^{-1}\left(\frac{29}{\sqrt{13}\sqrt{74}}\right)$$
$$\theta=20.7\underline{\underline{7}}°$$
$\theta=20.8°$ (correct to 1 decimal place)

4. (a) **Data:**



The diagram shows a circle, cente O and radius 4 cm. The sector AOB subtends angle $\frac{\pi}{6}$ radians at the center. Area of $\Delta AOB = \frac{1}{2}r^2 \sin\theta$.

Required to Calculate: The area of the shaded segment.

Calculation:

Area of the shaded segment = Area of sector AOB – Area of triangle AOB

$$= \left(\frac{\frac{\pi}{6}}{2\pi}\right)\left(\pi r^2\right) - \frac{1}{2}r^2 \sin\theta$$

$$= \left(\frac{1}{12} \times \pi (4)^2\right) - \frac{1}{2}(4)^2 \sin\left(\frac{\pi}{6}\right)$$

$$= \left(\frac{4\pi}{3} - 4\right) \text{cm}^2$$

(b) **Data:** $8\sin^2\theta = 5 - 10\cos\theta,\ 0° \le \theta \le 360°$.

Required to Find: θ, correct to one decimal place.

Solution:

$8\sin^2\theta = 5 - 10\cos\theta$

Recall: $\sin^2\theta + \cos^2\theta = 1$

$\therefore \sin^2\theta = 1 - \cos^2\theta$

Substituting this expression in the original equation, we obtain,

$8\left(1 - \cos^2\theta\right) = 5 - 10\cos\theta$

$8-8\cos^2\theta-5+10\cos\theta=0$

$-8\cos^2\theta+10\cos\theta+3=0$

$\times -1$

$8\cos^2\theta-10\cos\theta-3=0$

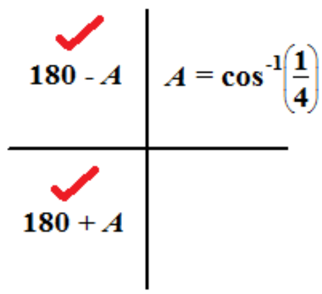
$(4\cos\theta+1)(2\cos\theta-3)=0$

$$\therefore \cos\theta=-\frac{1}{4} \text{ or } \frac{3}{2}$$

$-1\leq\cos\theta\leq 1 \ \ \forall\theta$

Hence, $\cos\theta=\frac{3}{2}$ has no real solutions.

Taking $\cos\theta=-\frac{1}{4}$



$\cos\theta=-\frac{1}{4}$ has solutions in quadrants 2 and 3.

$A=75.52°$

$\therefore \theta=180°-75.52°, 180°+75.52$

$=104.48°, 255.52°$

$=104.5°$ and $255.5°$ (correct to 1 decimal place)

(c) **Required to Prove:** $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}\equiv\tan\theta$

Proof:

Recall: $\sin 2\theta=2\sin\theta\cos\theta$ and $\cos 2\theta=\cos^2\theta-\sin^2\theta$ or $2\cos^2\theta-1$ or $1-2\sin^2\theta$.

Left hand side:

$$\frac{\sin\theta + 2\sin\theta\cos\theta}{1+\cos\theta + 2\cos^2\theta - 1} = \frac{\sin\theta(1+2\cos\theta)}{\cos\theta(1+2\cos\theta)}$$
$$= \frac{\sin\theta}{\cos\theta}$$
$$= \tan\theta$$

L.H.S. = R.H.S.

Q.E.D.

5. (a) **Required to Differentiate:** $(2x^2+3)\sin 5x$ with respect to *x*.

Solution:

Let $y = (2x^2+3)\sin 5x$

y is of the form $y = uv$ where

$u = 2x^2 + 3$ and $\frac{du}{dx} = 2(2x^{2-1}) = 4x$

$v = \sin 5x$ and $\frac{dv}{dx} = 5\cos 5x$

Let $t = 5x$

$\therefore v = \sin t$

$\frac{dv}{dx} = \frac{dv}{dt}\times\frac{dt}{dx}$ (Chain rule)

$= (\cos t)\times 5$

$= 5\cos 5x$

Recall: If $y = uv$, then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ (Product law)

Hence, $\frac{dy}{dx} = (\sin 5x)4x + (2x^2+3)\times 5\cos 5x$

$= 4x\sin 5x + 5(2x^2+3)\cos 5x$

So, $\frac{d}{dx}\{(2x^2+3)\sin 5x\} = 4x\sin 5x + 5(2x^2+3)\cos 5x$

(b) **Data:** $y = x^3 - 5x^2 + 3x + 1$

(i) **Required to Find:** The coordinates of all the stationary points on the curve.

Solution:

$y = x^3 - 5x^2 + 3x + 1$

The gradient function, $\frac{dy}{dx} = 3x^{3-1} - 5(2x^{2-1}) + 3$

$= 3x^2 - 10x + 3$

At a stationary point, $\frac{dy}{dx} = 0$.

Let $\frac{dy}{dx} = 0$

$$3x^2 - 10x + 3 = 0$$
$$(3x-1)(x-3) = 0$$
$$\therefore x = \frac{1}{3} \text{ and } x = 3$$

The x – coordinates of the stationary points are $\frac{1}{3}$ and 3.

When $x = \frac{1}{3}$
$$y = \left(\frac{1}{3}\right)^3 - 5\left(\frac{1}{3}\right)^2 + 3\left(\frac{1}{3}\right) + 1$$
$$= \frac{1}{27} - \frac{5}{9} + 2$$
$$= 1\frac{13}{27}$$

When $x = 3$
$$y = (3)^3 - 5(3)^2 + 3(3) + 1$$
$$= 27 - 45 + 9 + 1$$
$$= -8$$

$\therefore$ The stationary points on the curve are $\left(\frac{1}{3}, 1\frac{13}{27}\right)$ and $(3, -8)$.

The second derivative
$$\frac{d^2y}{dx^2} = 3\left(2x^{2-1}\right) - 10$$
$$= 6x - 10$$

When $x = \frac{1}{3}$ $\quad \frac{d^2y}{dx^2} = 6\left(\frac{1}{3}\right) - 10 \Rightarrow -\text{ve}$

$\therefore \left(\frac{1}{3}, 1\frac{13}{27}\right)$ is a maximum point.

When $x = 3$ $\quad \frac{d^2y}{dx^2} = 6(3) - 10 \Rightarrow +\text{ve}$

$\therefore (3, -8)$ is a minimum point.

OR

x	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
$\frac{dy}{dx}$	$+$	0	$-$

$\therefore$ At $x = \frac{1}{3}$, the stationary point is a maximum.

x	2.9	3	3.1
$\frac{dy}{dx}$	$-$	0	$+$

$\therefore$ At $x = 3$, the stationary point is a minimum.

(c) **Data:** A spherical balloon is filled with air at the rate of 200 cm^3s^{-1}.
Required to Calculate: The rate at which the radius is increasing, when the radius = 10 cm.
Calculation:

$V = \frac{4}{3}\pi r^3$ (V = Volume of the balloon, r = radius of the balloon)

$\frac{dV}{dt} = +200 \text{ cm}^3\text{s}^{-1}$ (+ $\Rightarrow$ and increasing rate)

$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$ (Chain rule)

$\frac{dV}{dr} = \frac{4}{3}\pi\left(3r^2\right)$

$= 4\pi r^2$

$\frac{dr}{dV} = \frac{1}{\frac{dV}{dr}} = \frac{1}{4\pi r^2}$

When $r = 10$ cm

$\frac{dr}{dt} = 200 \times \frac{1}{4\pi\left(10\right)^2}$

$= \frac{1}{2\pi} \text{ cms}^{-1}$

Note: $\frac{1}{2\pi} = +ve \Rightarrow$ and increasing rate.

So the radius of the balloon is increasing at the rate of $\frac{1}{2\pi}$ cms^{-1}.

6. (a) **Required to Evaluate:** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 3\theta \, d\theta$.

Solution:

$\int \cos 3\theta \, d\theta$

Let $t = 3\theta$ $\quad \frac{dt}{d\theta} = 3$ and $d\theta = \frac{dt}{3}$

$$\therefore \int \cos 3\theta \, d\theta \equiv \int \cos t \, \frac{dt}{3}$$

$$= \frac{\sin t}{3} + C \quad (C = \text{constant of integration})$$

$$= \frac{\sin 3\theta}{3} + C$$

Hence, $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 3\theta \, d\theta \equiv \left[\frac{\sin 3\theta}{3} + C\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

(The constant of integration, C, cancels off in a definite integral and may not be mentioned.)

$$= \left(\frac{\sin 3\left(\frac{\pi}{3}\right)}{3}\right) - \left(\frac{\sin 3\left(\frac{\pi}{6}\right)}{3}\right)$$

$$= \frac{1}{3}\sin(\pi) - \frac{1}{3}\sin\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{3}(0) - \frac{1}{3}(1)$$

$$= -\frac{1}{3}$$

(b) **Data:** $\frac{dy}{dx} = kx(x-1)$ for a curve (k = a constant). The gradient of the curve at $(2, 3)$ is 14.

(i) **Required to Calculate:** k

Calculation:

$\frac{dy}{dx} = 14$ at $x = 2$

$$\therefore 14 = k(2)^2 - k(2)$$
$$14 = 2k$$
$$k = 7$$

(ii) **Required to Calculate:** The equation of the curve.
Calculation:
The equation of the curve is

$$y = \int \left(7x^2 - 7x\right) dx$$

$$y = \frac{7x^{2+1}}{2+1} - \frac{7x^{1+1}}{1+1} + C \quad (C \text{ is the constant of integration})$$

$$y = \frac{7x^3}{3} - \frac{7x^2}{2} + C$$

The point $(2, 3)$ lies on the curve. Therefore the equation of the curve must be 'satisfied' when $x = 2$ and $y = 3$.

$$\therefore 3 = \frac{7(2)^3}{3} - \frac{7(2)^2}{2} + C$$

$$3 = \frac{56}{3} - \frac{28}{2} + C$$

$$C = 3 - \frac{56}{3} + \frac{28}{2}$$

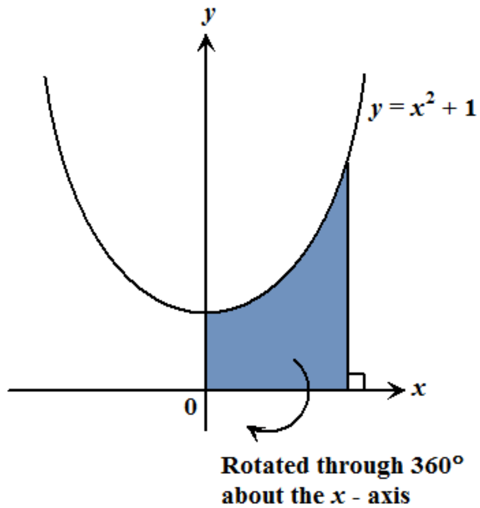
$$C = -\frac{5}{3}$$

$\therefore$ The equation of the curve is found to be $y = \frac{7x^3}{3} - \frac{7x^2}{2} - \frac{5}{3}$.

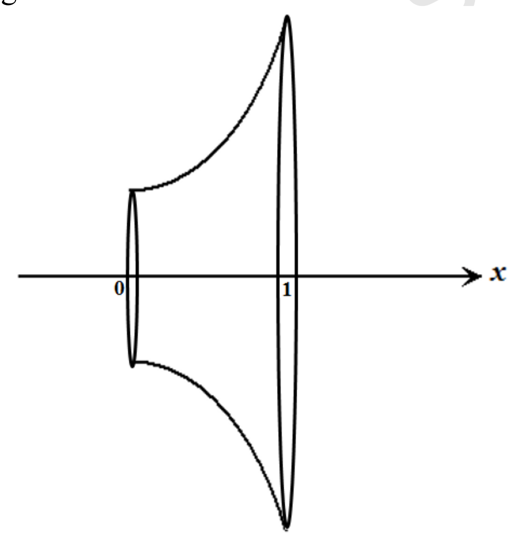
(c) **Data:** $y = x^2 + 1$

Required to Calculate: The volume of the solid generated when the region enclosed by the curve and the x – axis is rotated through 360° between $x = 0$ and $x = 1$.

Calculation:



A sketch of the solid generated looks like:



$$\text{Volume, } V = \pi\int_{x_2}^{x_1} y^2 \; dx$$

$$= \pi\int_0^1 \left(x^2+1\right)^2 \; dx$$

$$= \pi\int_0^1 \left(x^4+2x^2+1\right) dx$$

$$= \pi\left[\frac{x^5}{5}+\frac{2x^3}{3}+x\right]_0^1$$

$$= \pi\left\{\left(\frac{(1)^5}{5}+\frac{2(1)^3}{3}+1\right)-\left(\frac{(0)^5}{5}+\frac{2(0)^3}{3}+0\right)\right\}$$

$$= \pi\left(\frac{1}{5}+\frac{2}{3}+1\right) \text{ cubic units}$$

$$= \frac{28\pi}{15} \text{ cubic units}$$

7. (a) **Data:** The probability that a motorist stops at 1st, 2nd and 3rd traffic lights is 0.2, 0.5 and 0.8 respectively.

(i) **Required to Calculate:** The probability that the motorist stops at only one traffic light.

Calculation:

Le us define the 'stops' as

S_1 - the motorist stops at traffic light 1

S_2 - the motorists stops at traffic light 2

S_3 - the motorists stops at traffic light 3

$$P(S_1) = 0.2 \qquad P(S_2) = 0.5 \qquad P(S_3) = 0.8$$

$$\therefore P(S_1') = 1-0.2 \qquad P(S_2') = 1-0.5 \qquad P(S_3') = 1-0.8$$

$$= 0.8 \qquad = 0.5 \qquad = 0.2$$

P(Motorist stops at one traffic light)

$$= P\left(S_1 \text{ and } S_2' \text{ and } S_3' \text{ or } S_1' \text{ and } S_2 \text{ and } S_3' \text{ or } S_1' \text{ and } S_2' \text{ and } S_3\right)$$

$$= (0.2\times0.5\times0.2)+(0.8\times0.5\times0.2)+(0.8\times0.5\times0.8)$$

$$= 0.02+0.08+0.32$$

$$= 0.42$$

(ii) **Required to Calculate:** The probability that the motorist stops at at least two traffic lights.

Calculation:

P(Motorist stops at at least two traffic lights)

$$= P\left(S_1 \text{ and } S_2 \text{ and } S_3'\right)+P\left(S_1 \text{ and } S_2' \text{ and } S_3\right)+P\left(S_1' \text{ and } S_2 \text{ and } S_3\right)+$$

$$P\left(S_1 \text{ and } S_2 \text{ and } S_3\right)$$

$$= (0.2\times0.5\times0.2)+(0.2\times0.5\times0.8)+(0.8\times0.5\times0.8)+(0.2\times0.5\times0.8)$$

$$= 0.02+0.08+0.32+0.08$$

$$= 0.5$$

OR

P(Motorist stops at at least two traffic lights)
= 1 – P(Motorist stops at exactly one traffic light or does not stop at any of the three traffic lights)

$$=1-\{0.42+(0.2\times 0.5\times 0.8)\}$$
$$=1-(0.42+0.08)$$
$$=0.5$$

(b) **Data:** Table of values of a measure, x, and its frequency, f.
Required to Calculate: An estimation for the mean of x.
Calculation:

x	5 – 9	10 – 14	15 – 19	20 – 24
f	8	4	10	3

Since the values of x is grouped, then the mean, $\bar{x}=\frac{\sum fx}{\sum f}$, where x denotes the mid-class interval.

If the data was discrete or continuous, the mid-class interval would be the same.

Let us re-configure the table.
Assume that the data is discrete.
In this case, the lower class limit = the lower class boundary and the upper class limit = the upper class boundary.

Values of x	Mid Interval Value	f	fx
5 – 9	$\frac{5+9}{2}=7$	8	$8\times 7=56$
10 – 14	$\frac{10+14}{2}=12$	4	$4\times 12=48$
15 – 19	$\frac{15+19}{2}=17$	10	$10\times 17=170$
20 – 24	$\frac{20+24}{2}=22$	3	$3\times 22=66$
		$\sum f=25$	$\sum fx=340$

$$\bar{x} = \frac{\sum fx}{\sum f}$$
$$= \frac{340}{25}$$
$$= 13\frac{3}{5} \text{ or } 13.6$$

(c) **Data:** It rains on Monday.
The probability of rain following a day of rain is 25%. The probability of rain following a day of no rain is 12%.

(i) **Required to Draw:** A tree diagram to illustrate the data.
Solution:

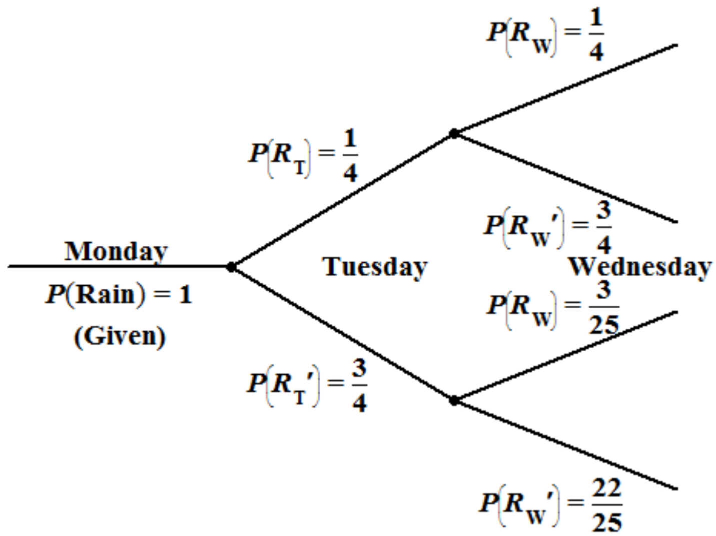
P(Rain, following a day of rain) $= 25\% = \frac{1}{4}$

Hence, P(No rain, following a day of rain $= 1 - \frac{1}{4} = \frac{3}{4}$ (Law of total probability)

P(Rain, following a day of no rain) $= 12\% = \frac{3}{25}$

Hence, P(No rain, following a day of no rain) $= 1 - \frac{3}{25} = \frac{22}{25}$(Law of total probability)

Let R_T = Rain on Tuesday
Let R_W = Rain on Wednesday



(ii) **Required to Calculate:** The probability of rain on Wednesday.
Calculation:

$$P(R_W) = P(R_T \text{ and } R_W) + P\left(R_T' \text{ and } R_W\right)$$

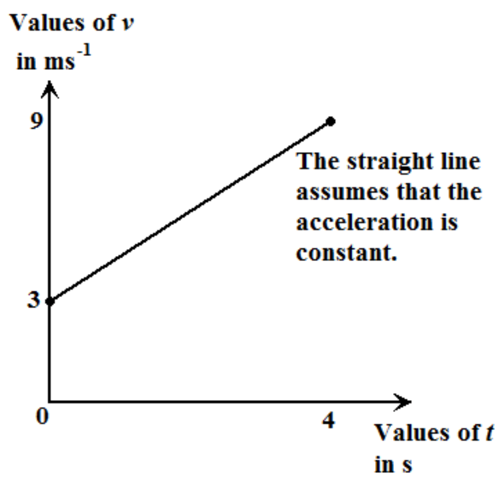
$$= \left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{3}{4} \times \frac{3}{25}\right)$$

$$= \frac{1}{16} + \frac{9}{100}$$

$$= \frac{61}{400} = 0.1525$$

8. (a) **Data:** A particle moves in a straight line with a velocity of 3 ms^{-1} at $t = 0$ s and 9 ms^{-1} at $t = 4$ s.

(i) **Required to Draw:** A velocity time graph of the motion of the particle.
Solution:



Note: If the acceleration is not constant, the line joining $(0, 3)$ to $(4, 9)$ would not be straight. Constant acceleration is assumed in this question.

(ii) **Required to Calculate:** The acceleration of the particle.
Calculation:
Assuming the acceleration is constant, the gradient of the straight line 'branch' will give the acceleration for the interval from $t = 0$ to $t = 4$.

$$\text{Gradient} = \frac{9-3}{4-0}$$
$$= \frac{6}{4}$$
$$= 1\frac{1}{2}\text{ ms}^{-2}$$

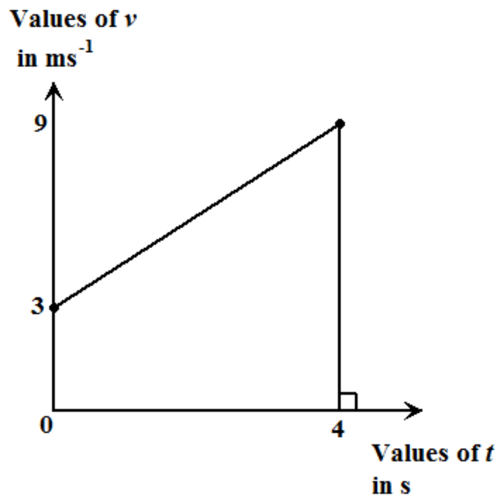
OR

Since the acceleration is constant, we may use the linear equation of motion $v = u + at$, where v = final velocity, u = initial velocity, a = acceleration and t = time.
Hence,

$$9 = 3 + a(4)$$
$$a = \frac{9-3}{4}\text{ ms}^{-2}$$
$$a = 1\frac{1}{2}\text{ ms}^{-2}$$

(iii) **Required to Calculate:** The increase in the displacement from $t = 0$ to $t = 4$.

Calculation:



The displacement, s, is obtained by calculating the area of the trapezium $= \frac{1}{2}(a+b)h$, where $h = 4$, $a = 3$ and $b = 9$.

$$s = \text{Area of trapezium}$$
$$= \frac{1}{2}(4)(3+9)$$
$$= 2 \times 12$$
$$= 24 \text{ m}$$

(b) **Data:** A particle moves in a straight line so that t s after passing O, its acceleration, a ms^{-2}, is given by $a = 3t - 1$ when $t = 2$, velocity, v, is 4 ms^{-1} and displacement, s, from O is 6 m.

(i) **Required to Calculate:** The velocity when $t = 4$.

Calculation:

$$v = \int a \; dt$$
$$v = \int (3t-1) \; dt$$
$$v = \frac{3t^2}{2} - t + C \quad \text{(where } C \text{ is the constant of integration)}$$

When $t = 2$, $v = 4$ (data)

$$\therefore 4 = \frac{3(2)^2}{2} - 2 + C$$
$$4 = 6 - 2 + C$$
$$C = 0$$
$$\therefore v = \frac{3t^2}{2} - t$$

When $t = 4$

$$v = \frac{3(4)^2}{2} - 4$$

$$= 24 - 4$$

$$= 20$$

$\therefore$ When $t = 4$, the velocity is 20 ms^{-1}.

(ii) **Required to Calculate:** The displacement when $t = 3$.

Calculation:

$$s = \int v \, dt$$

$$s = \int \left(\frac{3t^2}{2} - t \right) dt$$

$$s = \frac{3t^3}{3 \times 2} - \frac{t^2}{2} + K \quad (K \text{ is the constant of integration})$$

$$\text{s} = \frac{t^3}{2} - \frac{t^2}{2} + K$$

When $t = 2, s = 6$ (data)

$$\therefore 6 = \frac{(2)^3}{2} - \frac{(2)^2}{2} + K$$

$$6 = 4 - 2 + K$$

$$k = 4$$

$$\therefore s = \frac{t^3}{2} - \frac{t^2}{2} + 4$$

When $t = 3$

$$s = \frac{(3)^3}{2} - \frac{(3)^2}{2} + 4$$

$$= 13\frac{1}{2} - 4\frac{1}{2} + 4$$

$$= 13$$

$\therefore$ The displacement from O when $t = 3$ is 13 m.

END OF TEST