FAS-PASS Maths

CSEC ADDITIONAL MATHEMATICS MAY 2014

SECTION I

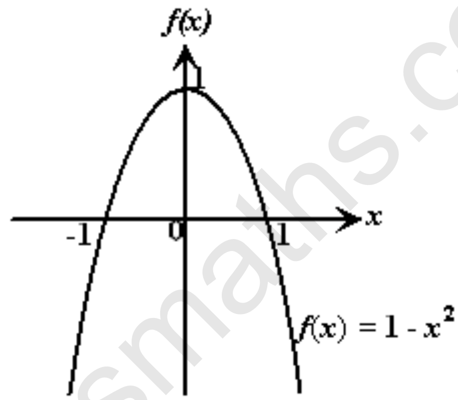
1. (a) (i) **Data:** $f : x \to 1 - x^2,\ x \in \Re$.

Required To Prove: f is not one to one

Proof:

$f : x \to 1 - x^2,\ x \in \Re$

A sketch is made of $f : x \to 1 - x^2,\ x \in \Re$

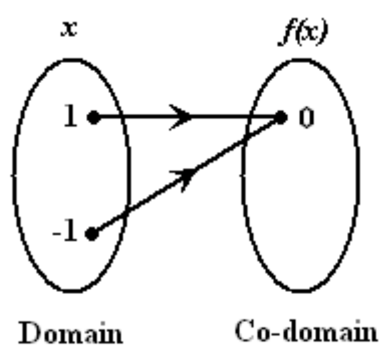


There can be more than one values of x that are mapped onto the same value of $f(x)$. In fact, with the exception of $x = 0$, there are always two values of x that are mapped onto the same value of $f(x)$.

For example,

$f(1) = 0$

$f(-1) = 0$



Hence, f is not one to one.

(ii) **Data:** $g: x \rightarrow \frac{1}{2}x - 3,\ x \in \Re$

a) **Required To Find:** $fg(x)$ and state its domain.

Solution:

$f: x \rightarrow 1 - x^2,\ x \in \Re$

$g: x \rightarrow \frac{1}{2}x - 3,\ x \in \Re$

$$\therefore fg(x) = 1 - \left(\frac{1}{2}x - 3\right)^2,\ x \in \Re$$

$$= 1 - \left(\frac{x^2}{4} - \frac{3x}{2} - \frac{3x}{2} + 9\right)$$

$$= 3x - 8 - \frac{x^2}{4}$$

$$fg(x) = 3x - 8 - \frac{x^2}{4}$$

Domain of $g(x)$ is $x \in \Re$.

Domain of $f(x)$ is $x \in \Re$.

$\therefore$ Domain of $fg(x)$ is $\Re \cap \Re$, which is $\Re$.

$$\therefore fg(x) = 3x - 8 - \frac{x^2}{4},\ x \in \Re.$$

b) **Required To Determine:** g^{-1} and to sketch g and g^{-1} on the same axes.

Solution:

$$g(x) = \frac{1}{2}x - 3$$

Let $y = \frac{1}{2}x - 3$

$$\therefore y + 3 = \frac{1}{2}x$$

$$x = 2(y + 3)$$

Replace y by x to obtain:

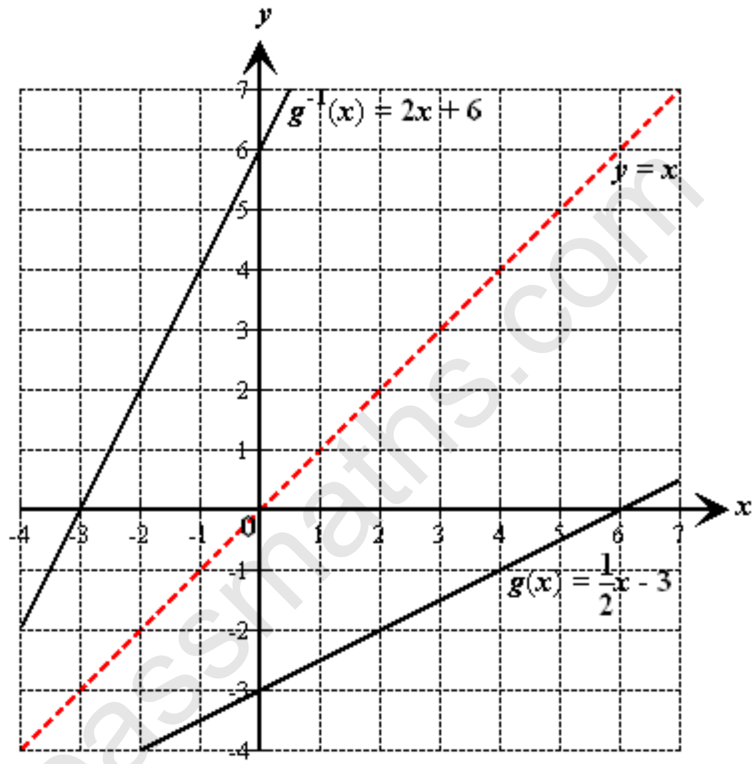
$$g^{-1}(x) = 2(x + 3)$$

$$= 2x + 6$$

Hence, $g(x)=\frac{1}{2}x-3$ and $g^{-1}(x)=2x+6$.

Recall that $g(x)\xrightarrow[y=x]{\text{Reflection in}} g^{-1}(x)$.

$g(x)$ and $g^{-1}(x)$ are sketched below.



(b) **Data:** When $2x^3+ax^2-5x-2$ is divided by $2x-1$, the remainder is -3.5.
Required To Determine: The value of the constant, *a*.
Solution:
We first recall the remainder and factor theorem.
If $f(x)$ is any polynomial and $f(x)$ is divided by $(ax+b)$ the remainder is $f\left(-\frac{b}{a}\right)$. If $f\left(-\frac{b}{a}\right)=0$, then $(ax+b)$ is a factor of $f(x)$.

Let $f(x)=2x^3+ax^2-5x-2$

$$\therefore f\left(-\frac{(-1)}{2}\right) = -3\frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = -3\frac{1}{2}$$

$$2\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) - 2 = -3\frac{1}{2}$$

$$\frac{1}{4} + \frac{1}{4}a - 2\frac{1}{2} - 2 = -3\frac{1}{2}$$

$$\frac{1}{4}a = -3\frac{1}{2} + 2\frac{1}{2} + 2 - \frac{1}{4}$$

$$\frac{1}{4}a = \frac{3}{4}$$

$$a = 3$$

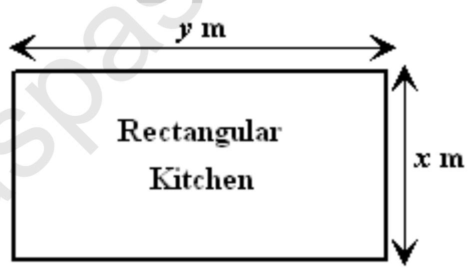
(c) **Data:** A kitchen of length y m and width x m.

Length of kitchen $= \frac{1}{2}$(Square of the width) and

Perimeter of the kitchen = 48 m.

Required To Calculate: The value of x and of y

Calculation:



Length of kitchen $= \frac{1}{2}$ (square of the width of the kitchen)

$$\therefore y = \frac{1}{2}x^2$$

Perimeter = 48 m

$$\therefore y + x + y + x = 48$$

$$2y + 2x = 48$$

$\div 2$

$$y + x = 24$$

Let

$y = \frac{1}{2}x^2 \qquad \ldots$ ❶

$y + x = 24 \qquad \ldots$ ❷

From ❷

$y = 24 - x$

Substituting $y = 24 - x$ into equation ❶

$$24 - x = \frac{1}{2}x^2$$

$\times 2$

$$48 - 2x = x^2$$

$$x^2 + 2x - 48 = 0$$

$$(x+8)(x-6) = 0$$

$$x = -8 \text{ or } 6$$

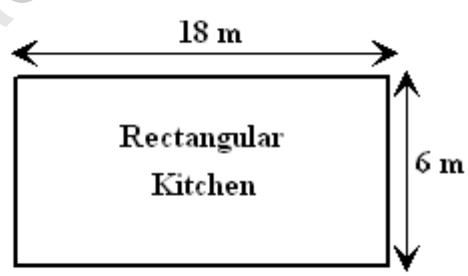
$x \neq$ negative, since the width of the kitchen cannot be a negative value.

$\therefore x = 6$ only

When $x = 6$

$$\begin{aligned} y &= 24 - x \\ &= 24 - 6 \\ &= 18 \end{aligned}$$

$\therefore$ The rectangular kitchen is 18 m by 6 m.



2. (a) **Data:** $f(x) = -2x^2 - 12x - 9$

(i) **Required To Express:** $f(x)$ in the form $k + a(x+h)^2$, where *k*, *a* and *h* are integers.

Solution:

$$\begin{aligned} -2x^2 - 12 - 9 &= -9 - (2x^2 + 12x) \\ &= -9 - 2(x^2 + 6x) \end{aligned}$$

One half the coefficient of $6x = \frac{1}{2}(6)$
$= 3$

$-2x^2 - 12x - 9 = * - 2(x+3)^2$, where $*$ is a number to be found.

$$\begin{aligned} -2(x+3)^2 &= -2(x+3)(x+3) \\ &= -2(x^2 + 6x + 9) \\ &= -2x^2 - 12x - 18 \end{aligned}$$

Hence,

$$\begin{aligned} * + (-18) &= -9 \\ \therefore * &= 9 \end{aligned}$$

$\therefore -2x^2 - 12x - 9 = 9 - 2(x+3)^2$

and which is of the form $k + a(x+h)^2$, where $k = 9 \in Z$, $a = -2 \in Z$ and $h = 3 \in Z$.

Alternative Method:

$$\begin{aligned} -2x^2 - 12x - 9 &= k + a(x+h)^2 \\ &= k + a(x^2 + 2hx + h^2) \\ &= ax^2 + 2ahx + (ah^2 + k) \end{aligned}$$

Equating coefficients:

$a = -2$

$$\begin{aligned} 2ah &= -12 \\ 2(-2)h &= -12 \\ h &= 3 \end{aligned}$$

$$\begin{aligned} ah^2 + k &= -9 \\ -2(3)^2 + k &= -9 \\ k &= 9 \end{aligned}$$

$\therefore f(x) = -2x^2 - 12x - 9 \equiv 9 - 2(x+3)^2$ and is of the form $k + a(x+h)^2$, where $k = 9 \in Z$, $a = -2 \in Z$ and $h = 3 \in Z$.

(ii) **Required To State:** The maximum value of $f(x)$.

Solution:

$$f(x) = 9 - 2(x+3)^2$$
$$\downarrow$$
$$\geq 0 \ \ \forall x$$

$\therefore$ The maximum value of $f(x) = 9 - 0$
$$= 9$$

Alternative Method:

$$f(x) = -2x^2 - 12 - 9$$
$$f'(x) = -2(2x) - 12$$
$$= -4x - 12$$

Let $f'(x) = 0$
$$-4x - 12 = 0$$
$$\therefore x = -3$$

The stationary value occurs when $f'(x) = 0$, that is at $x = -3$.

$$f(-3) = -2(-3)^2 - 12(-3) - 9$$
$$= -18 + 36 - 9$$
$$= 9$$

$f''(x) = -4\ (<0) \Rightarrow f(x) = 9$ is the maximum value of $f(x)$.

(iii) **Required To Determine:** The value of x for which $f(x)$ is a maximum.

Solution:

$$f(x) = -2x^2 - 12x - 9$$
$$= 9 - 2(x+3)^2$$
$$\geq 0 \ \ \forall x$$

$f(x)$ has a maximum of $9 - 0$ and this occurs when $-2(x+3)^2 = 0$ and $x = -3$

OR

AS SHOWN BEFORE

The stationary value of $f(x)$ occurs at $x=-3$ and $f(-3)$ gives a maximum value of 9.
Hence the maximum value of $f(x)$ occurs at $x=-3$.

(b) **Required To Solve:** For x in $3+5x-2x^2 \le 0$.

Solution:

$3+5x-2x^2 \le 0$

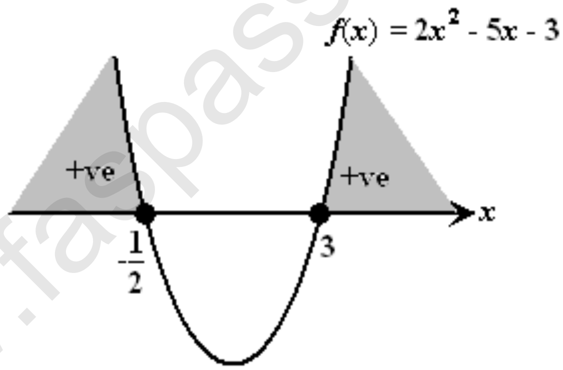
$\times -1$

$2x^2-5x-3 \ge 0$

$(2x+1)(x-3) \ge 0$

Hence, $y=2x^2-5x-3$ cuts the horizontal axis at $-\frac{1}{2}$ and 3.

The coefficient of $x^2>0 \Rightarrow$ the quadratic curve has a minimum point.



The solution set of x for which $3+5x-2x^2 \le 0$ is the same for $2x^2-5x-3 \ge 0$.
The range of values is $\{x : x \ge 3\} \cup \left\{x : x \le -\frac{1}{2}\right\}$ as illustrated in the diagram above.

(c) **Data:** Series is given by $0.2+0.02+0.002+0.0002+\ldots$

(i) **Required To Prove:** That the given series is geometric.
Proof:
Let us look at the terms of the series and the number of each of the terms.

No. of	1	2	3	4	…

term					
Term	0.2	0.02 $=0.2\times10^{-1}$	0.002 $=0.2\times10^{-2}$	0.0002 $=0.2\times10^{-3}$	…

This is of the form:

$$a \qquad ar \qquad ar^2 \qquad ar^3 \qquad \ldots$$

where $a=0.2$ and $r=10^{-1}$.

Hence, T_n, the n^{th} term is of the form $T_n = ar^{n-1}$, where n is the number of the term.

Hence, the series is a geometric progression with 1st term, $a=0.2$ and the common ratio, $r=10^{-1}$.

(ii) **Required To Calculate:** The sum to infinity of the series.
Calculation:

Recall, the sum to infinity of the geometric series, $S_\infty = \dfrac{a}{1-r}$, $|r|<1$.

$$\therefore S_\infty = \frac{0.2}{1-10^{-1}}$$
$$= \frac{0.2}{1-0.1}$$
$$= \frac{0.2}{0.9}$$
$$= \frac{2}{9} \text{ (as a fraction in exact form)}$$

SECTION II

3\. (a) (i) **Data:** The lines with equations $x+3y=6$ and $kx+2y=12$ are perpendicular.
Required To Calculate: The value of k
Calculation:
Let us consider the line with equation
$x+3y=6$
$3y=-x+6$
$y=-\dfrac{1}{3}x+2$ and is of the form $y=mx+c$, where $m=-\dfrac{1}{3}$ is the gradient.

$kx+2y=12$

$2y=-kx+12$

$y=-\frac{k}{2}x+6$ and is of the form $y=mx+c$, where $m=-\frac{k}{2}$ is the gradient.

Hence,

$-\frac{1}{3}\times-\frac{k}{2}=-1$ (the product of the gradients of perpendicular lines $=-1$)

$\frac{k}{6}=-1$

$k=-6$

(ii) **Data:** The center of a circle is at the point of intersection of the lines given in 3(a)(i).
The radius of the circle is 5 cm.
Note: The radius should be given as 5 units.

Required To Determine: The equation of the circle
Solution:
Let

$x+3y=6 \quad \ldots❶$

$-6x+2y=12 \ldots❷$

From ❷

$-6x+2y=12$

$2y=12+6x$

$y=6+3x$

Substitute $y=6+3x$ into ❶

$x+3(6+3x)=6$

$x+18+9x=6$

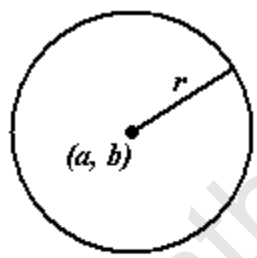
$10x=-12$

$x=-1\frac{1}{5}$

When $x=-1\frac{1}{5}$

$$y = 6 + 3\left(-1\frac{1}{5}\right)$$
$$= 6 - 3\frac{3}{5}$$
$$= 2\frac{2}{5}$$

$\therefore$ The center of the circle is $\left(-1\frac{1}{5},\ 2\frac{2}{5}\right)$.



The equation of the circle with center (a, b) and radius r, is
$(x-a)^2 + (y-b)^2 = r^2$.

$\therefore$ Equation of the circle is

$$(x - (-1\tfrac{1}{5}))^2 + (x - 2\tfrac{2}{5})^2 = (5)^2$$

$$= (x + \tfrac{6}{5})^2 + (x - \tfrac{12}{5})^2 = (5)^2$$

$$= x^2 + \frac{12}{5}x + \frac{36}{25} + y^2 - \frac{24}{5}y + \frac{144}{25} = 25$$

x 25

$$25x^2 + 25y^2 + 60x - 120y + 36 + 144 = 625$$

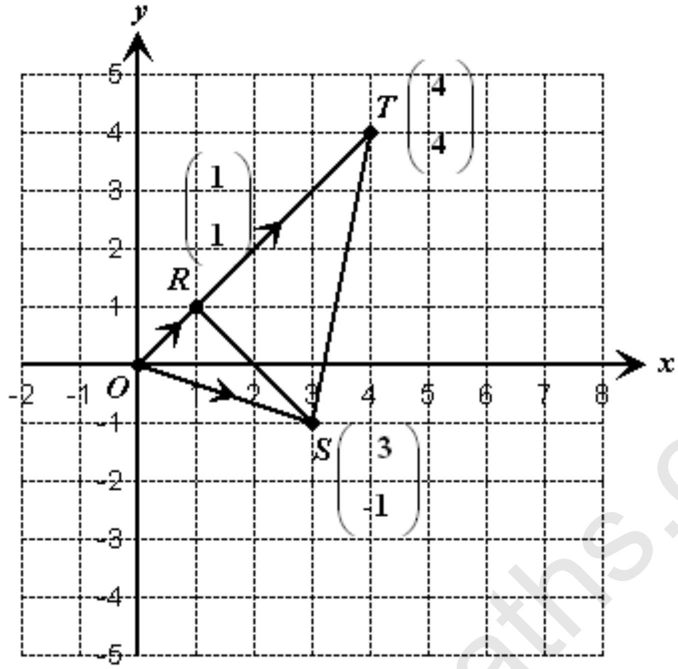
$$25x^2 + 25y^2 + 60x - 120y = 445$$

$$25x^2 + 25y^2 + 60x - 120y - 445 = 0$$

(b) **Data:** $OR = \begin{pmatrix}1\\1\end{pmatrix}$, $OS = \begin{pmatrix}3\\-1\end{pmatrix}$ and $OR = \begin{pmatrix}4\\4\end{pmatrix}$.

(i) **Required To Prove:** $T\hat{R}S = 90°$.

Proof:



If $T\hat{R}S = 90°$, then $RT \,.\, RS = 0$ by the dot product law.

$a\,.\,b = |a||b|\cos\theta$

If $\theta = 90°$, then $a\,.\,b = 0$.

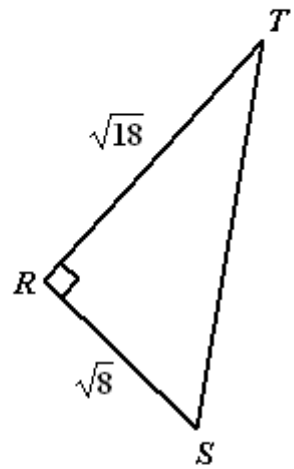
$$RT = RO + OT$$
$$= -\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}4\\4\end{pmatrix}$$
$$= \begin{pmatrix}3\\3\end{pmatrix}$$

$$RS = RO + OS$$
$$= -\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}3\\-1\end{pmatrix}$$
$$= \begin{pmatrix}2\\-2\end{pmatrix}$$

$$RT\,.\,RS = \begin{pmatrix}3\\3\end{pmatrix} . \begin{pmatrix}2\\-2\end{pmatrix}$$
$$= (3\times 2) + (3\times -2)$$
$$= 0$$

Q.E.D.

(ii) **Required To Calculate:** The length of the hypotenuse of triangle TRS.
Calculation:



$$RT = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

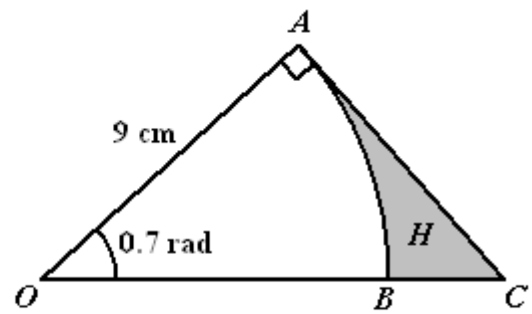
$$RS = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$\left|RT\right| = \sqrt{(3)^2 + (3)^2}$$
$$= \sqrt{18}$$

$$\left|RS\right| = \sqrt{(2)^2 + (-2)^2}$$
$$= \sqrt{8}$$

$$TS^2 = \left(\sqrt{18}\right)^2 + \left(\sqrt{8}\right)^2 \quad \text{Pythagoras' Theorem}$$
$$TS^2 = 18 + 8$$
$$= 26$$
$$TS = \sqrt{26} \text{ units}$$

4. (a) **Data:** Diagram showing a sector *OAB* of a circle with center *O*, radius 9 cm and angle at O = 0.7 radians.



(i) **Required To Calculate:** The area of sector *OAB*.
Calculation:

Area of sector $OAB = \frac{1}{2}(9)^2 \times 0.7$
$= 28.35 \text{ cm}^2$

(ii) **Required To Calculate:** The area of the shaded region, *H*.
Calculation:
Angle AOC = 90^0
(The angle made by the tangent to a circle and a radius, at the point of contact, is a right angle

$\frac{AC}{9} =$ tan (0.7 radians)

$AC = 7.580$ cm

Area of the triangle *OAB* = ½ (9 x 7.580) cm^2
$= 34.11 \text{ cm}^2$
$\therefore$ Area of the shaded region, *H* = Area of triangle *OAC* – Area of sector *OAB*
$= (34.11- 28.35) \text{ cm}^2$
$= 5.76 \text{ cm}^2$

(b) **Data:** $\sin\frac{\pi}{6} = \frac{1}{2}, \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Required To Prove: $\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}\left(\sqrt{3}\cos x - \sin x\right)$, where x is acute.

Proof: By the compound angle formula

$$\cos\left(x+\frac{\pi}{6}\right)=\cos x\cos\frac{\pi}{6}-\sin x\sin\frac{\pi}{6}$$
$$=(\cos x)\frac{\sqrt{3}}{2}-(\sin x)\frac{1}{2}$$
$$=\frac{1}{2}\left(\sqrt{3}\cos x-\sin x\right)$$

Q.E.D.

(c) **Required To Prove:** $\dfrac{\tan\theta\sin\theta}{1-\cos\theta}\equiv 1+\dfrac{1}{\cos\theta}$

Proof:

Left hand side:

$$\frac{\tan\theta\sin\theta}{1-\cos\theta}=\frac{\frac{\sin\theta}{\cos\theta}.\sin\theta}{1-\cos\theta}$$
$$=\frac{\sin^2\theta}{\cos\theta(1-\cos\theta)}$$

Recall: $\sin^2\theta+\cos^2\theta=1$

$$\therefore \sin^2\theta=1-\cos^2\theta$$
$$=(1-\cos\theta)(1+\cos\theta)$$

Hence,

$$\frac{\tan\theta\sin\theta}{1-\cos\theta}=\frac{(1+\cos\theta)\cancel{(1-\cos\theta)}}{\cos\theta\cancel{(1-\cos\theta)}}$$
$$=\frac{1+\cos\theta}{\cos\theta}$$
$$=\frac{1}{\cos\theta}+\frac{\cos\theta}{\cos\theta}$$
$$=\frac{1}{\cos\theta}+1 \quad \text{(Right hand side)}$$

Q.E.D.

SECTION III

5. (a) **Data:** A curve has equation $y=3+4x-x^2$ and the point $P(3,\ 6)$ lies on the curve.

Required To Find: The equation of the tangent to the curve at P

Solution:

$$y=3+4x-x^2$$

Gradient function, $\frac{dy}{dx} = 4 - 2x$

$\therefore$ The gradient of the tangent at $P = 4 - 2(3)$

$= -2$

Hence, the equation of the tangent at P is

$$\frac{y-6}{x-3} = -2$$

$$y - 6 = -2(x-3)$$

$$y - 6 = -2x + 6$$

$$y = -2x + 12$$

$2x + y - 12 = 0$ which is of the form $ax + by + c = 0$, where $a = 2 \in Z$, $b = 1 \in Z$ and $c = -12 \in Z$.

(b) **Data:** $f(x) = 2x^3 - 9x^2 - 24x + 7$

(i) **Required To Find:** All the stationary points of $f(x)$.

Solution:

$y = 2x^3 - 9x^2 - 24x + 7$

$\therefore$ Gradient function, $\frac{dy}{dx} = 2\left(3x^{3-1}\right) - 9(2x) - 24$

$= 6x^2 - 18x - 24$

At a stationary point, $\frac{dy}{dx} = 0$.

Let

$6x^2 - 18x - 24 = 0$

$\div 6$

$x^2 - 3x - 4 = 0$

$(x+1)(x-4) = 0$

$\therefore$ Stationary points occur at $x = -1$ and $x = 4$.

$$f(-1) = 2(-1)^3 - 9(-1)^2 - 24(-1) + 7$$
$$= -2 - 9 + 24 + 7$$
$$= 20$$

$$f(4) = 2(4)^3 - 9(4)^2 - 24(4) + 7$$
$$= 128 - 144 - 96 + 7$$
$$= -105$$

$\therefore$ Stationary points are $(-1, 20)$ and $(4, -105)$.

(ii) **Required To Determine:** The nature of each of the stationary points of $f(x)$.

Solution:

$$\frac{d^2y}{dx^2} = 6(2x) - 18$$
$$= 12x - 18$$

When $x = -1$

$$\frac{d^2y}{dx^2} = 12(-1) - 18$$
$$= -30 \text{ (Negative)}$$

$\therefore (-1,\ 20)$ is a maximum point.

When $x = 4$

$$\frac{d^2y}{dx^2} = 12(4) - 18$$
$$= 30 \text{ (Positive)}$$

$\therefore (4,\ -105)$ is a minimum point.

6. (a) **Required To Evaluate:** $\int_2^4 x(x^2 - 2)\ dx$

Solution:

$$\int_2^4 \left\{x(x^2-2)\right\}\ dx = \int_2^4 (x^3 - 2x)\ dx$$
$$= \left[\frac{x^4}{4} - x^2\right]_2^4$$

The constant of integration is omitted in a definite integral as it cancels out.

$$\int_2^4 \left\{x(x^2-2)\right\}\ dx = \left\{\frac{(4)^4}{4} - (4)^2\right\} - \left\{\frac{(2)^4}{4} - (2)^2\right\}$$
$$= (64 - 16) - (4 - 4)$$
$$= 64 - 16$$
$$= 48$$

(b) **Required To Evaluate:** $\int_0^{\frac{\pi}{3}} (4\cos x + 2\sin x)\ dx$

Solution:

$$\int_0^{\frac{\pi}{3}} (4\cos x + 2\sin x)\, dx = \left[4(\sin x) + 2(-\cos x)\right]_0^{\frac{\pi}{3}}$$

$$= \left[4\sin x - 2\cos x\right]_0^{\frac{\pi}{3}}$$

$$= \left(4\sin\left(\frac{\pi}{3}\right) - 2\cos\left(\frac{\pi}{3}\right)\right) - \left(4\sin(0) - 2\cos(0)\right)$$

$$= \left(4 \times \frac{\sqrt{3}}{2} - 2 \times \frac{1}{2}\right) - (4 \times 0 - 2 \times 1)$$

$$= \left(2\sqrt{3} - 1\right) - (-2)$$

$$= 2\sqrt{3} + 1$$

(c) **Data:** A curve with gradient function $\frac{dy}{dx} = 6x^2 - 1$ passes through the point $P(2, -5)$.

(i) **Required To Determine:** The equation of the curve.
Solution:

$$\frac{dy}{dx} = 6x^2 - 1$$

$\therefore$ Equation of the curve is $\int (6x^2 - 1)\, dx$

$$y = \frac{6x^3}{3} - x + C \quad \text{(where } C \text{ is a constant)}$$

$$y = 2x^3 - x + C$$

Since $(2, -5)$ lies on the curve

$$-5 = 2(2)^3 - 2 + C$$

$$-5 = 16 - 2 + C$$

$$C = -19$$

$\therefore$ Equation of the curve is $y = 2x^3 - x - 19$.

(ii) **Required To Calculate:** The area bounded by the curve, the x – axis and the verticals $x = 3$ and $x = 4$.
Calculation:
The region whose area is required lies entirely above the x – axis.

Area required $= \int_3^4 (2x^3 - x - 19)\, dx$

$$= \left[\frac{2x^4}{4} - \frac{x^2}{2} - 19x \right]_3^4$$

$$= \left\{ \frac{2(4)^4}{4} - \frac{(4)^2}{2} - 19(4) \right\} - \left\{ \frac{2(3)^4}{4} - \frac{(3)^2}{2} - 19(3) \right\}$$

$$= \{128 - 8 - 76\} - \left\{ 40\frac{1}{2} - 4\frac{1}{2} - 57 \right\}$$

$$= 44 - (-21)$$

$$= 65 \text{ square units}$$

SECTION IV

7\. (a) **Data:** A class has 60 students. 27 study Mathematics, 20 study Biology and 22 do not study either Mathematics or Biology.

(i) **Required To Calculate:** The probability that a student chosen at random studies both Mathematics and Biology.

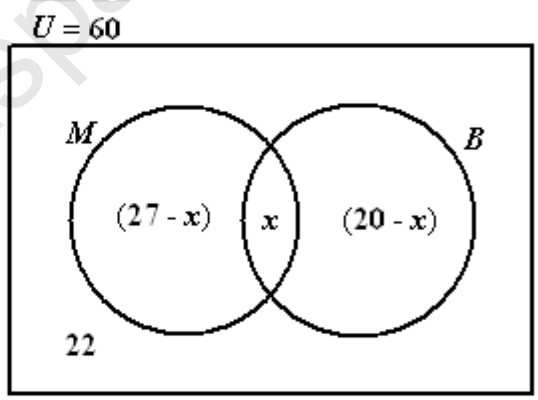
Calculation:

Creating a Venn diagram to illustrate the data:

Let:

$M = \{\text{Students who study Mathematics}\}$

$B = \{\text{Students who study Biology}\}$



Let x be the number of students who study both Mathematics and Biology.

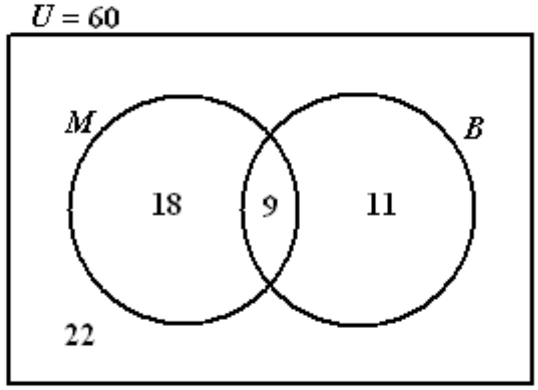
Hence,

$$(27 - x) + x + (20 - x) + 22 = 60$$

$$69 - x = 60$$

$$x = 9$$

$\therefore$ 9 students study both Mathematics and Biology.



The probability that a student chosen at random studies both Mathematics and Biology

$$= \frac{\text{No. of students studying both Mathematics and Biology}}{\text{Total no. of students}}$$

$$= \frac{9}{60}$$

$$= \frac{3}{20}$$

(ii) **Required To Calculate:** The probability that a student chosen at random studies Biology only.

Calculation:

The probability that a student chosen at random studies Biology only

$$= \frac{\text{No. of students studying Biology only}}{\text{Total no. of students}}$$

$$= \frac{11}{60}$$

(b) **Data:** Two ordinary 6 sided dice are tossed.
S represents the sum of their scores.
An incomplete sample space diagram for *S* is given.

(i) **Required To Complete:** The sample space diagram.

Solution:

6	$6+1=7$	$6+2=8$	$6+3=9$	$6+4=10$	$6+5=11$	$6+6=12$
5	$5+1=6$	$5+2=7$	$5+3=8$	$5+4=9$	$5+5=10$	$5+6=11$
4	$4+1=5$	$4+2=6$	$4+3=7$	$4+4=8$	$4+5=9$	$4+6=10$
3	$3+1=4$	$3+2=5$	$3+3=6$	$3+4=7$	$3+5=8$	$3+6=9$
2	$2+1=3$	$2+2=4$	$2+3=5$	$2+4=6$	$2+5=7$	$2+6=8$
1	$1+1=2$	$1+2=3$	$1+3=4$	$1+4=5$	$1+5=6$	$1+6=7$
	1	**2**	**3**	**4**	**5**	**6**

Score on Die 1			**Sum, *S***			
6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
Score on Die 2	**1**	**2**	**3**	**4**	**5**	**6**

(ii) a) **Required To Find:** $P(S>9)$

Solution:

$$P(S>9)=\frac{\text{Number of scores }>9}{\text{Total number of scores}}$$

$$=\frac{1+2+3}{6\times 6}$$

$$=\frac{6}{36}$$

$$=\frac{1}{6}$$

b) **Required To Find:** $P(S \leq 4)$

Solution:

$$P(S \leq 4) = \frac{\text{Number of scores } \leq 4}{\text{Total number of scores}}$$
$$= \frac{3+2+1}{6 \times 6}$$
$$= \frac{6}{36}$$
$$= \frac{1}{6}$$

(iii) **Data:** D is the difference between the scores on the dice. An incomplete table for d and $P(D = d)$ is given.

Required To Find: The value of a, b and c in the table, that is, to complete the table.

Solution:

For $d = 1$ the scores on the dice could be

6 – 5	5 – 6
5 – 4	4 – 5
4 – 3	3 – 4
3 – 2	2 – 3
2 – 1	1 – 2

Total number of scores in which the difference is 1 is 10

$$\therefore P(D = d = 1) = \frac{10}{36}$$
$$= \frac{5}{18}$$

For $d = 3$ the scores on the dice could be

6 – 3	3 – 6
5 – 2	2 – 5
4 – 1	1 – 4

Total number of scores in which the difference is 3 is 6

$$\therefore P(D = d = 3) = \frac{6}{36}$$
$$= \frac{1}{6}$$

For $d = 5$ the scores on the dice could be

1 – 6
6 – 1

Total number of scores in which the difference is 5 is 2

$$\therefore P(D = d = 5) = \frac{2}{36}$$
$$= \frac{1}{18}$$

$$\therefore a = \frac{5}{18}, b = \frac{1}{6} \text{ and } c = \frac{1}{18}.$$

(c) **Data:** Stem and leaf diagram showing the scores of 51 people in an aptitude test.

(i) **Required To Calculate:** The median and the quartiles for the data.
Calculation:
There are 51 scores in the data.
The middle score of 51 scores is the 26$^{\text{th}}$ score.
From the diagram, the 26$^{\text{th}}$ score in ascending or descending degree of magnitude is 71.
Hence, the median score is 71.

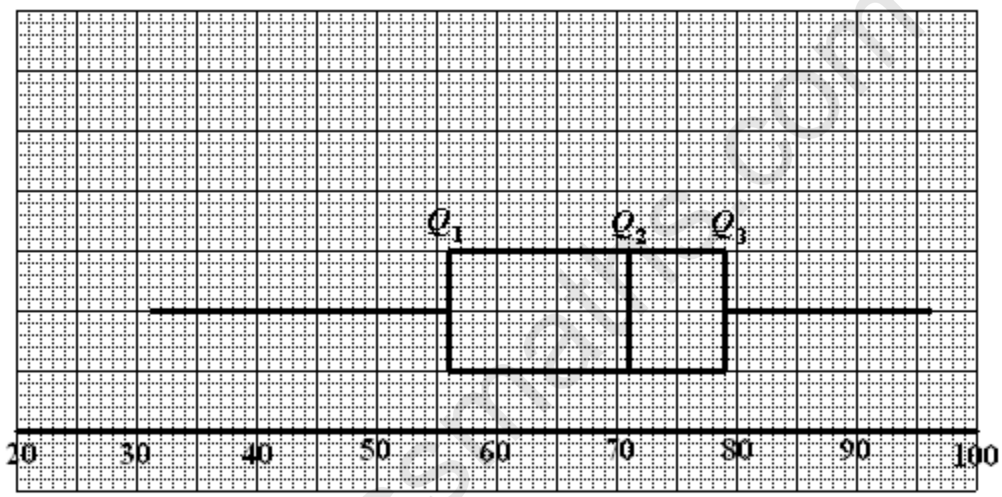
$\frac{1}{4}$ of $51 = 12\frac{3}{4}$
Hence, the 13$^{\text{th}}$ value is the lower quartile, Q_1.
$Q_1 = 56$ (obtained from the diagram)

$\frac{3}{4}$ of $51 = 38\frac{1}{4}$
Hence, the 39$^{\text{th}}$ value is the upper quartile, Q_3.
$Q_3 = 79$

The interquartile range is $Q_3 - Q_1 = 79 - 56$
$= 23$

The semi-interquartile range is $\frac{1}{2}(Q_3 - Q_1) = \frac{1}{2}(23)$
$= 11.5$

(ii) **Required To Construct:** A box and whisker plot to illustrate the data and to comment on the data.
Solution:
The median of the data is the 26th value, which is 71.
There are 25 scores before the median.
The 13th score is the lower median, which is 56.
There are 25 scores after the median score.
$26+13=39$ and hence the 39th score is the upper median. This is 79.



Scale 2 cm ≡ 10 marks/units

Comments
$\frac{1}{4}$ of the scores were less than 56.
$\frac{1}{4}$ of the scores were more than 79.
$\frac{1}{4}$ of the scores were between 56 and 71.
$\frac{1}{4}$ of the scores were between 71 and 79.

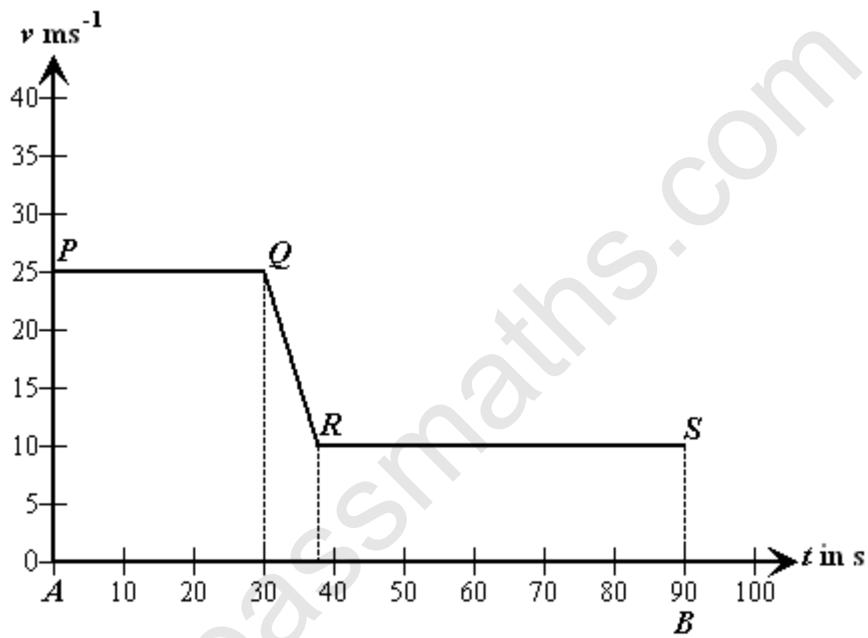
Interquartile range = 23

$$Q_1 - 1\frac{1}{2}\text{I.Q.R} = 56 - 1\frac{1}{2}(23)$$
$$= 21.5$$

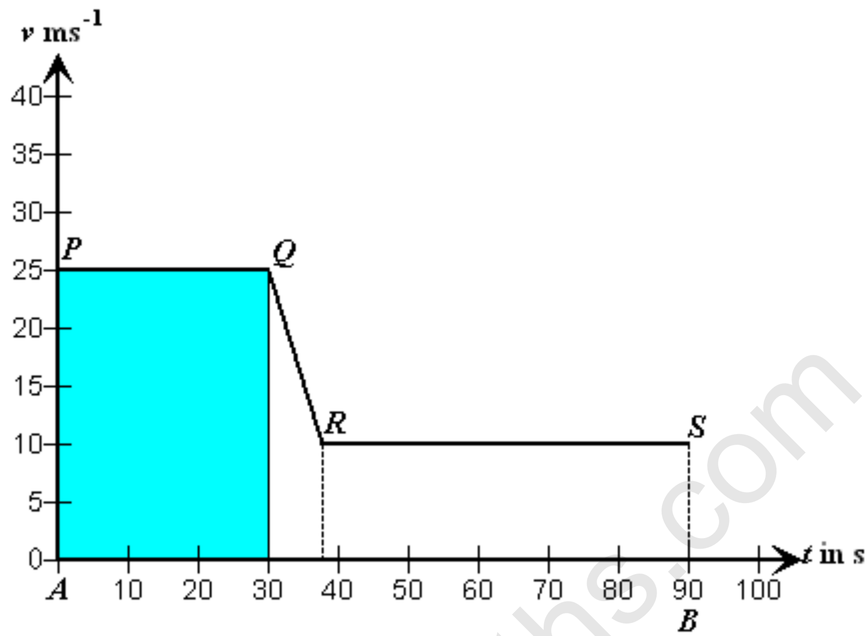
$$Q_3 + 1\frac{1}{2}\text{I.Q.R.} = 113.5$$

There are no values < 21.5 or > 113.5
Hence, there are no outliers.
The whisker to the left of the box is longer than the whisker to the right of the box. Hence, there are more extreme values towards the negative end and so the distribution is said to be negatively skewed.

8. (a) **Data:** A velocity – time graph showing the motion of a car travelling 1410 m from A to B.



(i) **Required To Calculate:** The distance covered by the car before it decelerates.
Note: A car cannot travel. It is better to have said 'the distance covered by the car'.
Calculation:
The 'branches' of the graph are named PQ, QR and RS for convenience and are as shown on the diagram.
The car moves at a constant 25 ms^{-1} for 30 seconds before starting to decelerate.



$\therefore$ Distance covered = Area under the 'branch'

$= 30 \times 25$

$= 750$ m

OR

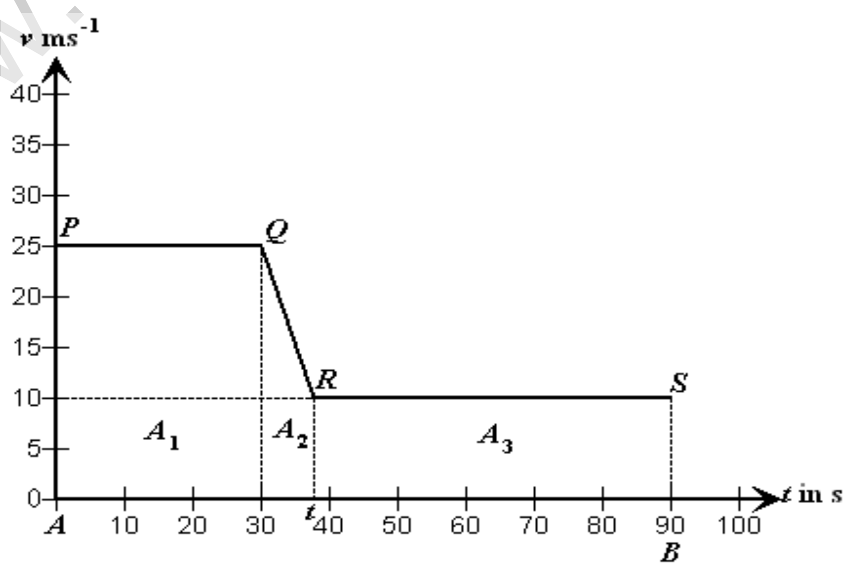
Distance $=$ Constant Speed $\times$ Time

$= 25 \text{ ms}^{-1} \times 30 \text{ s}$

$= 750$ m

(ii) **Required To Calculate:** The deceleration of the car as it goes from 25 ms^{-1} to 10 ms^{-1}.

Calculation:



The regions A_1, A_2 and A_3 are as shown on the diagram
The distance covered by the car is the area under the graph.
From $t=30$ to $t=90$ the distance can be obtained by the sum of the area of the regions A_2 and A_3.

Area of $A_1 = 750$

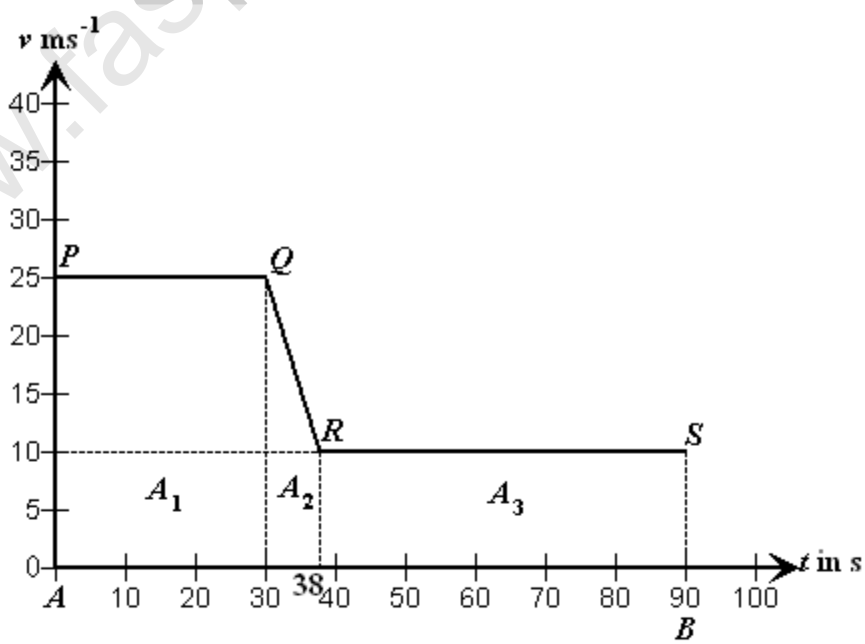
$$\therefore \text{Area of } \left(A_2 + A_3\right) = 1410 - 750$$
$$= 660$$

$$\text{Area of } A_2 == \frac{1}{2}\left(25+10\right)\times\left(t-30\right)$$

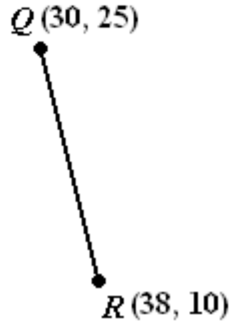
$$\text{Area of } A_3 = \left(90-t\right)\times 10$$

Hence,

$$\frac{1}{2}\left(35\right)\left(t-30\right)+10\left(90-t\right)=660$$
$$17\frac{1}{2}t-525+900-10t=660$$
$$7\frac{1}{2}t=285$$
$$t=38$$



Hence, the acceleration of the car is the gradient of the branch QR.



Acceleration $= \dfrac{10-25}{38-30}$

$= -\dfrac{15}{8}$ ms^{-2}

$\therefore$ The deceleration is $\dfrac{15}{8}$ ms^{-2}.

OR

From Q to R:
Initial velocity, u = 25 ms^{-1}
Final velocity, v = 10 ms^{-1}
Time = 8 s
The acceleration, a, is constant

$v = u + at$

$\therefore 10 = 25 + a(8)$

$a = -\dfrac{15}{8}$

$\therefore$ The deceleration is $\dfrac{15}{8}$ ms^{-2}.

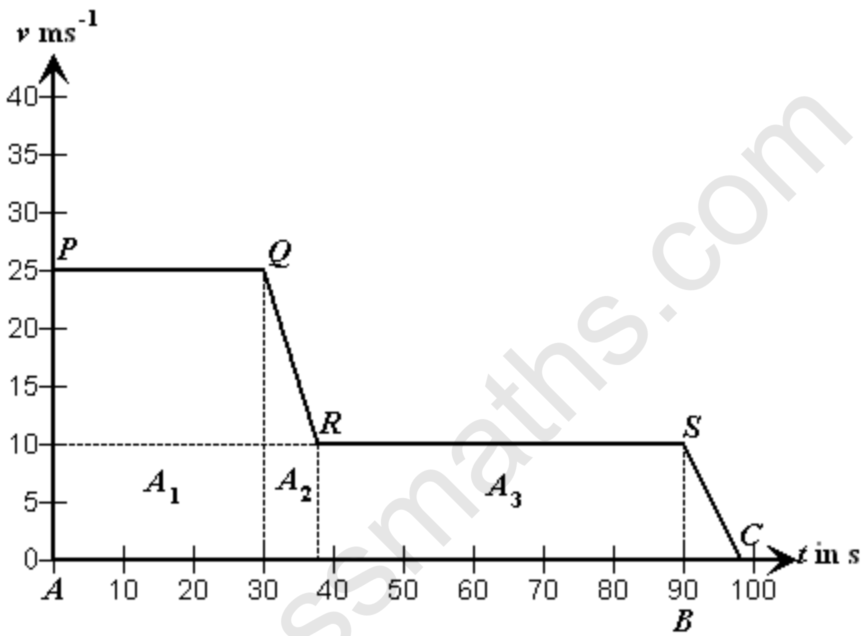
(iii) **Required To Find:** The period of time for which the car maintained the speed of 10 ms^{-1}.
Solution:
The car maintains the speed of 10 ms^{-1} from R to S, that is, from $t = 38$ to $t = 90$.

$\therefore$ Car maintains a speed of 10 ms^{-1} for $90 - 38 = 52$ s.

(iv) **Data:** The car comes to rest at C, decelerating uniformly from B and covering a further 30 m.

Required To Calculate: The average velocity of the car from A to C

Calculation:



In the branch from S to C, the area of triangle SBC = 30 (since 30 m is covered).

$SB = 10$

$$\therefore \frac{BC \times 10}{2} = 30$$

$$BC = 6$$

$\therefore$ Time at C = 96

OR

$u = 10 \text{ ms}^{-1}$
$v = 0 \text{ ms}^{-1}$
$s = 30$ m

$$v^2 = u^2 + 2as$$

$$0 = 10^2 + 2a(30)$$

$$\therefore a = -\frac{100}{60}$$

$$= -\frac{5}{3}$$

$$v = u + at$$

$$0 = 10 + \left(-\frac{5}{3}t\right)$$

$$t = 6$$

From A to C:

$$\frac{\textit{Total distance covered}}{\textit{Total time taken}} = \frac{1410 + 30}{96}$$

$$= \frac{1440}{96}\ \text{ms}^{-1}$$

$$= 15\ \text{ms}^{-1}$$

(b) **Data:** A particle starts from rest, at P, and travels in a straight line. The particle comes to rest at Q, 10 seconds later. Velocity is v ms^{-1}.

$v = 0.72t^2 - 0.096t^3 \qquad 0 \le t \le 5$

$v = 2.4t - 0.24t^2 \qquad 5 \le t \le 10$

(i) **Required To Calculate:** The time when velocity is at a maximum.

Calculation:

Let acceleration at time, t, by a ms^{-2}.

$$a = \frac{dv}{dt}$$

$$\frac{dv}{dt} = 0.72(2t) - 0.096\left(3t^2\right) \qquad 0 \le t \le 5$$

$$= 1.44t - 0.288t^2$$

Let $\dfrac{dv}{dt} = 0$

$$t(1.44 - 0.288t) = 0$$

$$t = 0 \text{ or } 5$$

When $t = 0$ the particle is at rest.

$\therefore t = 5$, when $a = 0$.

OR

$$\frac{dv}{dt} = 2.4 - 0.24(2t) \qquad 5 \le t \le 10$$

When $\dfrac{dv}{dt} = a = 0$

Maximum velocity occurs when the acceleration is 0

$$2.4-0.24(2t)=0$$
$$t=\frac{2.4}{0.48}$$
$$=5$$

$\therefore$ Maximum velocity occurs at $t=5$.

(ii) **Required To Calculate:** The maximum velocity.
Calculation:
When $t=5$

$$v=0.72(5)^2-0.096(5)^3 \qquad 0\le t\le 5$$
$$=6 \text{ ms}^{-1}$$

OR

$$v=2.4(5)-0.24(5)^2 \qquad 5\le t\le 10$$
$$=6 \text{ ms}^{-1}$$

(iii) **Required To Calculate:** The distance moved by the particle from P to where the particle attains its maximum velocity.
Calculation:
The distance required is the distance moved by the particle from $t=0$ to $t=5$.

$$v=0.72t^2-0.096t^3 \qquad 0\le t\le 5$$
Let s be the distance from O after time, t.
$$s=\int\left(0.72t^2-0.096t^3\right)dt$$
$$=(0.72)\frac{t^3}{3}-\frac{0.096t^4}{4}+C \quad \text{(where } C \text{ is the constant of integration)}$$

$s=0$ at $t=0$,
$\therefore C=0$

$$\therefore s=0.24t^3-0.024t^4$$

When $v=0$
$$0.72t^2-0.096t^3=0$$
$$t^2(0.72-0.096t)=0$$
$$t=0 \text{ or } 7\frac{1}{2}$$

Hence, the particle does not come to instantaneous rest in the period $t = 0$ to $t = 5$.

$\therefore$ From $t = 0$ to $t = 5$, distance covered $= 0.24(5)^3 - 0.024(5)^4$
$= 15$ m