CSEC ADDITIONAL MATHEMATICS MAY 2013

SECTION I

1. (a) **Data:** $f(x) = x^3 - x^2 - 14x + 24$

(i) **Required To Prove:** $(x+4)$ is a factor of $f(x)$.

Proof:

Recall: The remainder and factor theorem.

If $f(x)$ is any polynomial and $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$. If $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$.

Hence, if $(x+4)$ is a factor of $f(x)$, then $f(-4)$ would be equal to 0.

$$\begin{aligned} f(-4) &= (-4)^3 - (-4)^2 - 14(-4) + 24 \\ &= -64 - 16 + 56 + 24 \\ &= -80 + 80 \\ &= 0 \end{aligned}$$

$\therefore (x+4)$ is a factor of $f(x)$.

(ii) **Required To Determine:** The other linear factors of $f(x)$.

Solution:

$$\begin{array}{r} x^2 - 5x + 6 \\ x+4 \overline{\big)\, x^3 - x^2 - 14x + 24} \\ \underline{-x^3 + 4x^2} \\ -5x^2 - 14x + 24 \\ \underline{--5x^2 - 20x} \\ 6x + 24 \\ \underline{-6x + 24} \\ \underline{0} \end{array}$$

$x^2 - 5x + 6 = (x-2)(x-3)$

Hence, the other linear factors of $f(x)$ are $(x-2)$ and $(x-3)$.

(b) **Data:** $f(x) = \dfrac{2x-1}{x+2}$, $x \neq -2$

(i) **Required To Find:** $f^{-1}(x)$

Solution:

Let $y=\frac{2x-1}{x+2}$

Making x the subject,

$$y(x+2)=2x-1$$
$$xy+2y=2x-1$$
$$xy-2x=-1-2y$$
$$x(y-2)=-1-2y$$
$$x=\frac{-1-2y}{y-2}$$

$\times -1$

$$x=\frac{1+2y}{2-y}$$

Replace y by x to obtain:

$$f^{-1}(x)=\frac{1+2x}{2-x},\ x\neq 2$$

(ii) **Data:** $g(x)=x+1$

Required To Find: $fg(x)$

Solution:

$$f(x)=\frac{2x-1}{x+2}$$
$$\therefore fg(x)=\frac{2(x+1)-1}{(x+1)+2}$$
$$=\frac{2x+2-1}{x+1+2}$$
$$=\frac{2x+1}{x+3},\ x\neq -3$$

$\therefore fg(x)=\frac{2x+1}{x+3}$ in its simplified form.

(c) **Data:** $5^{3x-2}=7^{x+2}$

Required To Prove: $x=\frac{2(\log 5+\log 7)}{(\log 125-\log 7)}$

Proof:

$$5^{3x-2}=7^{x+2}$$

Taking logs to the same base:

$$\log\left(5^{3x-2}\right)=\log\left(7^{x+2}\right)$$

By the power law of logs:
$$(3x-2)\log 5=(x+2)\log 7$$

Expanding:
$$3x\log 5-2\log 5=x\log 7+2\log 7$$
$$3x\log 5-x\log 7=2\log 5+2\log 7$$
$$x(3\log 5-\log 7)=2(\log 5+\log 7)$$
$$x(\log 5^3-\log 7)=2(\log 5+\log 7)$$
$$x=\frac{2(\log 5+\log 7)}{\log 5^3-\log 7}$$
$$x=\frac{2(\log 5+\log 7)}{\log 125-\log 7}$$

Q.E.D.

2. (a) **Data:** $f(x)=3x^2+6x-1$

(i) **Required To Express:** $f(x)$ in the form $a(x+h)^2+k$, where a, h and k are constants.

Solution:
$$a(x+h)^2+k=a(x+h)(x+h)+k$$
$$=a(x^2+2hx+h^2)+k$$
$$=ax^2+2ahx+ah^2+k$$
$$\therefore 3x^2+6x-1=ax^2+2ahx+(ah^2+k)$$

Equating coefficients
$$a=3$$

$$2ah=6$$
$$\therefore 2(3)h=6$$
$$\therefore h=1$$

$$-1=ah^2+k$$
$$\therefore -1=3(1)^2+k$$
$$-1=3+k$$
$$\therefore k=-4$$

Hence, $3x^2+6x-1=3(x+1)^2-4$ and which is of the form, $a(x+h)^2+k$, where $a=3,\ h=1$ and $k=-4$ and a, h and k are constants.

Alternative Method:

$3x^2+6x-1=3\left(x^2+2x\right)-1$

One half the coefficient of $2x$ is $\frac{1}{2}(2)=1$.

$\therefore 3\left(x^2+2x\right)-1=3(x+1)^2+*$, where * is a number to be determined.

Now, $3(x+1)^2$

$$=3\left(x^2+2x+1\right)$$
$$=3x^2+6x+3$$

So, $3+*=-1$

$\therefore *=-4$

$\therefore 3x^2+6x-1$ can be written as $3(x+1)^2-4$, which is of the form $a(x+h)^2+k$, where $a=3$, $h=1$ and $k=-4$ and a, h and k are constants.

(ii) **Required To State:** The minimum value of $f(x)$.

Solution:

$$f(x)=3x^2+6x-1$$
$$=3\underbrace{(x+1)^2}_{\geq 0,\ \forall x}-4$$

$\therefore$ The minimum value of $f(x)=0-4$

$$=-4$$

(iii) **Required To Determine:** The value of x for which $f(x)$ is a minimum.

Solution:

$$f(x)=3x^2+6x-1$$
$$=3(x+1)^2-4$$

The minimum value of $f(x)$ occurs when

$3(x+1)^2=0$

$\div 3$

$(x+1)^2=0$ and

$x+1=0$

$\therefore x=-1$

Therefore, the minimum value of $f(x)$ is – 4 and occurs at $x=-1$.

Alternative method to (ii) and (iii):

$f(x)=3x^2+6x-1$

At a stationary value of $f(x)$, its 1st derivative, $f'(x)=0$.

$f'(x)=3(2x)+6$

$\quad\quad =6x+6$

Let $f'(x)=0$

$6x+6=0$

$\therefore x=-1$

The stationary value of $f(x)$ occurs at $x=-1$.

$f(-1)=3(-1)^2+6(-1)-1$

$\quad\quad =3-6-1$

$\quad\quad =-4$

$f''(x)=6>0 \Rightarrow f(x)$ is minimum at $x=-1$ and has a value of – 4.

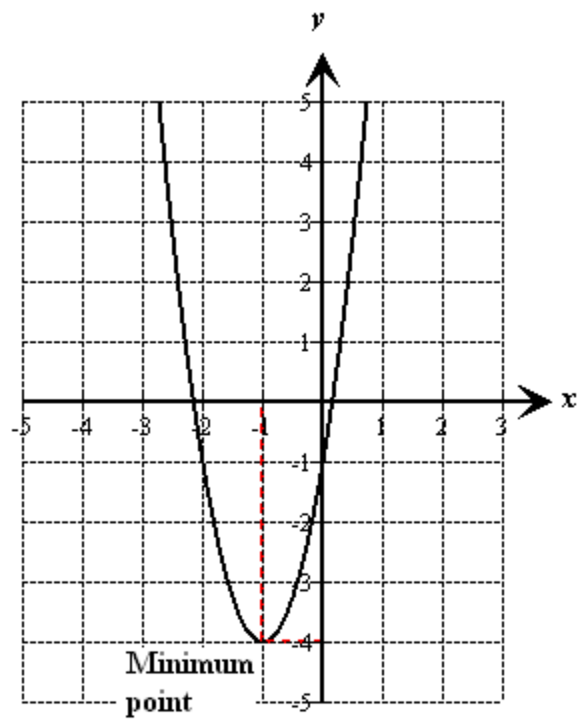
Alternative method to (ii) and (iii):

Draw $f(x)=3x^2+6x-1$

x	$f(x)$
Choose values of x	Calculate the corresponding values of $f(x)$

Draw the graph of $f(x)$ on clearly labelled axes. Read off the minimum value and the value of x at which it occurs. This will occur at the one turning point which is a minimum point.

The diagram drawn below is a sketch which illustrates this.

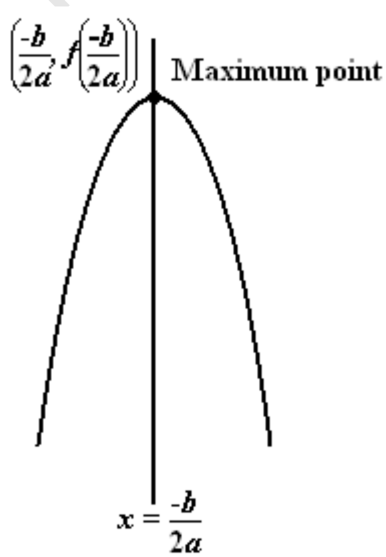


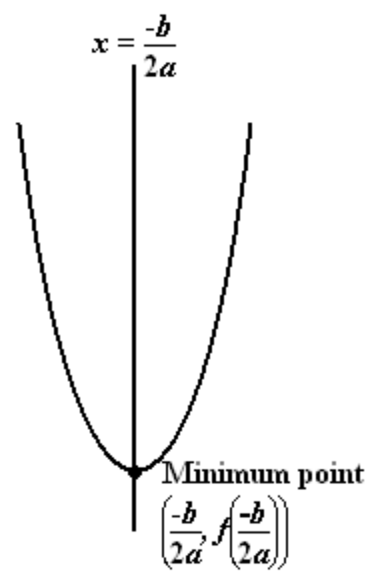
The minimum point is $(-1, -4)$.

$\therefore f(x)$ has a minimum value of -4 at $x = -1$.

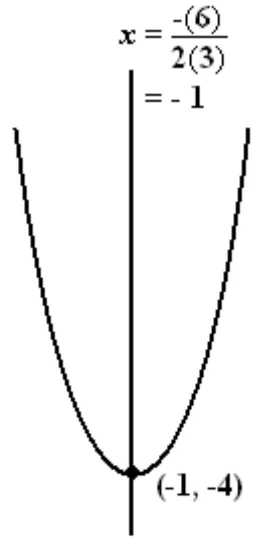
Alternative method to (ii) and (iii):

Recall: If $y = ax^2 + bx + c$, the quadratic curve has either a maximum or a minimum point and is symmetrical about a vertical axis with equation $x = \frac{-b}{2a}$.





$f(x)=3x^2+6x-1$ is of the form ax^2+bx+c, where
$a=3\ (>0) \Rightarrow f(x)$ has a minimum point.
The value of $b=6$.



$f(x)=3x^2+6x-1$

The axis of symmetry has equation $x=\dfrac{-(6)}{2(3)}$
$=-1$

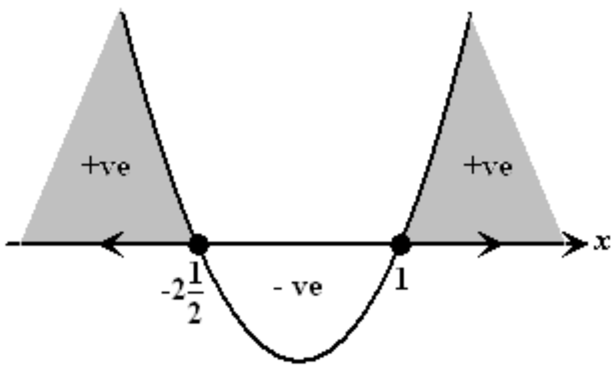
When $x=-1$
$f(-1)=3(-1)^2+6(-1)-1$
$=3-6-1$
$=-4$

$\therefore$ Coordinates of the minimum point is $(-1, -4)$.
$\therefore$ Minimum value of $f(x)$ is -4 and occurs at $x=-1$.

(b) **Data:** $2x^2+3x-5\geq 0$.
Required To Calculate: The values of *x* that satisfy the inequation
Calculation:
Let
$2x^2+3x-5=0$
$(2x+5)(x-1)=0$
Hence if we let $y=2x^2+3x-5$, the graph of *y* cuts the $x-$ axis at 1 and $-2\dfrac{1}{2}$.

The coefficient of x^2 is positive and therefore, $y = 2x^2 + 3x - 5$ has a minimum point.

If we sketch $y = 2x^2 + 3x - 5$, the graph would look like,



From the graph:

$y = 2x^2 + 3x - 5 \geq 0$ for $\{x : x \geq 1\} \cup \left\{x : x \leq -2\frac{1}{2}\right\}$.

(c) **Data:** Series is $\frac{1}{4} + \frac{2}{4^2} + \frac{1}{4^3} + \frac{2}{4^4} + \ldots$

Required To Calculate: The sum to infinity of the series

Calculation:

The series $\frac{1}{4} + \frac{2}{4^2} + \frac{1}{4^3} + \frac{2}{4^4} + \ldots$ can be rewritten as the sum of two separate series, both of which are geometric series. These are

$$\left(\frac{1}{4} + \frac{1}{4^3} + \frac{1}{4^5} + \ldots\right) + \left(\frac{2}{4^2} + \frac{2}{4^4} + \frac{2}{4^6} + \ldots\right)$$

Let us first consider the first geometric series

$\frac{1}{4} + \frac{1}{4^3} + \frac{1}{4^5} + \ldots = \frac{1}{4} + \frac{1}{4} \times \frac{1}{4^2} + \frac{1}{4} \times \frac{1}{\left(4^2\right)^2} + \ldots$ which is of the form

$a + ar + ar^2 + \ldots$, where the first term $a = \frac{1}{4}$ and the common ratio $r = \frac{1}{4^2}$.

In this geometric progression, $|r| < 1$,

Hence

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{\frac{1}{4}}{1-\frac{1}{4^2}}$$
$$= \frac{4}{15}$$

Let use consider the second geometric series

$$\frac{2}{4^2}+\frac{2}{4^4}+\frac{2}{4^6}+\ldots = \frac{2}{4^2}+\frac{2}{4^2}\times\frac{1}{4^2}+\frac{2}{4^2}\times\frac{1}{\left(4^2\right)^2}+\ldots$$

This is a geometric progression with first term $a=\frac{2}{4^2}$ and common ratio $r=\frac{1}{4^2}$.

This is also a geometric progression with $|r|<1$.

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{\frac{2}{4^2}}{1-\frac{1}{4^2}}$$
$$= \frac{\frac{1}{8}}{\frac{15}{16}}$$
$$= \frac{2}{15}$$

$\therefore$ The sum to infinity of the given series $\frac{1}{4}+\frac{2}{4^2}+\frac{1}{4^3}+\frac{2}{4^4}+\ldots$

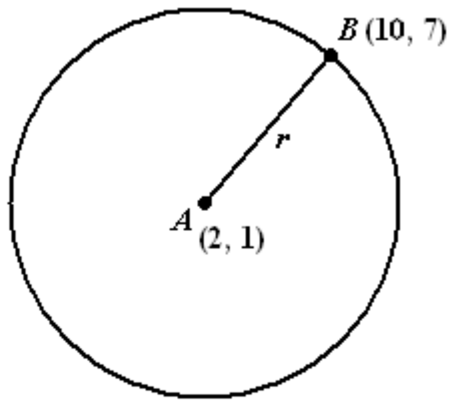
$$= \left(\frac{1}{4}+\frac{1}{4^3}+\frac{1}{4^5}+\ldots\right)+\left(\frac{2}{4^2}+\frac{2}{4^4}+\frac{2}{4^6}+\ldots\right)$$
$$= \frac{4}{15}+\frac{2}{15}$$
$$= \frac{6}{15}$$
$$= \frac{2}{5}$$

SECTION II

3. (a) (i) **Data:** A circle with centre A, $(2, 1)$ and which passes through the point $B\ (10, 7)$.

Required To Calculate: The equation of the circle in the form $x^2+y^2+hx+gy+k=0$, where h, g and $k \in Z$.

Calculation:



The length of the radius $=\sqrt{(10-2)^2+(7-1)^2}$

$=10$ units

The center of the circle is (2, 1)

$\therefore$ Equation of the circle is

$$(x-2)^2+(y-1)^2=(10)^2$$

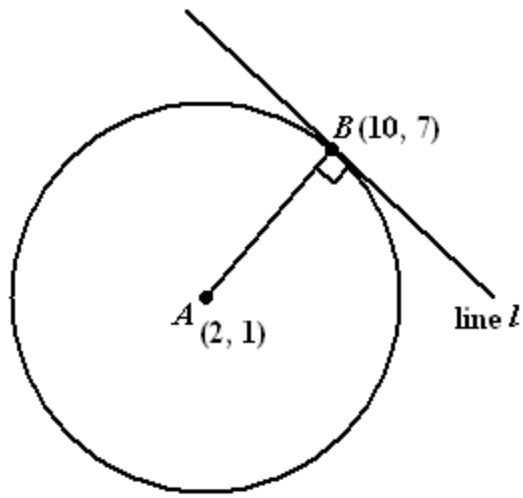
$$x^2-4x+4+y^2-2y+1=100$$

$x^2+y^2-4x-2y-95=0$ which is of the form $x^2+y^2+hx+gy+k=0$, where $h=-4\in Z$, $g=-2\in Z$ and $k=-95\in Z$.

(ii) **Data:** The line, l, is a tangent to the circle at B.

Required To Calculate: The equation of l

Calculation:



To find the equation of *l*, we need:

i. a point on *l*, for example, $B\left(10,\,7\right)$.
ii. The gradient of *l*.

We can find the gradient of *l* by two ways.

First we find the gradient of $AB = \dfrac{7-1}{10-2}$

$$= \frac{6}{8}$$

$$= \frac{3}{4}$$

$\therefore$ The gradient of $l = -\dfrac{4}{3}$ (The product of the gradients of perpendicular lines is – 1 AND the angle made by the tangent to a circle and a radius, at the point of contact, is a right angle)

OR

$x^2 + y^2 - 4x - 2y + 105 = 0$
Differentiate implicitly w.r.t. x

$$2x + 2y\frac{dy}{dx} - 4 - 2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{4-2x}{2y-2}$$

$$= \frac{2-x}{y-1}$$

The gradient of the tangent at B, that is the gradient of $l = \frac{2-10}{7-1}$

$$= \frac{-8}{6}$$

$$= -\frac{4}{3}$$

The equation of l is

$$\frac{y-7}{x-10} = \frac{-4}{3}$$

$$3(y-7) = -4(x-10)$$

$$3y - 21 = -4x + 40$$

$$3y + 4x - 61 = 0$$

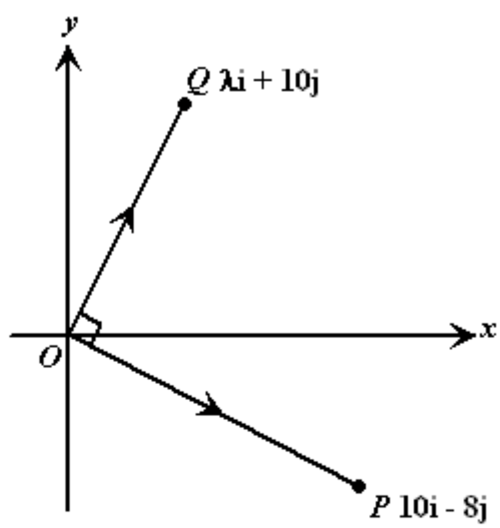
$$3y = -4x + 61$$

$$y = -\frac{4}{3}x + 20\frac{1}{3}$$

(b) **Data:** P and Q have position vectors $10\mathbf{i} - 8\mathbf{j}$ and $\lambda\mathbf{i} + 10\mathbf{j}$, where λ is a constant and are such that OP and OQ are perpendicular.

Required To Calculate: λ

Calculation:



If OP is perpendicular to OQ then

$OP \,.\, OQ = 0$

(Recalling the formula that $a.b = |a||b|\cos\theta$, where θ is the angle between *a* and *b*. If θ is 90^0 then $\cos\theta = 0$ and $a.b = 0$).

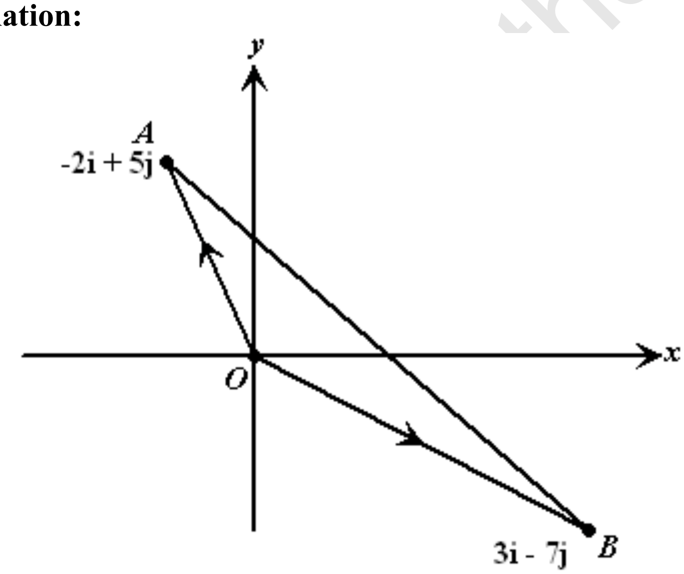
Hence,

$$(10 \times \lambda) + (-8 \times 10) = 0$$
$$10\lambda - 80 = 0$$
$$\lambda = \frac{80}{10}$$
$$\lambda = 8$$

(c) **Data:** $OA = -2\mathbf{i} + 5\mathbf{j}$ and $OB = 3\mathbf{i} - 7\mathbf{j}$.

Required To Calculate: The unit vector in the direction of AB

Calculation:



$$AB = AO + OB \quad \text{(Triangle law)}$$
$$= -(-2\mathbf{i} + 5\mathbf{j}) + 3\mathbf{i} - 7\mathbf{j}$$
$$= 5\mathbf{i} - 12\mathbf{j}$$

Any vector in the direction of AB can be expressed in the form, $\alpha(5\mathbf{i} - 12\mathbf{j})$, where α is a scalar.

Let the unit vector be, $\alpha(5\mathbf{i} - 12\mathbf{j})$ where α is to be determined.

Since the required vector is a unit vector, then it has a modulus or magnitude of 1.

$$\therefore |5\alpha . i - 12\alpha . j| = 1$$

$$\sqrt{(5\alpha)^2 + (12\alpha)^2} = 1$$

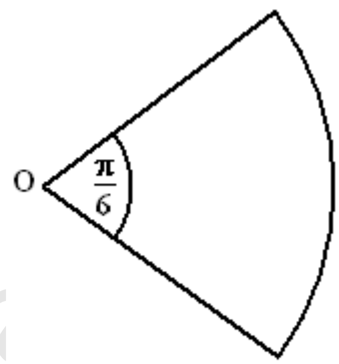
$$\sqrt{169\alpha^2} = 1$$

$$13\alpha = 1$$

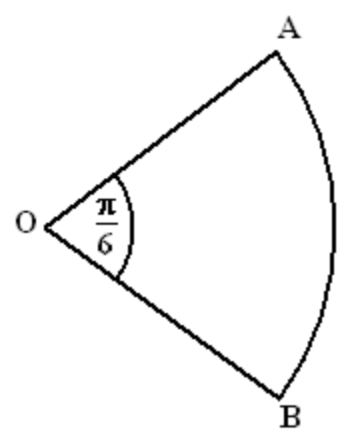
$$\alpha = \frac{1}{13}$$

$\therefore$ The unit vector in the direction of AB is $\frac{1}{13}(5\mathbf{i} - 12\mathbf{j})$ or $\frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j}$.

4. (a) **Data:** Sector of a circle with angle at the center $O = \frac{\pi}{6}$. Perimeter of sector $= \frac{5}{6}(12 + \pi)$ cm.



Required To Calculate: The area of the sector
Calculation:
We name the sector *AOB* and let the radius be *r* cm.



Perimeter of the sector = Length of radius OA + Length of arc AB + Length of radius BO

$$= r + (r \times \frac{\pi}{6}) + r$$

$$= 2r + \frac{\pi r}{6} \text{ cm}$$

Hence,

$$2r + \frac{\pi r}{6} = \frac{5}{6}(12+\pi)$$

$$r\left(2+\frac{\pi}{6}\right) = \frac{5}{6}(12+\pi)$$

$$r = \frac{\frac{5}{6}(12+\pi)}{\left(2+\frac{\pi}{6}\right)}$$

$$= \frac{10+\frac{5\pi}{6}}{2+\frac{\pi}{6}}$$

$$= \frac{60+5\pi}{12+\pi}$$

$$= \frac{5(12+\pi)}{12+\pi}$$

$$= 5 \text{ cm}$$

Area of the sector $= \frac{1}{2}r^2 \times \left(\frac{\pi}{6}\right)$

$$= \frac{1}{2} \times (5)^2 \times \frac{\pi}{6}$$

$$= \frac{25\pi}{12} \text{ cm}^2 \text{ in exact form.}$$

(b) **Data:** $2\cos^2\theta + 3\sin\theta = 0$ for $0° \le \theta \le 360°$.
Required To Calculate: θ
Calculation:

$2\cos^2\theta + 3\sin\theta = 0$

Recall: $\sin^2\theta + \cos^2\theta = 1$

$$\therefore \cos^2\theta = 1 - \sin^2\theta$$

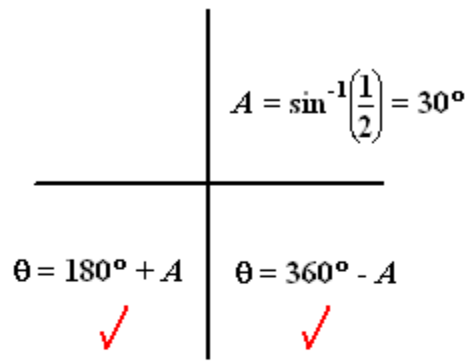
Therefore

$$2\left(1-\sin^2\theta\right)+3\sin\theta=0$$
$$2-2\sin^2\theta+3\sin\theta=0$$
$$2\sin^2\theta-3\sin\theta-2=0$$
$$\left(2\sin\theta+1\right)\left(\sin\theta-2\right)=0$$
$$\sin\theta=2 \text{ or } -\frac{1}{2}$$

The maximum value of sin θ = 1

$\therefore \sin\theta=2$ has no real solutions.

When $\sin\theta=-\frac{1}{2}$, θ lies in quadrants 3 and 4.



$$\therefore \theta=180°+30°$$
$$=210°$$

$$\theta=360°-30°$$
$$=330°$$

$\therefore \theta=210°$ or $330°$ for the given range.

(c) **Data:** $\tan\left(\theta-\alpha\right)=\frac{1}{2}$ and $\tan\theta=3$

Required To Calculate: The acute angle α.

Calculation:

$$\tan\left(\theta-\alpha\right)=\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha} \quad \text{(Compound angle formula)}$$

Hence,

$$\frac{3-\tan\alpha}{1+3\tan\alpha}=\frac{1}{2}$$

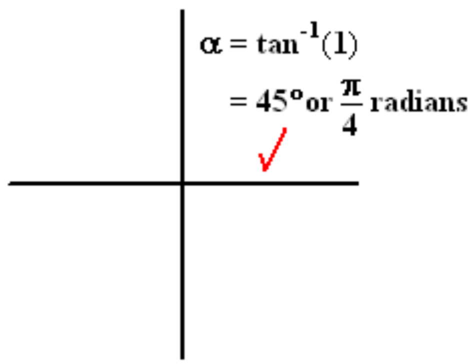
$$6-2\tan\alpha = 1+3\tan\alpha$$

$$5 = 5\tan\alpha$$

$$\tan\alpha = 1$$

$$\alpha = \tan^{-1}(1)$$

$$= 45° \text{ or } \frac{\pi}{4} \text{ radians}$$



The acute angle α is $45°$ or $\frac{\pi}{4}$ radians.

SECTION III

5. (a) **Data:** $y = x^3 - 3x^2 + 2$

(i) **Required To Find:** The coordinates of the stationary points of y.
Solution:

At a stationary point, the gradient function, $\frac{dy}{dx}=0$.

$$\frac{dy}{dx} = 3x^{3-1} - 3\left(2x^{2-1}\right) + 0$$

$$= 3x^2 - 6x$$

Let $\frac{dy}{dx}=0$.

$$3x^2 - 6x = 0$$

$$3x(x-2) = 0$$

$\therefore x = 0$ and 2 are the $x-$coordinates of the stationary points on the curve.

When $x=2$

$$y=(2)^3-3(2)^2+2$$
$$=8-12+2$$
$$=-2$$

When $x=0$

$$y=(0)^3-3(0)^2+2$$
$$=2$$

$\therefore$ The stationary points are $(2,-2)$ and $(0,2)$.

(ii) **Required To Find:** The second derivative of *y* and hence the nature of each of the stationary points.

Solution:

$$\frac{dy}{dx}=3x^2-6x$$

$$\frac{d^2y}{dx^2}=3\left(2x^{2-1}\right)-6 \quad \text{(The second derivative)}$$
$$=6x-6$$

When $x=2$

$$\frac{d^2y}{dx^2}=6(2)-6$$
$$=6 \quad \text{(Positive)}$$

$\therefore (2,-2)$ is a minimum point.

When $x=0$

` $$\frac{d^2y}{dx^2}=6(0)-6$$
$$=-6 \quad \text{(Negative)}$$

$\therefore (0,2)$ is a maximum point.

The stationary points on the curve are $(0,2)$, which is a maximum point and $(2,-2)$, which is a minimum point.

(b) **Data:** $y=(5x+3)^3\sin x$

Required To Differentiate: *y* w.r.t. *x*.

Solution:

$y=(5x+3)^3\sin x$ is of the form $y=uv$, where

$u = (5x+3)^3$

Let $t = 5x+3$

$\therefore u = t^3$

$\frac{du}{dx} = \frac{du}{dt} \times \frac{dt}{dx}$ (Chain rule)

$= 3t^2 \times 5$

$= 15t^2$

Re-substituting, we get

$\frac{du}{dx} = 15(5x+3)^2$

$v = \sin x$

$\frac{dv}{dx} = \cos x$

Recall:

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ (Product law)

$= (\sin x) \times 15(5x+3)^2 + (5x+3)^3 \times \cos x$

$= 15(5x+3)^2 \sin x + (5x+3)^3 \cos x$

$= (5x+3)^2 \{15\sin x + (5x+3)\cos x\}$ in its simplest form.

6. (a) **Required To Find:** $\int (5x^2+4)\, dx$

Solution:

$\int (5x^2+4)\, dx = \frac{5x^{2+1}}{2+1} + 4x + C$ (C is the constant of integration)

$= \frac{5x^3}{3} + 4x + C$

(b) **Required To Evaluate:** $\int_0^{\frac{\pi}{2}} (3\sin x - 5\cos x)\, dx$

Solution:

$\int_0^{\frac{\pi}{2}} (3\sin x - 5\cos x)\, dx = \left[3(-\cos x) - 5(\sin x)\right]_0^{\frac{\pi}{2}}$

NOTE: The constant of integration is omitted, since it cancels off in a definite integral.

$$\int_0^{\frac{\pi}{2}} (3\sin x - 5\cos x)\, dx = [-3\cos x - 5\sin x]_0^{\frac{\pi}{2}}$$

$$= \left\{-3\cos\left(\frac{\pi}{2}\right) - 5\sin\left(\frac{\pi}{2}\right)\right\} - \{-3\cos(0) - 5\sin(0)\}$$

$$= (-3(0) - 5(1)) - (-3(1) - 5(0))$$

$$= -5 + 3$$

$$= -2$$

(c) **Data:** A curve passes through the points $P(0, 8)$ and $Q(4, 0)$. $\frac{dy}{dx} = 2 - 2x$.

Required To Find: The area of the finite region bounded by the curve in the first quadrant

Solution:

It is helpful to first find the equation of the curve so as to create a diagram illustrating the finite region. This is essentially to observe if the region lies entirely above or below or partially above and partially below the x – axis.

The equation of the curve

$$y = \int (2 - 2x)\, dx$$

$$y = 2x - \frac{2x^{1+1}}{1+1} + C \qquad (C \text{ is the constant of integration})$$

$$y = 2x - x^2 + C$$

$Q(0, 8)$ lies on the curve.

Therefore $0 = 2(0) - (0)^2 + C$

and $C = 0$

The equation of the curve is $y = 2x - x^2 + 8$.

When $y = 0$

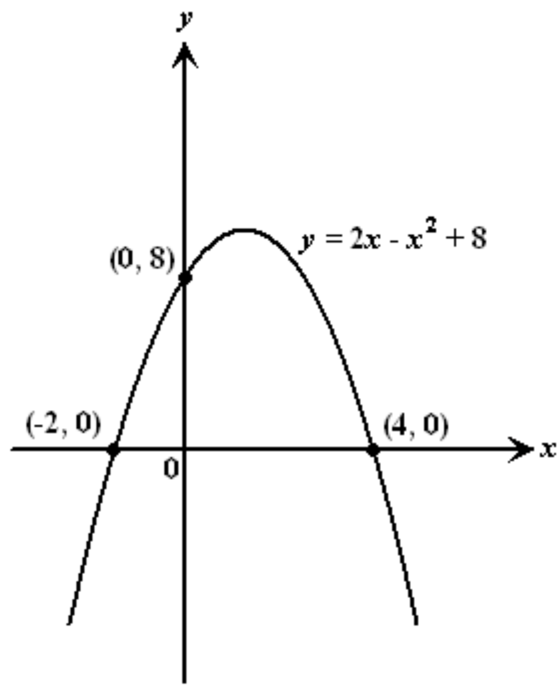
$$2x - x^2 + 8 = 0$$

$$(4 - x)(x + 2) = 0$$

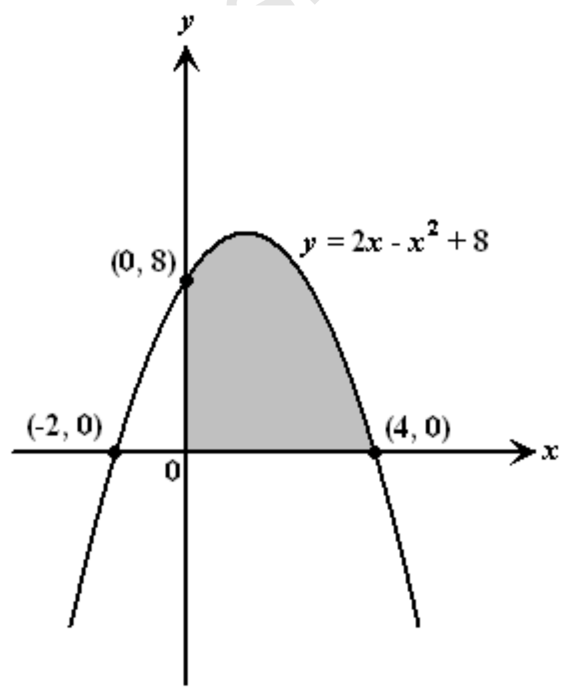
The curve cuts the x – axis at 4 and at – 2.

The coefficient of $x^2 > 0 \Rightarrow$ The quadratic curve has a maximum point.

The curve $y = 2x - x^2 + 8$ looks like:



The finite region which lies in the first quadrant and whose area is required, is shown shaded.



Area of the shaded region $= \int_0^4 \left(2x - x^2 + 8\right) dx$

$$= \left[x^2 - \frac{x^3}{3} + 8x \right]_0^4$$

$$= \left\{ (4)^2 - \frac{(4)^3}{3} + 8(4) \right\} - \left\{ (0)^2 - \frac{(0)^3}{3} + 8(0) \right\}$$

$$= \left(16 - 21\frac{1}{3} + 32 \right) - (0)$$

$$= 26\frac{2}{3} \text{ square units (exactly)}$$

SECTION IV

7. (a) **Data:** Incomplete tree diagram showing the gender and method of payment of people buying petrol.

(i) **Required To Complete:** The tree diagram given.

Solution:

Key:
F – Female
M – Male
C – Cash purchase
O – Other purchase

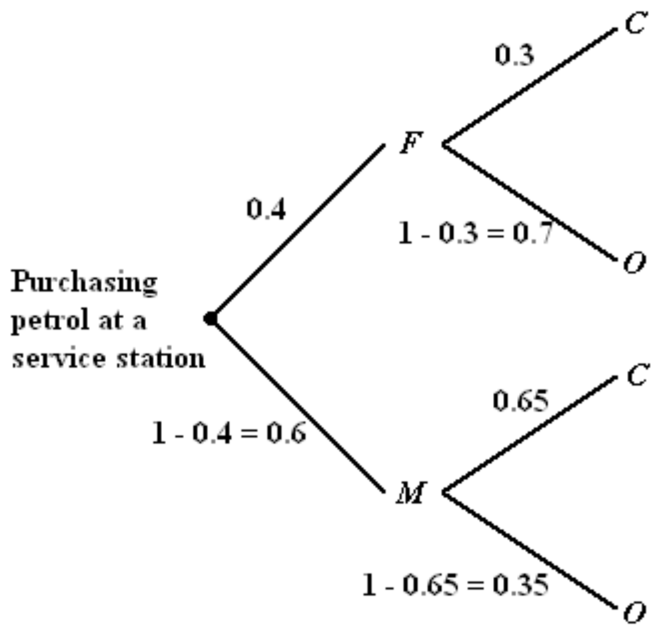
$$P(F \text{ paying cash}) = \frac{30}{100}$$
$$= 0.3$$

$$P(F \text{ paying by some other form}) = 1 - 0.3$$
$$= 0.7$$

$$P(M \text{ paying cash}) = \frac{65}{100}$$
$$= 0.65$$

$$P(M \text{ paying by some other form}) = 1 - 0.65$$
$$= 0.35$$

Key:
F – Female
M – Male
C – Cash purchase
O – Other purchase



(ii) **Required To Calculate:** The probability that a customer pays for petrol with cash.

Calculation:

$$P(\text{Customer pays with cash}) = P(\text{Male pays with cash OR Female pays with cash})$$
$$= (0.6 \times 0.65) + (0.4 \times 0.3)$$
$$= 0.39 + 0.12$$
$$= 0.51$$

(iii) **Data:** Event *T*: Customer is *F*, GIVEN that the customer pays with *C*.
Event *V*: Customer is *M* and does not pay with *C*.
Required To Calculate: The more likely event to occur between *T* or *V*.
Calculate:
Let *F* be the event that the customer is female and let *C* be the event that the customer pays with cash.
$P(T) = P(F \text{ given that } C)$, the conditional probability is expressed as

$$P(F/C) = \frac{P(F \cap C)}{P(C)} \quad \text{(By definition)}$$

$$= \frac{0.4 \times 0.3}{(0.4 \times 0.3) + (0.6 \times 0.65)}$$

$$= \frac{0.12}{0.12 + 0.39}$$

$$= \frac{0.12}{0.51}$$

$$= \frac{12}{51}$$

$$= \frac{4}{14}$$

$$\therefore P(T) = \frac{4}{17}$$

$$P(V) = P(M \text{ and } C')$$

$$= 0.6 \times 0.35$$

$$= 0.21$$

$$P(V) = \frac{21}{100}$$

$$P(T) = \frac{4}{17}$$

$$= \frac{400}{1700}$$

$$P(V) = \frac{21}{100}$$

$$= \frac{357}{1700}$$

$$P(T) > P(V)$$

$\therefore T$ is more likely to occur that V.

(b) **Data:** Marks obtained by 30 students in an exam.

(i) **Required To State:** One advantage of using a stem and leaf diagram versus a box and whisker plot.

Solution:

In a stem and leaf diagram the value of each individual data point can be easily read off.

OR

The data is arranged compactly and the stem is not repeated for multiple data values.

OR

Stem and leaf diagrams are useful for highlighting outliers.

(ii) **Required To Construct:** A stem and leaf diagram to show the data.
Solution:

Stem	**Leaf**								
4	0	1	5						
5	0	0	1	1	3	6	6	8	8
6	3	3	6	6	9				
7	2	4	5	5	6				
8	0	1	3	5	9				
9	2	4	9						

Key:
4 | 5 = 45
Leaf unit = 1.0
Stem unit = 10

(iii) **Required To Determine:** The median mark
Solution:
There are 30 marks.
$\therefore$ There are two middle marks are the 15^{th} and the 16^{th} marks.
15^{th} mark = 66
16^{th} mark = 66

$$\therefore \text{Median mark } = \frac{66+66}{2} = 66$$

(iv) **Required To Calculate:** The semi inter-quartile range of the marks.
Calculation:

$$\frac{1}{4} \text{ of } 30 = 7\frac{1}{2}$$

$\therefore$ Lower quartile is the 8^{th} mark = 53

$$\frac{3}{4} \text{ of } 30 = 22\frac{1}{2}$$

$\therefore$ Upper quartile is the 23^{rd} mark = 80

$$\text{Inter-quartile range} = 80 - 53$$
$$= 27$$

$$\text{Semi inter-quartile range} = \frac{1}{2}(\text{Inter-quartile range})$$
$$= \frac{1}{2}(27)$$
$$= 13.5$$

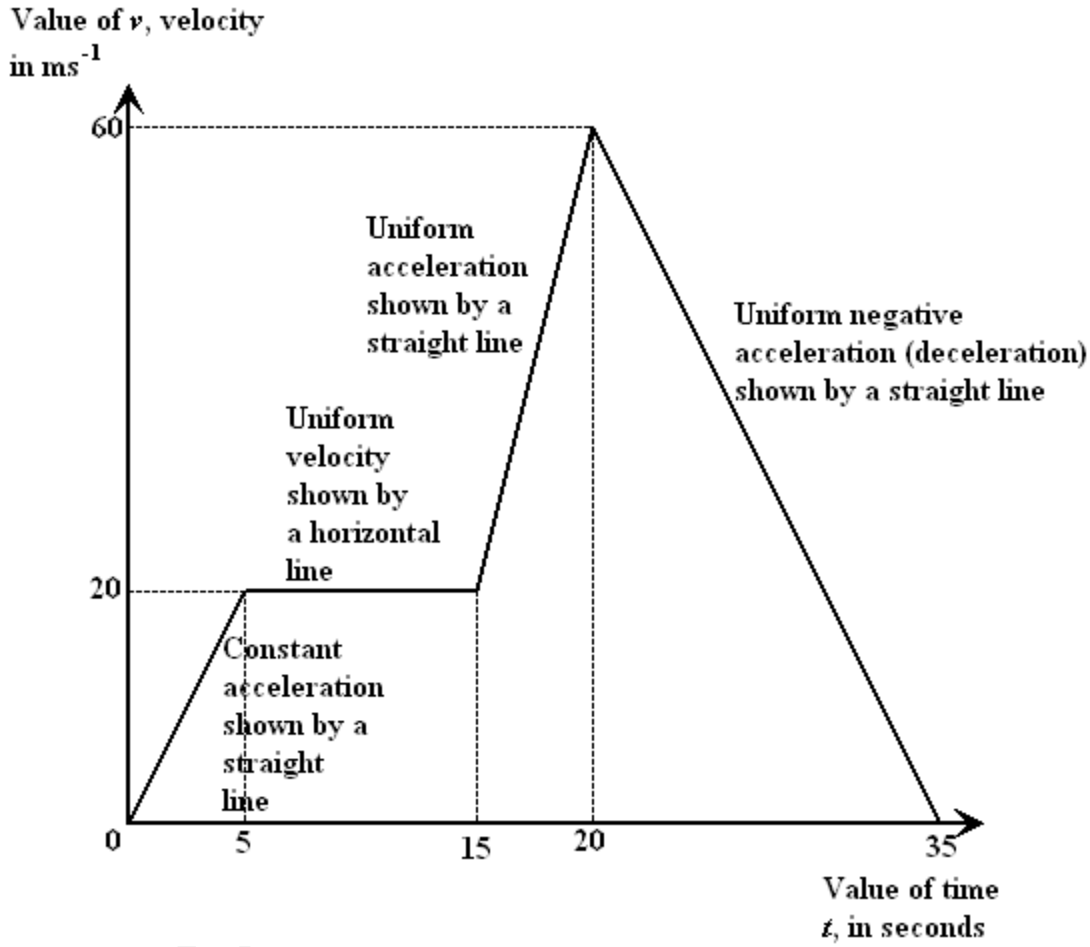
(v) **Required To Determine:** The probability that two students chosen at random from the class both scored less than 50 on the exam.
Solution:
The number of students who scored less than 50 is 3.

$$P(\text{First student chosen scored less than } 50) = \frac{3}{30}$$
$$P(\text{Second student chosen scored less than } 50) = \frac{3-1}{30-1}$$
$$= \frac{2}{29}$$

$$P(\text{Two students chosen scored less than } 50) = \frac{3}{30} \times \frac{2}{29}$$
$$= \frac{1}{145}$$

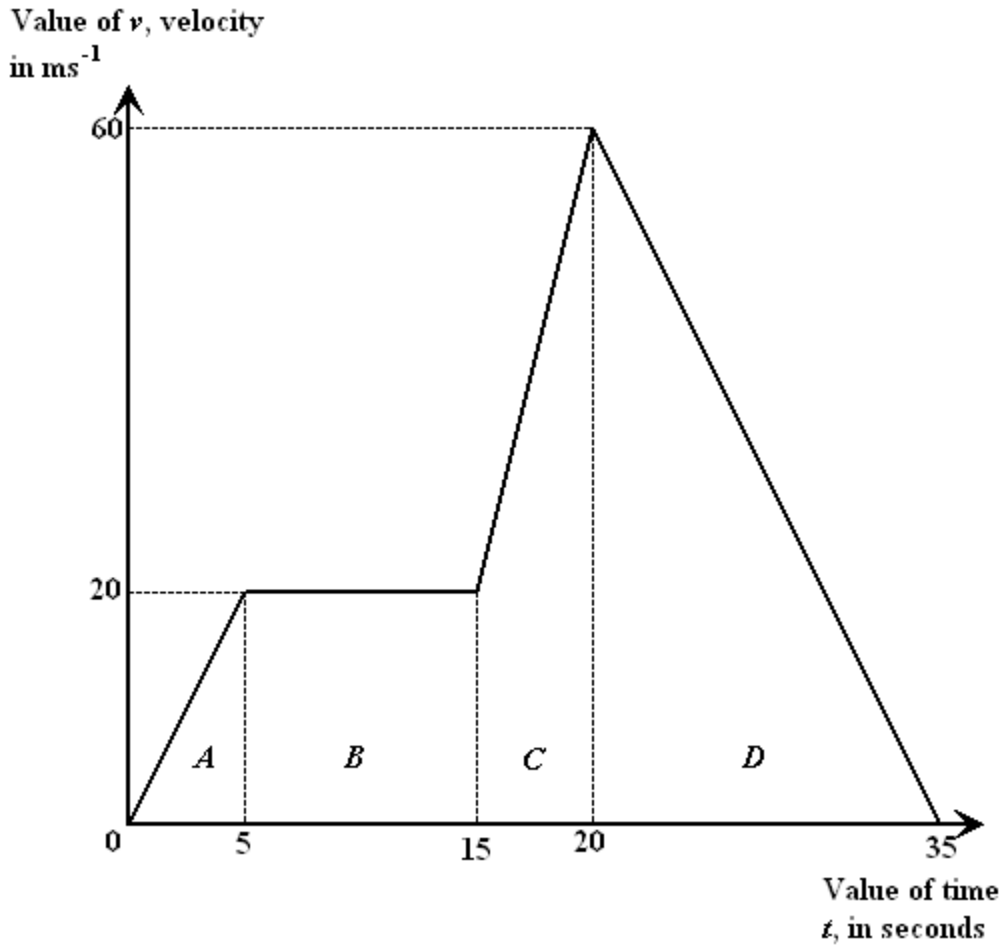
8. (a) **Data:** A particle starts from rest and accelerates uniformly to 20 ms^{-1} in 5 seconds. The particle has the same velocity for a further 10 seconds. The particle accelerates uniformly to reach 60 ms^{-1} in a further 5 seconds. The particle decelerates uniformly for 15 seconds before it comes to rest.

(i) **Required To Draw:** A velocity – time graph to illustrate the motion of the particle.
Solution:
We choose a convenient scale.



(ii) a) **Required To Determine:** The total distance, in metres, travelled by the particle

Solution:



The distance covered by the particle can be found by finding the area under the line graph.
We divide this region into four separate regions *A*, *B*, *C* and *D* for convenience, as shown.

Area of triangle $A = \frac{5 \times 20}{2}$
$= 50$

Area of rectangle $B = 20 \times (15 - 5)$
$= 200$

Area of trapezium $C = \frac{1}{2} \times (20 + 60) \times (20 - 15)$
$= 200$

Area of triangle $D = \frac{60 \times (35-20)}{2}$
$= 450$

Total distance travelled $= 50 + 200 + 200 + 450$
$= 900$ m

b) **Required To Determine:** The average velocity of the particle for the entire journey.
Solution:

$$\text{Average velocity} = \frac{\text{Total distance covered}}{\text{Total time taken}}$$
$$= \frac{900}{35}$$
$$= 25\frac{5}{7} \text{ ms}^{-1}$$

(b) **Data:** The velocity of a particle at time, *t*, is given by $v = 3t^2 - 18t + 15$.

(i) **Required To Calculate:** The values of *t* when the particle is instantaneously at rest.
Calculation:
At instantaneous rest, $v = 0$.
Let
$3t^2 - 18t + 15 = 0$
$\div 3$
$t^2 - 6t + 5 = 0$
$(t-1)(t-5) = 0$
$t = 1 \text{ or } 5$

$\therefore$ The particle is at instantaneous rest when $t = 1$ and $t = 5$.

(ii) **Required To Calculate:** The distance travelled by the particle between 1 second and 3 seconds.
Calculation:
Let the distance from *O*, at time *t*, be given by *s* units.
$s = \int v \, dt$
$s = \int (3t^2 - 18t + 15) \, dt$
$s = \frac{3t^{2+1}}{2+1} - \frac{18t^{1+1}}{1+1} + 15t + C$ (where C is the constant of integration)
$s = t^3 - 9t^2 + 15t + C$

When $t = 0,\ s = 0$

$\therefore 0 = (0)^3 - 9(0)^2 + 15(0) + C$

$C = 0$

$s = t^3 - 9t^2 + 15t$

When $t = 1$

$s = (1)^3 - 9(1)^2 + 15(1)$

$= 7$ units (No units of distance were given in the question)

The particle has stopped when $t = 2$

When $t = 2$

$s = (2)^3 - 9(2)^2 + 15(2)$

$= 8 - 36 + 30$

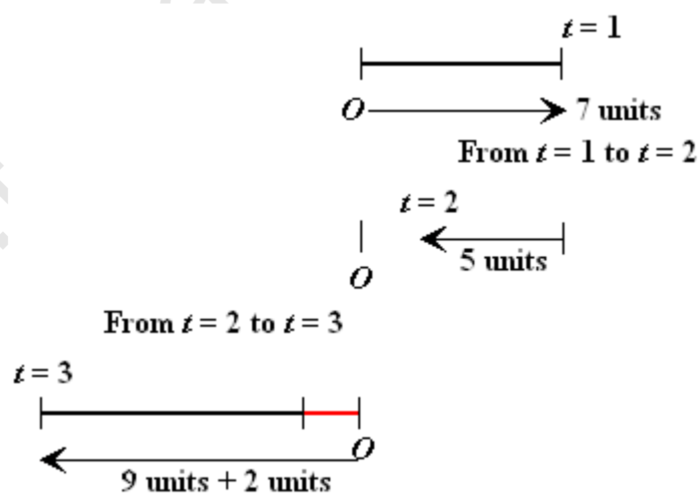
$= 2$ units

When $t = 3$

$s = (3)^3 - 9(3)^2 + 15(3)$

$= 27 - 81 + 45$

$= -9$ units



At $t = 1$, the particle is 7 units from *O*.
From $t = 1$ to $t = 2$, the particle moved 5 units in the opposite direction.
From $t = 2$ to $t = 3$, the particle moved $(2 + 9)$ units further in the same direction as it did in the previous phase.

$\therefore$ Total distance covered or travelled for $t=1$ to $t=3$ is
$5+2+9=16$ units.

(iii) a) **Required To Calculate:** $\frac{dv}{dt}$ when $t=2$.

Calculation:

$$v=3t^2-18t+15$$

$$\frac{dy}{dt}=3(2t)-18$$
$$=6t-18$$

When $t=2$

$$\frac{dv}{dt}=6(2)-18$$
$$=-6 \text{ units s}^{-2}$$

b) **Required To Calculate:** $\frac{dv}{dt}$ when $t=3$.

Calculation:

When $t=3$

$$\frac{dv}{dt}=6(3)-18$$
$$=0 \text{ units s}^{-2}$$

(iv) a) **Required To Interpret:** The value of the result in 8 (b) (iii) a).

Solution:

$\frac{dv}{dt}$ = acceleration of the particle at time, *t*.

In 8 (b) (iii) a), acceleration $=-6$ units s^{-2}.

$\therefore$ The particle has a negative acceleration or a deceleration. The particle is decreasing in velocity.

b) **Required To Interpret:** The value of the result in 8 (b) (iii) b).

Solution:

In 8 (b) (iii) b), acceleration = 0 units s^{-2}.

This implies that the particle has no acceleration and is therefore moving with constant velocity.