FAS-PASS Maths

CSEC ADDITIONAL MATHEMATICS MAY 2012

SECTION I

1. (a) **Data:** $f(x)=x^3+1,\ 0\le x\le 3$ and $g(x)=x+5,\ x\in\Re$.

(i) **Required To Determine:** $g(f(x))$.

Solution:

$$g(f(x))=gf(x)$$
$$=(x^3+1)+5$$
$$=x^3+6$$
$$\therefore g(f(x))=x^3+6$$

(ii) **Required To State:** The range of $g(f(x))$.

Solution:

Domain of $f(x)$ is $0\le x\le 3$.

Domain of $g(x)$ is $x\in\Re$.

$\therefore$ Domain of $g(f(x))=\{0\le x\le 3\}\cap\{\Re\}$

$=\{0\le x\le 3\}$ or $\{x:0\le x\le 3\}$

When $x=0$ $g(f(x))=6$

When $x=3$ $g(f(x))=33$

Hence the range of $g(f(x))$ is $6\le g(f(x))\le 33$

(iii) **Required To Determine:** The inverse of $g(f(x))$.

Solution:

Let

$$y=g(f(x))$$
$$y=x^3+6$$
$$y-6=x^3$$
$$x^3=y-6$$
$$x=\sqrt[3]{y-6}$$

Replace y by x

$$y=\sqrt[3]{x-6}$$
$$\therefore (gf)^{-1}(x)=\sqrt[3]{x-6}$$

or $(x-6)^{1/3}$

(b) **Data:** $x+2$ is a factor of $f\left(x\right)=2x^3-3x^2-4x+a$.
Required To Calculate: The value of a.

Calculation:
Recall: The remainder and factor theorem states that if $f\left(x\right)$ is any polynomial and $f\left(x\right)$ is divided by $\left(x-a\right)$, the remainder is $f\left(a\right)$. If $f\left(a\right)=0$, then $\left(x-a\right)$ is a factor of $f\left(x\right)$.

Hence,

$$f\left(-2\right)=0$$

$$\therefore 2\left(-2\right)^3-3\left(-2\right)^2-4\left(-2\right)+a=0$$

$$-16-12+8+a=0$$

$$a=20$$

(c) **Data:** $3^{2x}-9\left(3^{-2x}\right)=8$
Required To Calculate: x
Calculation:

$$3^{2x}-9\left(3^{-2x}\right)=8$$

$$3^{2x}-\frac{9}{3^{2x}}-8=0$$

$\times 3^{2x}$

$$\left(3^{2x}\right)^2-8\left(3^{2x}\right)-9=0$$

$$\left(3^{2x}-9\right)\left(3^{2x}+1\right)=0$$

$3^{2x}-9=0$ OR $3^{2x}+1=0$

$3^{2x}=9$ $\quad$ $3^{2x}=-1$

When $3^{2x}=9$

$3^{2x}=3^2$

Equating indices

$2x=2$

$x=\frac{2}{2}$

$x=1$

$\therefore x=1$ only

When $3^{2x}=-1$, x has no real solutions.

(d) (i) **Required To Express:** $x^3 = 10^{x-3}$ in the form $\log_{10} x = ax + b$.

Solution:

$x^3 = 10^{x-3}$

Taking $\log_{10}$ or lg

$\log_{10}\left(x^3\right) = \log_{10}\left(10^{x-3}\right)$

$3\log_{10} x = (x-3)\log_{10} 10$

$\log_{10} 10 = 1$

$\therefore 3\log_{10} x = (x-3)\times 1$

$\div 3$

$\log_{10} x = \frac{1}{3}x - 1$

This is of the form, $\log_{10} x = ax + b$, where $a = \frac{1}{3}$ and $b = -1$.

(ii) **Required To State:** The value of the gradient of a graph of $\log_{10} x$ vs x.

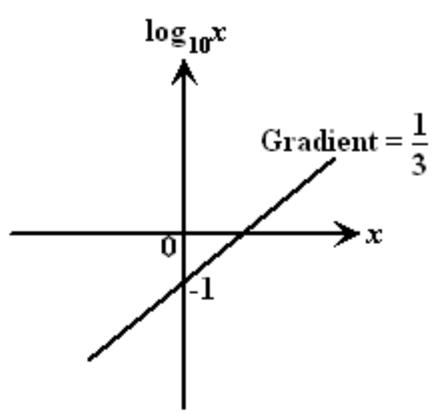
Solution:

$\log_{10} x = \frac{1}{3}x - 1$ is of the form of a straight line $Y = mX + C$, where

$Y = \log_{10} x$, $m = \frac{1}{3}$, $X = x$ and $C = -1$.

Hence, if $\log_{10} x$ vs x is drawn, a straight line will be obtained with a gradient of $\frac{1}{3}$.

A sketch of the graph would look like:



2. (a) **Data:** $x^2 - 4x + 6 = 0$ has roots α and β.

Required To Calculate: The value of $\alpha^2 + \beta^2$.

Calculation:

If $ax^2+bx+c=0$ where $(a,\ b,\ c\in\Re)$. Then

$\div a$

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

If α and β are roots of the equation, then

$$(x-\alpha)(x-\beta)=0$$

$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

And

$$x^2+\frac{b}{a}x+\frac{c}{a}=x^2-(\alpha+\beta)x+\alpha\beta$$

Equating coefficients

$\alpha+\beta=\dfrac{-b}{a}$ and $\alpha\beta=\dfrac{c}{a}$

$x^2-4x+6=0$ is of the form $ax^2+bx+c=0$, where $a=1$, $b=-4$ and $c=6$.

Since the roots are α and β, then

$$\alpha+\beta=\frac{-(-4)}{1}$$
$$=4$$

and

$$\alpha\beta=\frac{6}{1}$$
$$=6$$

$$(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta$$

$$\therefore \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$
$$=(4)^2-2(6)$$
$$=16-12$$
$$=4$$

Hence, $\alpha^2+\beta^2=4$

(b) **Data:** $\dfrac{2x-5}{3x+1}>0$

Required To Find: The range of values of x

Solution:

$$\frac{2x-5}{3x+1}>0,\ x\neq -\frac{1}{3}$$

$\times(3x+1)^2$

$(2x-5)(3x+1)>0$

Solving the range of values of x by sketching $y=(2x-5)(3x+1)$

When $y=0$

$2x-5=0$

$2x=5$

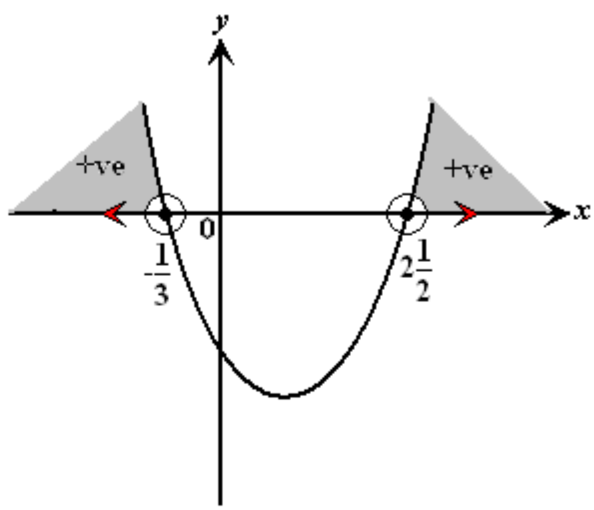
$x=\frac{5}{2}$ or $2\frac{1}{2}$

$3x+1=0$

$3x=-1$

$x=-\frac{1}{3}$

The coefficient of $x^2>0$, therefore the quadratic curve has a minimum point and cuts the horizontal axis at 2 ½ and $-\frac{1}{3}$.



Hence, $\frac{2x-5}{3x+1}>0$ for the domain $\left\{x : x > 2\frac{1}{2}\right\} \cup \left\{x : x < -\frac{1}{3}\right\}$ as shown shaded in the above sketch.

(c) **Data:** A customer repays a loan, increasing the payment by \$$x$ per month. Payment in 5th month = \$50 and payment in 9th month = \$70.
Required To Calculate: The total amount of money repaid at the end of the 24th month

Calculation:
Presenting the payments in a table and letting the 1st payment be $*a*.

Month Number	1	2	3	4	5	…	9	…	24
Amount repaid in $	a	$a+x$	$a+2x$	$a+3x$	$a+4x$	…	$a+8x$	…	$a+23x$

This is in arithmetic progression with first term = *a* and common difference = *x*.

Let
$a+4x=50 \quad \ldots❶$
$a+8x=70 \quad \ldots❷$

Equation ❷ – Equation ❶
$4x=20$
$x=\frac{20}{4}$
$x=5$

When $x=5$, substitute into Equation ❶.
$a+4(5)=50$
$a+20=50$
$a=50-20$
$a=30$

Hence, the 1st payment is $30 = *a* and the common difference, *x*, is $5, which is the increase in payment each month.

The amount paid in total at the end of the 24th month is the sum of the first 24 terms of the arithmetic progression.

Recall:
$S_n=\frac{n}{2}\{2a+(n-1)d\}$, where
$n=$ number of terms
$a=$ first term
$d=$ common difference

$$\therefore S_{24} = \frac{24}{2}\left\{2(30)+(24-1)5\right\}$$
$$= 12\left\{60+23(5)\right\}$$
$$= \$2100$$

$\therefore$ The total amount repaid = \$2 100.

SECTION II

3. (a) **Data:** Circle with equation $x^2+y^2-4x+6y=87$.

(i) **Required To Prove:** The line with equation $x+y+1=0$ passes through the centre of the circle.

Solution:

$$x^2+y^2-4x+6y=87$$

Re-writing the equation as $x^2+y^2+2(-2)x+2(3)y+(-87)=0$ and which is of the form $x^2+y^2+2gx+2fy+c=0$. This is in the general equation of a circle with center $(-g,-f)$ and radius $\sqrt{g^2+f^2-c}$.

In the above equation, $g=-2,\ f=3$ and $c=-87$.

$\therefore$ Centre of circle $=\left(-(-2),\,-(3)\right)$
$=(2,\,-3)$

To prove the point lies on the line $x+y+1=0$, we substitute $x=2$ and $y=-3$ in the equation $x+y+1=0$.

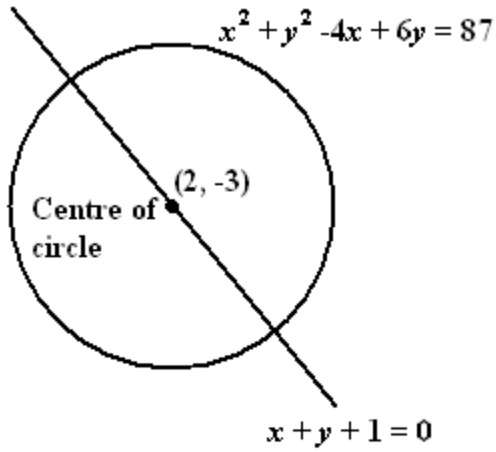
$2+(-3)+1=0$

$0=0$ (True)

$\therefore (2,\,-3)$ lies on the line.

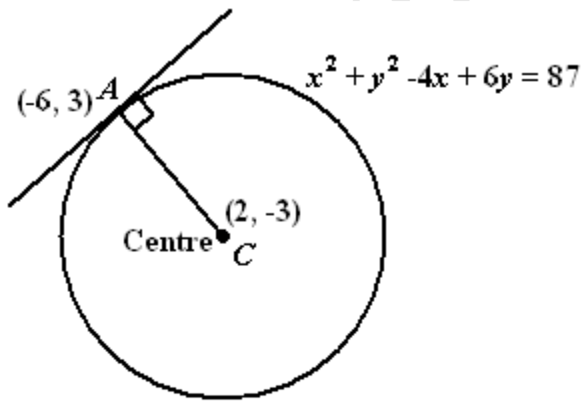
Hence, the line $x+y+1=0$ passes through the centre of the given circle.

A sketch of the circle and the line looks like:



(ii) **Required To Find:** The equation of the tangent to the circle at the point $A(-6, 3)$.

Solution:



Let the centre of the circle be *C*.

The gradient of the radius, $AC = \frac{3-(-3)}{-6-2}$

$= \frac{6}{-8}$

$= -\frac{3}{4}$

The gradient of the tangent at $A = \frac{-1}{-\frac{3}{4}}$

$= \frac{4}{3}$

(The product the gradient of perpendicular lines $= -1$)

(The angle made by the radius of a circle and a tangent, at the point of contact, is 90°).

The equation of the tangent at A is

$$\frac{y-3}{x-(-6)}=\frac{4}{3}$$

$$3y-9=4x+24$$

$$3y=4x+33 \text{ or}$$

$$y=\frac{4}{3}x+11$$

Alternative method:

Recall: The gradient of the tangent to a circle is $\frac{-(g+x)}{f+y}$.

Hence the gradient of the tangent at $A=\frac{-(-2+(-6))}{3+3}$

$$=\frac{4}{3}$$

The equation of tangent at A is

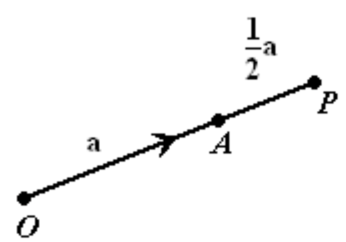
$$\frac{y-3}{x+6}=\frac{4}{3}$$

$$3y=4x+33 \text{ or}$$

$$y=\frac{4}{3}x+11$$

(b) **Data:** $\mathbf{OA}=\mathbf{a}$, $\mathbf{OB}=\mathbf{b}$ and $\mathbf{AP}=\frac{1}{2}\mathbf{OA}$, where $\mathbf{a}=\begin{pmatrix}2\\1\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}3\\2\end{pmatrix}$.

(i) **Required To Write:** $\mathbf{BP}$ in terms of **a** and **b**.
Solution:



$$\mathbf{OP} = \mathbf{OA} + \mathbf{AP}$$
$$= \mathbf{a} + \frac{1}{2}\mathbf{a}$$
$$= 1\frac{1}{2}\mathbf{a}$$

$$\mathbf{BP} = \mathbf{BO} + \mathbf{OP}$$
$$= -\mathbf{b} + 1\frac{1}{2}\mathbf{a} \text{ or}$$
$$= 1\frac{1}{2}\mathbf{a} - \mathbf{b} \text{ (in terms of } \mathbf{a} \text{ and } \mathbf{b}\text{)}$$

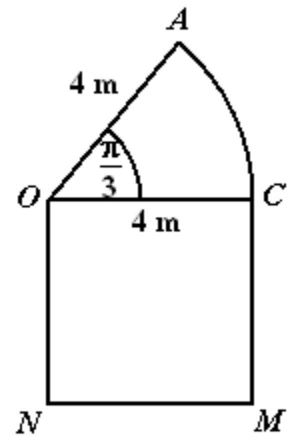
(ii) **Required To Find:** $|\mathbf{BP}|$

Solution:

$$\mathbf{BP} = 1\frac{1}{2}\mathbf{a} - \mathbf{b}$$
$$= 1\frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 1\frac{1}{2} \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$$

$$\therefore |\mathbf{BP}| = \sqrt{(0)^2 + \left(-\frac{1}{2}\right)^2}$$
$$= \sqrt{0 + \frac{1}{4}}$$
$$= \sqrt{\frac{1}{4}}$$
$$= \frac{1}{2} \text{ unit}$$

4. (a) **Data:**



(i) **Required To Calculate:** The area of the shape *OACMN*.

Calculation:

Area of the sector $OAC = \frac{1}{2}(4)^2 \times \frac{\pi}{3}$

$= \frac{8\pi}{3}\ \text{m}^2$

Area of the square $ONMC = 4 \times 4$

$= 16\ \text{m}^2$

Hence the area of the shape $OACMN = \left(16 + \frac{8\pi}{3}\right)\ \text{m}^2$

(ii) **Required To Calculate:** The perimeter of the shape *OACMNO.*
Calculation:

The perimeter is the sum of the lengths of the lines *OA, AC, CM, MN* and *NO*

$$\text{Perimeter} = OA + AC + CM + MN + NO$$

$$= 4 + \left(4 \times \frac{\pi}{3}\right) + 4 + 4 + 4$$

$$= \left(16 + \frac{4\pi}{3}\right)\ \text{m}$$

(b) **Data:** $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

Required To Evaluate: The exact value of $\cos\left(\frac{7\pi}{12}\right)$.

Solution:

$$\cos\left(\frac{7\pi}{12}\right) \equiv \cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$$

$$= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \quad \text{(Compound angle formula)}$$

$$= \frac{1}{2}\times\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$

OR

Dividing both the numerator and denominator by $\sqrt{2}$.

$$\frac{1}{2}.\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}}$$

$$\frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

(c) **Required To Prove:** $\frac{1}{\sec\theta+\tan\theta} \equiv \frac{1-\sin\theta}{\cos\theta}$

Proof:

Left hand side:

Recall: $\sec\theta = \frac{1}{\cos\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$$\therefore \frac{1}{\sec\theta+\tan\theta} = \frac{1}{\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}\times\frac{\cos\theta}{\cos\theta}$$

$$= \frac{\cos\theta}{1+\sin\theta}$$

$$\times\frac{1-\sin\theta}{1-\sin\theta}$$

$$= \frac{\cos\theta(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}$$

$$= \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta}$$

Recall: $\sin^2\theta+\cos^2\theta = 1$ and $1-\sin^2\theta = \cos^2\theta$

The L.H.S reduces to

$$= \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta}$$

$$= \frac{1-\sin\theta}{\cos\theta} \quad \text{(Right hand side)}$$

Q.E.D

SECTION III

5. (a) **Required To Differentiate:** $\frac{3x+4}{x-2}$

Solution:

Let $y = \frac{3x+4}{x-2}$, which is of the form $y = \frac{u}{v}$, where

$u = 3x+4,\ \frac{du}{dx} = 3$

and

$v = x-2,\ \frac{dv}{dx} = 1$

Recall the formula

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad \text{(Quotient law)}$$

$$\therefore \frac{d}{dx}\left(\frac{3x+4}{x-2}\right) = \frac{(x-2)3-(3x+4)1}{(x-2)^2}$$

$$= \frac{3x-6-3x-4}{(x-2)^2}$$

$$= \frac{-10}{(x-2)^2} \text{ in its simplest form.}$$

(b) **Data:** The point $P\,(2, 10)$ lies on the curve $y = 3x^2+5x-12$.

(i) **Required To Find:** The equation of the tangent to the curve at P.

Solution:

$y = 3x^2+5x-12$

Gradient function, $\frac{dy}{dx} = 3(2x)+5$

$= 6x+5$

$\therefore$ The gradient of the tangent at $P = 6(2)+5$

$$=17$$

The equation of the tangent at P is

$$\frac{y-10}{x-2}=17$$
$$y-10=17(x-2)$$
$$y=17x-24$$

(ii) **Required To Find:** The equation of the normal to the curve at P.
Solution:

Gradient of the normal at $P=-\frac{1}{17}$

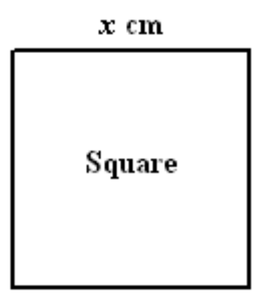
(The product of the gradient of perpendicular lines is -1)

Equation of the normal at P is

$$\frac{y-10}{x-2}=-\frac{1}{17}$$
$$17y-170=-x+2$$
$$17y=-x+172 \text{ (or any other equivalent form)}$$

(c) **Data**: The length of the side of a square is increasing at rate of 4 cms^{-1}
Required To Find: The rate of increase of the area, when the length of the side is 5 cm
Solution:

Let the square be of side x cm.
Let area = A cm^2.



$$\frac{dx}{dt}=4\,\text{cms}^{-1} \quad \text{(Data)}$$
$$A=x\times x$$
$$=x^2$$
$$\frac{dA}{dt}=\frac{dA}{dx}\times\frac{dx}{dt} \quad \text{(Chain rule)}$$
$$\frac{dA}{dx}=2x$$
$$\therefore \frac{dA}{dt}=2x\times 4$$

When $x=5$

$$\frac{dA}{dt}=2(5)\times 4 \quad \text{(The rate of increase of the area)}$$
$$=40\ \text{cm}^2\text{s}^{-1}$$

6. (a) **Required To Evaluate:** $\int_1^2 (16-7x)^3\ dx$.

Solution:

Let $t = 16 - 7x$

$$\frac{dt}{dx} = -7$$

$$\therefore dx = \frac{dt}{-7}$$

When $x = 2$

$$t = 16 - 7(2)$$
$$= 2$$

When $x = 1$

$$t = 16 - 7(1)$$
$$= 9$$

$$\therefore \int_1^2 (16-7x)^3\ dx \equiv \int_9^2 t^3\ \frac{dt}{-7}$$
$$= \left[\frac{t^4}{4(-7)}\right]_9^2$$
$$= \left[\frac{t^4}{-28}\right]_9^2$$
$$= \frac{(2)^4}{-28} - \frac{(9)^4}{-28}$$
$$= \frac{16}{-28} + \frac{6561}{28}$$
$$= \frac{6545}{28}$$
$$= \frac{935}{4}$$
$$= 233\frac{3}{4}$$

(b) **Data:** $Q\ (4,\ 8)$ lies on a curve for which $\frac{dy}{dx} = 3x - 5$.

Required To Determine: The equation of the curve

Solution:

$\frac{dy}{dx} = 3x - 5$

The equation of the curve is

$y = \int (3x - 5)\ dx$

$y = \frac{3x^2}{2} - 5x + C$, where C is a constant.

$(4, 8)$ lies on the curve.

Substituting $x = 4$ and $y = 8$.

$\therefore 8 = \frac{3(4)^2}{2} - 5(4) + C$

$8 = 24 - 20 + C$

$C = 4$

$\therefore$ The equation of the curve is $y = \frac{3x^2}{2} - 5x + 4$.

(c) **Required To Calculate:** The area between the curve $y = 2\cos x + 3\sin x$ and the $x-$axis from $x = 0$ to $\frac{\pi}{3}$.

Calculation:

The region whose area is required lies entirely above the $x-$axis and in the first quadrant.

$$\text{Area} = \int_0^{\frac{\pi}{3}} (2\cos x + 3\sin x)\ dx$$

$$= \left[2(\sin x) - 3(\cos x)\right]_0^{\frac{\pi}{3}}$$

$$= \left(2\sin\frac{\pi}{3} - 3\cos\frac{\pi}{3}\right) - (2\sin 0 - 3\cos 0)$$

$$= \left(2\times\frac{\sqrt{3}}{2} - 3\times\frac{1}{2}\right) - (2(0) - 3(1))$$

$$= \sqrt{3} - 1\frac{1}{2} + 3$$

$$= \left(\sqrt{3} + 1\frac{1}{2}\right) \text{ square units (exactly)}$$

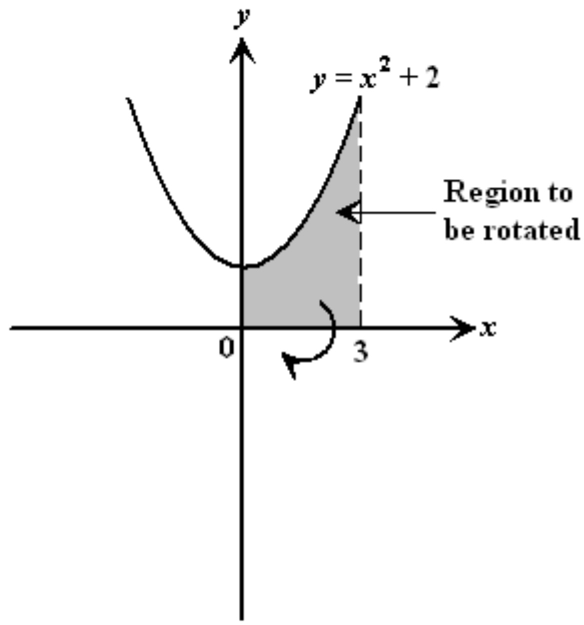
$$\approx 3.23\underline{\underline{2}}$$

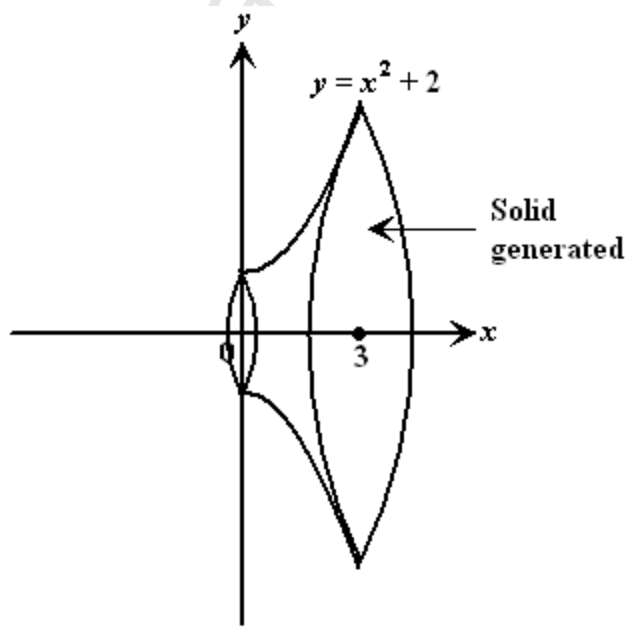
$$\approx 3.23 \text{ square units (to 2 decimal places or 3 significant figures)}$$

(d) **Required To Calculate:** The volume generated when the region bounded by $y = x^2 + 2$, the $x-$axis and the verticals $x = 0$ and $x = 3$ is rotated through $360°$ about the $x-$axis.

Calculation:

Volume, $V = \pi \int_{x_1}^{x_2} y^2 \, dx$





Volume, $V = \pi \int_0^3 \left(x^2+2\right)^2 \, dx$

$$= \pi \int_0^3 \left(x^4 + 4x^2 + 4\right) dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{4x^3}{3} + 4x \right]_0^3$$

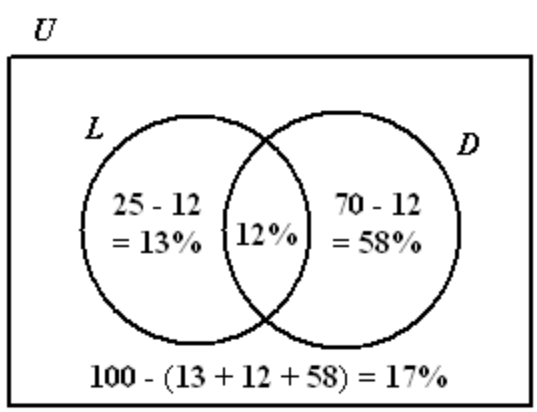
$$= \pi \left\{ \frac{(3)^5}{5} + \frac{4(3)^3}{3} + 4(3) - (0) \right\}$$

$$= \pi \left\{ \frac{243}{5} + 36 + 12 \right\}$$

$$= 96\frac{3}{5}\pi \text{ cubic units}$$

SECTION IV

7. (a) **Data:** 25% of people in a town own laptops. 70% of people in the town own desktop computers. 12% own both laptops and desktop computers.
Required To Determine: The proportion that does not own either a laptop or a desktop computer
Solution:
Creating a Venn diagram to illustrate the data
Let:
$U = \{\text{People in the survey}\}$
$L = \{\text{People who own laptops}\}$
$D = \{\text{People who own desktop computers}\}$

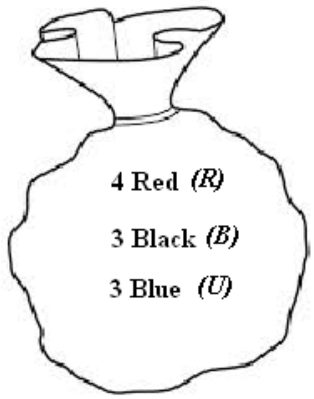


$\therefore$ The percentage of people who do not own either a laptop or a desktop computer = 17%.

This proportion is $\dfrac{17}{100}$ of the number of people in the survey.

NOTE: The term 'own neither' as written on the examination paper is better stated as 'do not own either'.

(b) **Data:** Bag of 4 red, 3 black and 3 blue marbles.



Three marbles are drawn at random from the bag.

(i) **Required To Find:** The probability that the marbles are of the same colour.

Solution:

$P(\text{Same colour}) = P(R\,\&\,R\,\&\,R)$ or $P(B\,\&\,B\,\&\,B)$ or $P(U\,\&\,U\,\&\,U)$

$$\begin{aligned}
&\quad\;\; 1^{\text{st}} \qquad\qquad 2^{\text{nd}} \qquad\qquad 3^{\text{rd}}\\
&= \frac{4}{(4+3+3)} \times \frac{3}{(4+3+3)-1} \times \frac{2}{(4+3+3-1)-1}\\
&\quad + \frac{3}{(4+3+3)} \times \frac{2}{(4+3+3)-1} \times \frac{1}{(4+3+3-1)-1}\\
&\quad + \frac{3}{(4+3+3)} \times \frac{2}{(4+3+3)-1} \times \frac{1}{(4+3+3-1)-1}\\
&= \left(\frac{4}{10}\times\frac{3}{9}\times\frac{2}{8}\right) + \left(\frac{3}{10}\times\frac{2}{9}\times\frac{1}{8}\right) + \left(\frac{3}{10}\times\frac{2}{9}\times\frac{1}{8}\right)\\
&= \frac{(4\times3\times2)+(3\times2\times1)+(3\times2\times1)}{10\times9\times8}\\
&= \frac{1}{20}
\end{aligned}$$

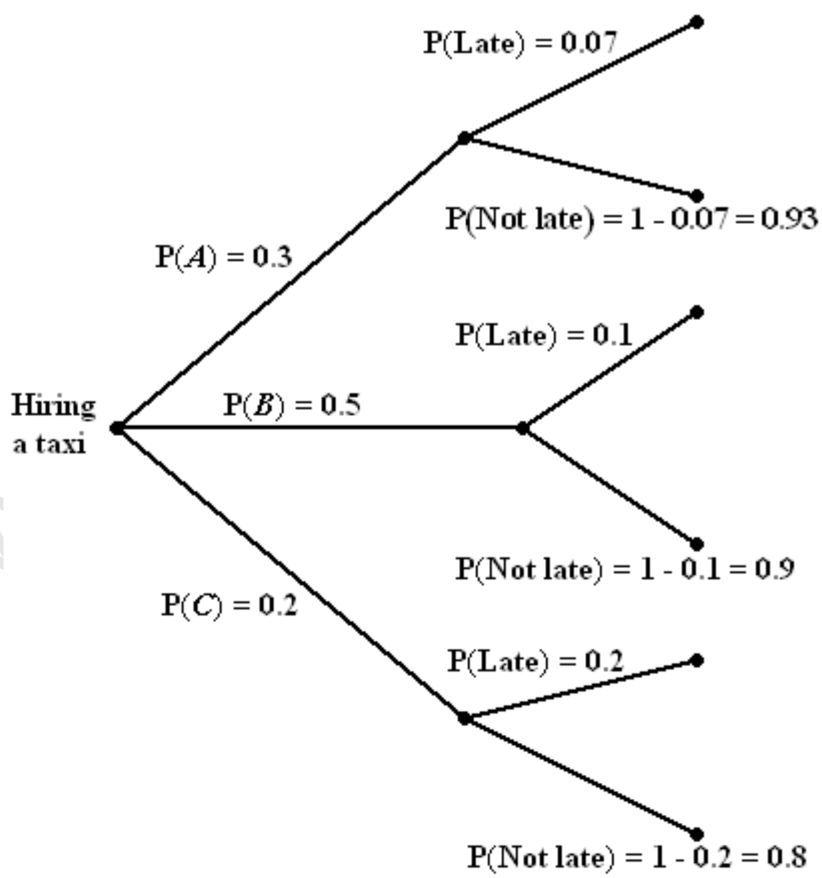
(ii) **Required To Find:** The probability that the selection contains exactly one red marble.

Solution:
P (Exactly one red marble) = P (Red and not Red and not Red)
= P (R and R' and R')

$$=\frac{4}{10}\times\frac{3+3}{(4+3+3)-1}\times\frac{3+3-1}{(4+3+3-1)-1}$$
$$=\frac{4}{10}\times\frac{6}{9}\times\frac{5}{8}$$
$$=\frac{1}{6}$$

(c) **Data:** The probability of hiring a taxi from garage A, B or C is $P(A)=0.3$, $P(B)=0.5$ and $P(C)=0.2$.
$P(\text{Late from } A)=0.07$, $P(\text{Late from } B)=0.1$ and $P(\text{Late from } C)=0.2$.

(i) **Required To Illustrate:** The information given on a tree diagram.
Solution:



(ii) a) **Required To Determine:** The probability that a taxi chosen at random, will arrive late.

Solution:

$$P(\text{Taxi will arrive late})$$
$$= P(A \text{ is chosen and taxi is late})$$
$$\text{or } P(B \text{ is chosen and taxi is late})$$
$$\text{or } P(C \text{ is chosen and taxi is late})$$
$$= (0.3 \times 0.07) + (0.5 \times 0.1) + (0.2 \times 0.2)$$
$$= \frac{111}{1000}$$

b) **Required To Determine:** The probability that a taxi chosen at random, will come from garage C given that it is late.

Solution:

$$P(\text{Taxi comes from } C \text{ given that it is late})$$

Let C be the event that a taxi comes from garage C.
Let L be the event that the taxi arrives line.
Required to calculate $P(C/L)$.

$$P(C/L) = \frac{P(C \cap L)}{P(L)} \quad \text{(Conditional probability)}$$

$$P(C \cap L) = P(\text{Taxi is late and comes from } C)$$
$$= 0.2 \times 0.2$$

$$P(L) = 0.111$$

$$P(C/L) = \frac{0.2 \times 0.2}{0.111}$$
$$= \frac{40}{111}$$

8. (a) **Data:**

Phase 1	A car starts from A. $\therefore$ Initial velocity, $u = 0$ ms^{-1}. Final velocity, $v = 24$ ms^{-1}. Acceleration, a, is constant = 6 ms^{-2}.
Phase 2	Speed of 24 ms^{-1} is kept for 5 s. $\therefore$ Acceleration = 0 ms^{-2} for 5 s.
Phase 3	The car comes to rest (velocity = 0 ms^{-1}) by acceleration = -3 ms^{-2}, which is a deceleration.

(i) **Required To Draw:** A velocity-time graph to illustrate the motion of the car.

Solution:

Considering the car's motion in the three separate phases as indicated above.

Phase 1

$u = 0,\ v = 24,\ a = 6$

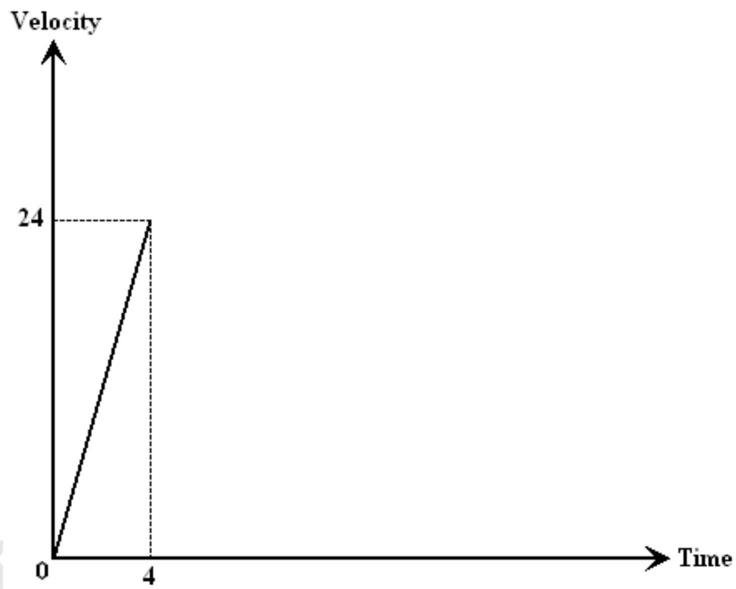
Recall:

$v = u + at$ (The equation of linear motion)

$\therefore 24 = 0 + 6(t)$

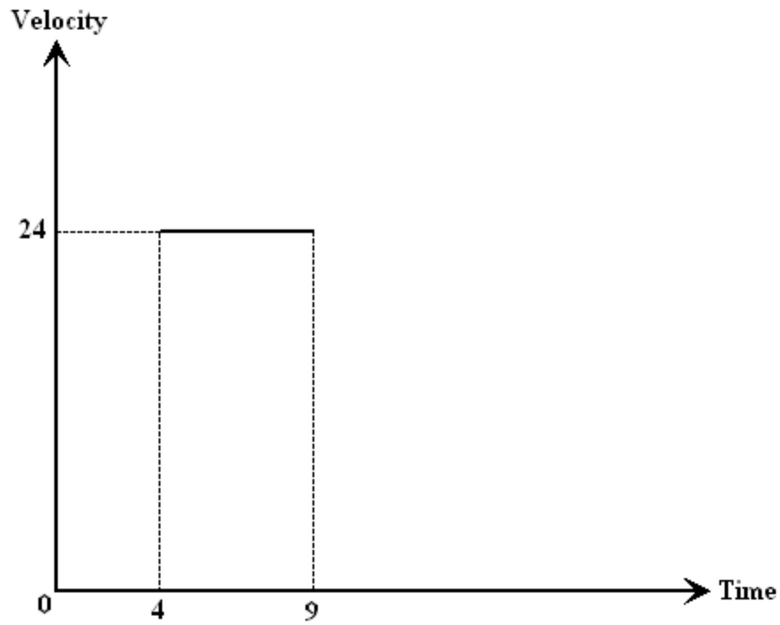
$\therefore t = 4$

For this phase:



The straight line has a gradient of 6 and indicates a constant acceleration of 6 ms^{-2}.

Phase 2



A horizontal line indicates an acceleration of 0 ms^{-2} for 5 seconds.
A horizontal line on a velocity-time graph indicates zero acceleration or constant velocity.

Phase 3

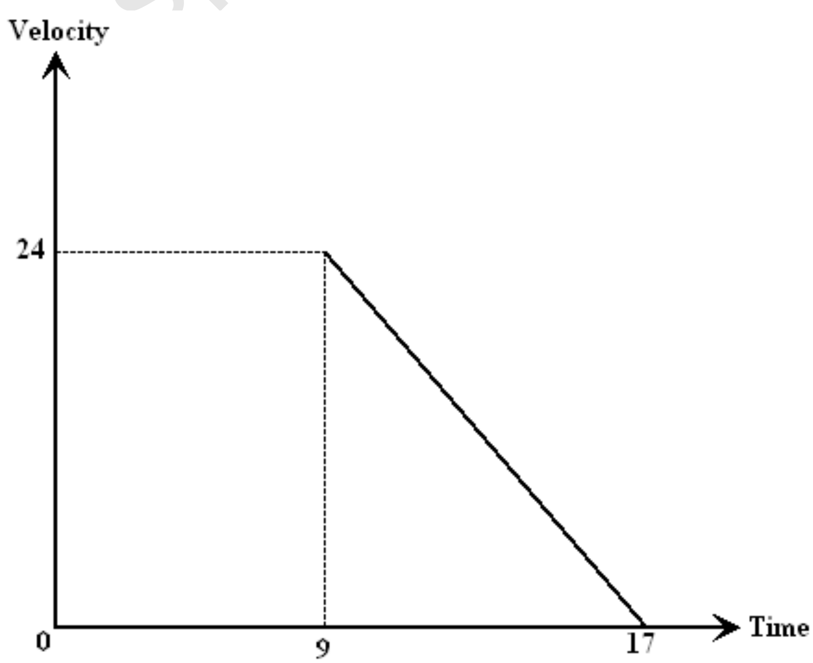
$u = 24,\ v = 0,\ a = -3$

$v = u + at$

$\therefore 0 = 24 + (-3)t$

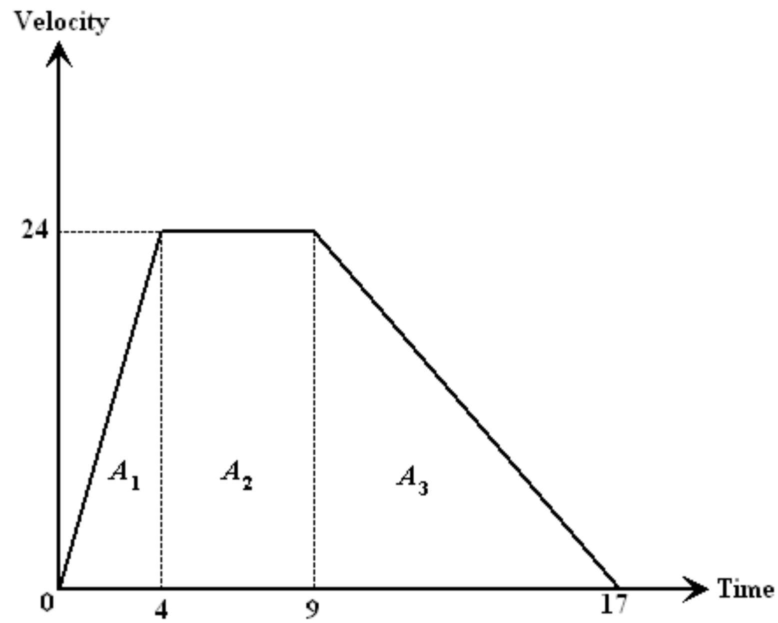
$\therefore t = 8$

$\therefore$ Car comes to rest in 8 seconds.



A straight line indicates constant acceleration.
A straight line with a gradient of – 3, that is, a negative gradient indicates a deceleration.

Put all three branches of the phase 1, 2 and 3 together to construct the velocity-time graph.



(ii) **Required To Determine:** The total distance covered by the car.
Solution:
The total distance covered can be found by the area bounded by the lines (branches) and the horizontal axis.
The total area is divided into regions A_1, A_2 and A_3, as shown in the above diagram.

Area of triangle $A_1 = \frac{4 \times 24}{2}$
$= 48 \text{ m}$

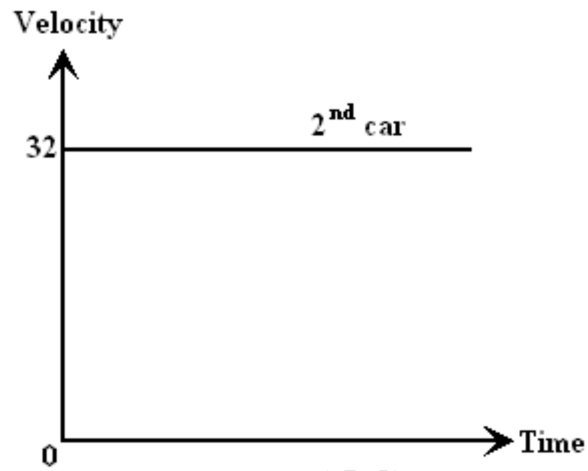
Area of rectangle $A_2 = 24 \times (9 - 4)$
$= 120 \text{ m}$

Area of triangle $A_3 = \frac{(17 - 9) \times 24}{2}$
$= 96 \text{ m}$

$\therefore$ Total distance covered $= 48 + 120 + 96 \quad = 264 \text{ m}$

FAS-PASS Maths

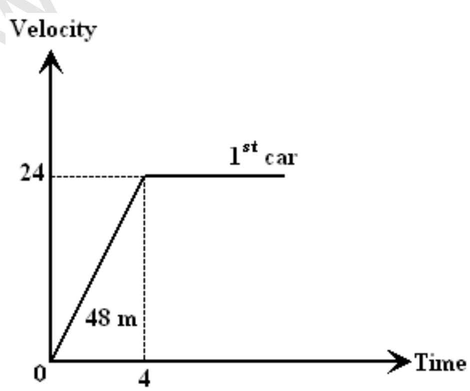
(iii) **Data:** A second car, moving at a constant velocity of 32 ms^{-1} drives past point A, 3 seconds after the first car left point A.
Required To Calculate: The length of time after the first car started that this second car meets it
Calculation:
Speed of the second car = 32 ms^{-1} (constant)

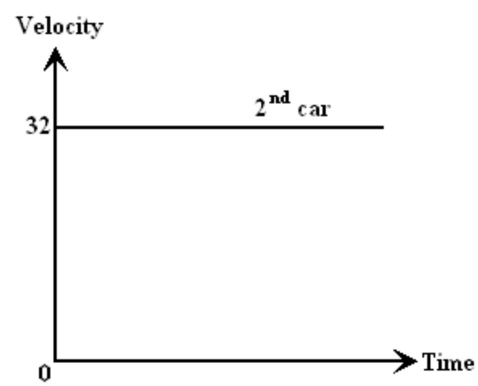


After the first 4 seconds, the first car covered $\frac{24\times 4}{2}=48\text{ m}$.

The second car started 3 seconds later and therefore would have covered only $32\times(4-3)=32\text{ m}$.

Hence, the second car did not meet the first car while the first car was accelerating (phase 1).
Perhaps they met while the first car was moving at a constant velocity. (This was given as an assumption at the end of the question but we can prove it as an exercise).





Let us suggest that the first car moved for t seconds at the constant speed of 24 ms^{-1} (phase 2) before the second car meets it.

Distance covered by the 1st car $=48+24t$ since it covers 48 m in its first phase and now moves at constant speed for t seconds

Distance covered by the 2nd car $=32\times(t+1)$ since it would have been moving for $(t+1)$ seconds

If t calculates to a 'real' answer, then the two cars would have met whilst the first car was in the phase 2 of its motion.

$$\therefore 48+24t=32t+32$$
$$16=8t$$
$$t=2$$

$t=2$ is a real answer.

$\therefore$ 1st car moved for $4+2=6$ seconds and the second car for 3 seconds before the two cars met.

(b) **Data:** A particle moves in a straight line with acceleration given by $a=(5t-1)$ ms^{-2} at any time in t seconds. When $t=2$ seconds, the particle has velocity 4 ms^{-1} and is 8 m from a fixed point O.

(i) **Required To Determine:** The velocity when $t=4$.

Solution:

Acceleration, $a=(5t-1)$ ms^{-2}

Let the velocity at time, t, be v ms^{-1}.

$$v=\int a\,dt$$
$$v=\int(5t-1)\,dt$$
$$v=\frac{5}{2}t^2-t+C \text{ (where } C \text{ is a constant)}$$

$v=4$, when $t=2$

NOTE: $t=2$ NOT $t=2$ seconds, since t was already given units in the question.

$$\therefore 4=\frac{5(2)^2}{2}-2+C$$
$$4=10-2+C$$
$$C=-4$$
$$v=\frac{5}{2}t^2-t+4$$

When $t=4$

$$v=\frac{5(4)^2}{2}-4-4$$
$$=40-8$$
$$=32\text{ ms}^{-1}$$

(ii) **Required To Determine:** The particles displacement from O when $t = 3$.

Solution:
Let the displacement from O at time t be s m.

$$s = \int v \, dt$$
$$= \int \left(\frac{5t^2}{2} - t - 4 \right) dt$$
$$= \frac{5t^3}{3 \times 2} - \frac{t^2}{2} - 4t + K \qquad (K = \text{constant})$$

$s = 8$ when $t = 2$

$$8 = \frac{5(2)^3}{6} - \frac{(2)^2}{2} - 4(2) + K$$
$$8 = 8\frac{1}{3} - 2 - 8 + K$$
$$\therefore K = 11\frac{1}{3}$$

$$s = \frac{5t^3}{6} - \frac{t^2}{2} - 4t + 11\frac{1}{3}$$

When $t = 3$

$$s = \frac{5(3)^3}{6} - \frac{(3)^2}{2} - 4(3) + 11\frac{1}{3}$$
$$= 22\frac{1}{2} - 4\frac{1}{2} - 12$$
$$= 17\frac{1}{3} \text{ m}$$