FAS-PASS Maths

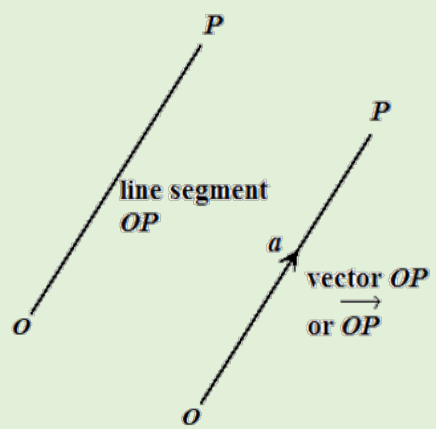
18: VECTORS

Scalar and Vector Quantities

To understand vectors, we must first be able to distinguish between the two measurable quantities, a scalar and a vector.

A **scalar** quantity is a measurable quantity that possesses only magnitude. It has no direction. Some examples of scalar quantities are distance, speed, area, volume, length, time and mass.

A **vector** quantity is a measurable quantity that possesses both magnitude and direction. Some examples of vectors are displacement, velocity, acceleration and weight. We use line segments to represent vectors and the direction of the arrow shows the direction of the vector. The difference between a scalar and a vector is illustrated below.



The vector $\overrightarrow{OP}$ is distinct from the scalar OP, whose line segment is not arrowed. The direction of the arrow shows the direction of the vector.

Vector Notation

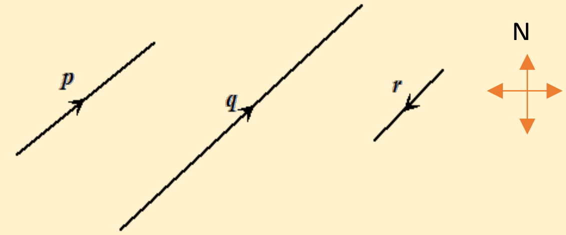
A vector is written or named as a line segment using the two end points, with an arrow at the top. This arrow indicates the direction of the vector. For example, the vector OP is written as $\overrightarrow{OP}$. In this case, the direction is from the point O to the point P. The point O is the tail of the vector and the point P is the head of the vector.

A vector may be symbolised by a letter, written in bold print, such as $\boldsymbol{a}$ or an underlined letter such as $\underline{\boldsymbol{a}}$ to indicate that a quantity is a vector. So, we may write $\overrightarrow{OP} = \boldsymbol{a}$.

Parallel and equal vectors

Vectors that have the same direction are parallel. For example, the vector, $\boldsymbol{p}$, 4 km North East is parallel to the vector, $\boldsymbol{q}$, 8 km North East. If, however, another vector, $\boldsymbol{r}$, is 2 km south-west, (which is opposite in direction to North East) then $\boldsymbol{r}$ is represented by an arrow drawn in the opposite direction, as shown below.

$\boldsymbol{p} = 4\ km\ NE$ $\quad$ $\boldsymbol{q} = 8\ km\ NE$ $\quad$ $\boldsymbol{r} = 2\ km\ SW$

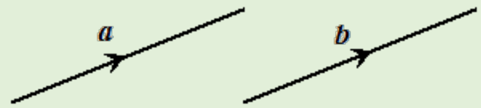


Note that all three vectors are parallel, but $\boldsymbol{p}$ and $\boldsymbol{q}$ are in the same direction while $\boldsymbol{r}$ is in the direction opposite to $\boldsymbol{p}$.

The vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are said to be *directly* parallel, while the vectors $\boldsymbol{p}$ and $\boldsymbol{r}$ are said to be *oppositely* parallel. Although the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are both in the same direction, their magnitudes are different. In our example, the magnitude of $\boldsymbol{p}$ is 4, the magnitude of $\boldsymbol{q}$ is 8 and the magnitude of $\boldsymbol{r}$ is 2.

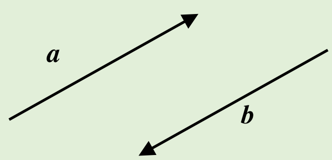
The vectors, $\boldsymbol{p}$ and $\boldsymbol{q}$ are directly parallel, so $\boldsymbol{q} = 2\boldsymbol{p}$, $\boldsymbol{p} = \frac{1}{2}\boldsymbol{q}$
The vectors, $\boldsymbol{p}$ and $\boldsymbol{r}$ are oppositely parallel, so $\boldsymbol{p} = -2\boldsymbol{r}$, $\boldsymbol{r} = -\frac{1}{2}\boldsymbol{p}$
The vectors, $\boldsymbol{r}$ and $\boldsymbol{q}$ are oppositely parallel, so $\boldsymbol{q} = -4\boldsymbol{r}$, $\boldsymbol{r} = -\frac{1}{4}\boldsymbol{q}$

Note if two vectors are **parallel**, one is a scalar multiple of the other. If two vectors are **equal**, they must have both the same magnitude and same direction. In other words, equal vectors represent the same movement at different positions on a plane.



Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same magnitude and also the same direction. Therefore $\boldsymbol{a} = \boldsymbol{b}$.

If two vectors are equal in magnitude but are opposite in direction, one is the inverse of the other.



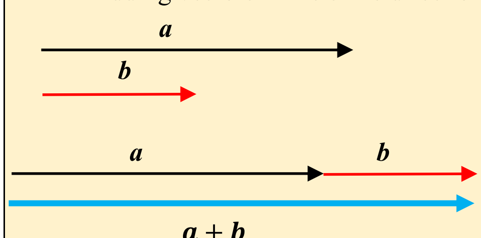
Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same magnitude but are opposite in direction. The vector $\boldsymbol{b}$ is the inverse (negative) of the vector $\boldsymbol{a}$ or $\boldsymbol{a}$ is the inverse of $\boldsymbol{b}$. Also, $\boldsymbol{a} = -\boldsymbol{b}$ or $\boldsymbol{b} = -\boldsymbol{a}$.

Adding Parallel vectors

Two or more vectors may be added or combined into a single vector called the **resultant**.

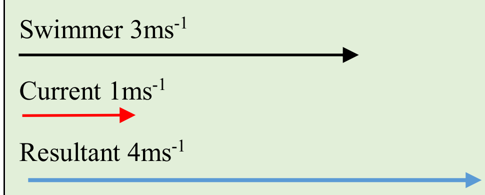
When both vectors are in the same direction, the resultant vector will be in the same direction as the separate or individual vectors. The magnitude of the resultant is obtained by adding the magnitudes of these individual vectors.

Adding vectors in the same direction

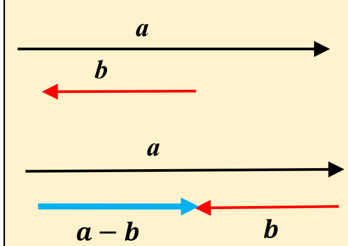


The two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are in the same direction. The resultant will be in the same direction as that of $\boldsymbol{a}$ and of $\boldsymbol{b}$. The magnitude of the resultant is the magnitude of $\boldsymbol{a}$ plus the magnitude of $\boldsymbol{b}$.

A practical example of parallel vectors is illustrated below. The swimmer is swimming in the same direction as the current (parallel vectors), hence the resultant is the sum of both velocities.

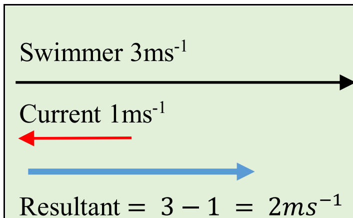


If the direction of the vectors is opposite to each other, then the resultant is still the sum, but in this case, one vector is negative so we end up subtracting the magnitudes.



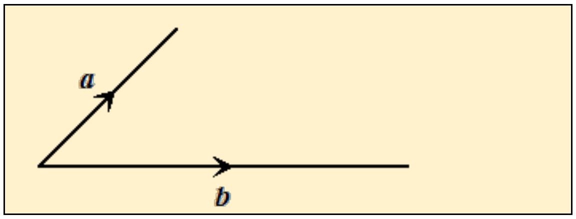
To add vectors in the opposite direction, we subtract magnitudes because $\boldsymbol{a} + (-\boldsymbol{b}) = \boldsymbol{a} - \boldsymbol{b}$.

If the swimmer was swimming against the current then the resultant would be the difference between the vectors.

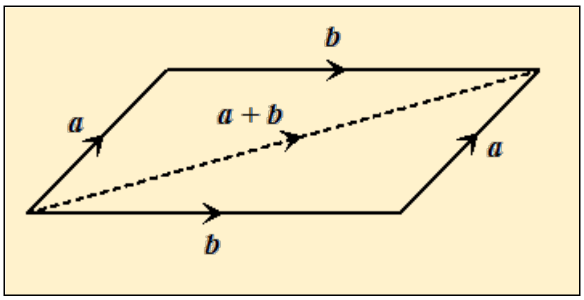


Adding non-parallel vectors

Vectors can be drawn from any point on a plane. Sometimes we are asked to add two vectors that are not parallel. In such cases, we use the property of equal vectors and conveniently shift one vector on the plane so that its starting point is at the endpoint of the other vector. Assuming two vectors are to be added and they start at the same point, like this:

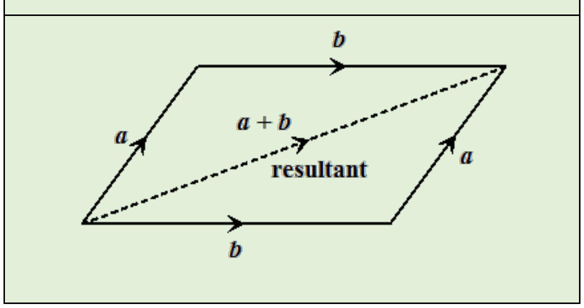


In order to add ***b*** to ***a***, we must apply the tail to head principle and shift ***b*** to the point where ***a*** ends. In so doing, we form a parallelogram with ***b*** and ***a*** as adjacent sides, as shown below.

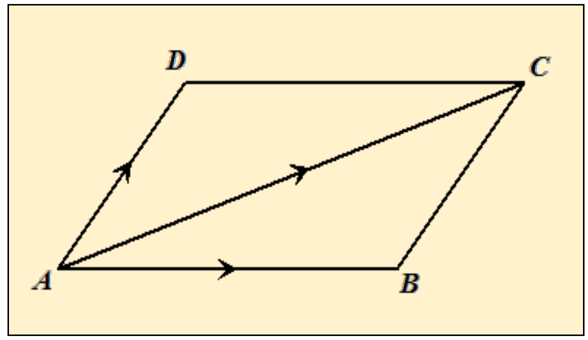


We are now in a position to determine the resultant vector. This is the vector that connects the starting point to the end point. If the vectors ***a*** and ***b*** are **not parallel**, then we may introduce the parallelogram law of vector addition to obtain their resultant.

Parallelogram law of vector addition
If two vectors are represented in magnitude and direction by the adjacent sides of a parallelogram, then the resultant is represented in magnitude and direction by the diagonal of the parallelogram taken from the starting point.



As illustrated above, when we use the parallelogram law to add two vectors that start from the same point, we shift one of the vectors to the end point of the other. In so doing, we formed a triangle whose third side is the resultant of the two vectors. This is illustrated in the diagram below.



By the parallelogram law:

$$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$$

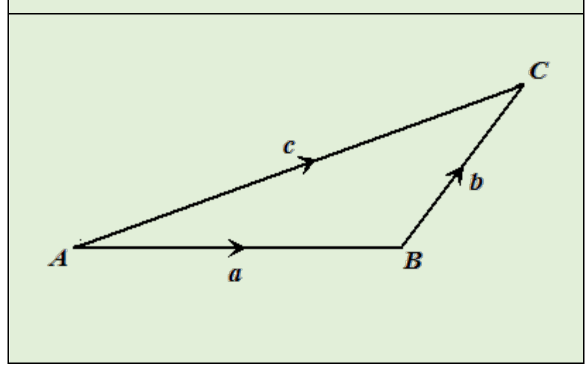
But, $\overrightarrow{BC} = \overrightarrow{AD}$ (opposite sides of a parallelogram)

Hence, we may replace $\overrightarrow{AD}$ by $\overrightarrow{BC}$ to get:

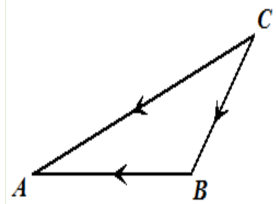
$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Notice we are now involving the three sides of a triangle ABC. This is called the triangle law for adding vectors.

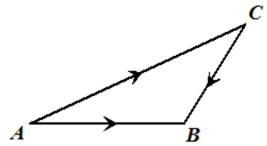
Triangle law of vector addition
If two vectors are represented in magnitude and direction by two sides of a triangle, taken in order, then the resultant is represented in magnitude and direction by the third side of the triangle, taken in the direction from the starting point.



Hence, we do not always need the parallelogram law to add two vectors, we can simply use the principle of adding vectors tail-to-head to find the resultant vector. If vectors are so positioned that the resultant can be obtained by completing a triangle, then we apply the triangle law. Observe the following applications of this law.



$\overrightarrow{CB} + \overrightarrow{BA} = \overrightarrow{CA}$



$\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$

If we observe any statement in which the triangle law is applied, we notice a pattern which allows us to predict the resultant without even drawing the triangle. This is illustrated below:

For a given triangle, ABC:

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

These two letters must be the same because both vectors share this vertex.

The resultant is always the connection between the starting point and the end point.

Example 1

Write down the resultant in each case:

(i) $\overrightarrow{PQ} + \overrightarrow{QR}$
(ii) $\overrightarrow{RP} + \overrightarrow{PQ}$
(iii) $\overrightarrow{QR} + \overrightarrow{RP}$

Solution

By observation of the pattern

(i) $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$ (Q is the common point, P is the starting point and R is the end point)

(ii) $\overrightarrow{RP} + \overrightarrow{PQ} = \overrightarrow{RQ}$ (P is the common point, R is the starting point and Q is the end point)

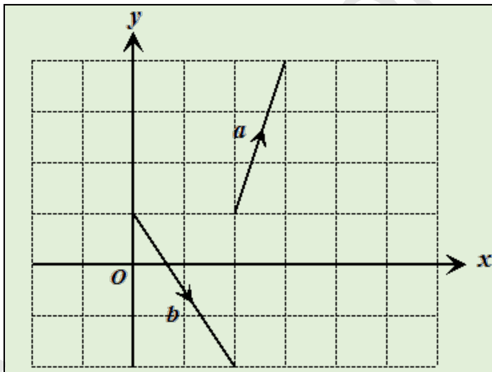
(iii) $\overrightarrow{QR} + \overrightarrow{RP} = \overrightarrow{QP}$ (R is the common point, Q is the starting point and P is the end point)

Vectors on the Cartesian Plane

So far, we have represented vectors using line segments with arrows to show their direction. When vectors are represented on the Cartesian Plane, we use another convention. We have used this convention in the study of motion geometry, where column matrices were used to describe a translation.

Column vectors

In this notation, a 2×1 matrix is used to represent the vector. The elements of the matrix or components of the vector are their displacements along the ***x*** and ***y***-axes.



The vectors, ***a*** and ***b*** can be described using column matrices, where

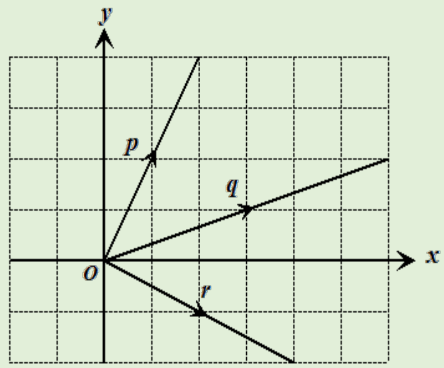
$\boldsymbol{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$. Notice in these two examples neither vector was drawn from O.

In general, the vector $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$ represents a displacement of ***a*** units along the ***x***-axis and a displacement of ***b*** units along the ***y***-axis.

Position vectors

All vectors are measured from a fixed point. Vectors on a Cartesian Plane can start at any point on the plane (as seen in the above example). Such vectors are known as free vectors. If, however, we start a vector at the origin, then such a vector is called a position vector.

In this sense, position vectors are not free vectors because they are tied to the origin. Their starting point is always the origin.



The vectors $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ are position vectors, where:

$\boldsymbol{p} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $\quad \boldsymbol{q} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ $\quad \boldsymbol{r} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

In general, if P (a, b), then the position vector, $\overrightarrow{OP} = \begin{pmatrix} a \\ b \end{pmatrix}$.

The inverse of $\overrightarrow{OP}$ is $\overrightarrow{PO} = -\overrightarrow{OP} = \begin{pmatrix} -a \\ -b \end{pmatrix}$

Example 2

The points A, B and C have position vectors

$\overrightarrow{OA} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ $\quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 12 \\ -2 \end{pmatrix}$

Express in the form $\begin{pmatrix} x \\ y \end{pmatrix}$ the vector

(i) $\overrightarrow{BA}$ $\quad$ (ii) $\overrightarrow{BC}$

Solution

(i) $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -\overrightarrow{OB} + \overrightarrow{OA}$

$= -\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} + \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

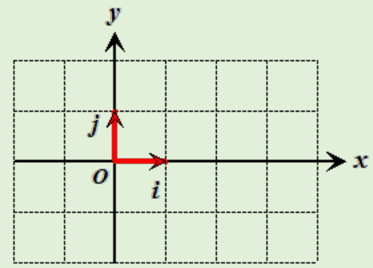
(ii) $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\overrightarrow{OB} + \overrightarrow{OC}$

$= -\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 12 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} + \begin{pmatrix} 12 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \end{pmatrix}$

Unit vectors

A unit vector is a vector whose length is one unit. A unit vector can have any direction, but its magnitude is always one.

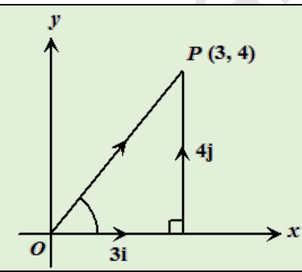
On the Cartesian plane, we can define two special unit vectors, $\boldsymbol{i}$ and $\boldsymbol{j}$, shown in the diagram below.



The unit vector $\boldsymbol{i}$ is one unit in length, parallel to the x-axis. The unit vector, $\boldsymbol{j}$ is one unit in length, parallel to the y-axis.

$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We can now define any position vector in terms of the unit vectors, $\boldsymbol{i}$ and $\boldsymbol{j}$.



If P is the point $(3, 4)$ then the vector $\overrightarrow{OP}$ can be written as $\overrightarrow{OP} = 3\mathbf{i} + 4\mathbf{j}$.

This is the unit vector notation.

We now have an alternative notation to represent a vector. This notation is convenient as it facilitates problems involving algebraic manipulations of vectors.

In general, we can represent vectors using either the unit vector notation or column matrix notation,

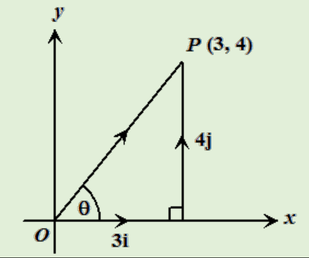
$a\mathbf{i} + b\mathbf{j} = \begin{pmatrix} a \\ b \end{pmatrix}$

The modulus of a vector

The modulus of a vector refers to its length. If we know the coordinates of a point, P, then we can find the length of the position vector, $\overrightarrow{OP}$ using Pythagoras' Theorem.

For example, to determine the length of the position vector $\overrightarrow{OP}$, where P (3, 4). We observe that OP is the hypotenuse of a right-angled triangle whose horizontal and vertical sides are 3 and 4 units respectively.

The diagram below illustrates the use of Pythagoras' Theorem in calculating the length of a vectors.



By Pythagoras' Theorem,

$\left|\overrightarrow{OP}\right| = \sqrt{(3)^2 + (4)^2} = 5$

In general, if $\overrightarrow{OP} = a\boldsymbol{i} + b\boldsymbol{j} = \binom{a}{b}$ then the magnitude or the modulus of $\overrightarrow{OP}$, is

$$\left|\overrightarrow{OP}\right| = \sqrt{a^2 + b^2}$$

The Direction of a Vector

The direction of a vector is the angle it makes with the positive direction of the x-axis. This is easily obtained by simple trigonometry.

In the above example, the direction of $\overrightarrow{OP}$ is determined by calculating θ using the tangent ratio.

Since $\tan\theta = \frac{4}{3}$, the direction of the vector $\overrightarrow{OP}$ is calculated by obtaining the value of θ from:

$$\theta = tan^{-1}\left(\frac{4}{3}\right) = 53.1^0$$

In general, if $\overrightarrow{OP} = a\boldsymbol{i} + b\boldsymbol{j} = \binom{a}{b}$ then the direction of $\overrightarrow{OP}$ is the angle that θ makes with the positive direction of the x-axis, where

$$\theta = tan^{-1}\left(\frac{b}{a}\right)$$

Adding of vectors - unit vector notation

When we are adding or subtracting vectors, using the unit vector notation, we add the coefficients of $\boldsymbol{i}$ and $\boldsymbol{j}$ as illustrated below.

If $a = a_1\mathbf{i} + b_1\mathbf{j}$ and

$b = a_2\mathbf{i} + b_2\mathbf{j}$

Then $a + b = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j}$

So too, $a - b = (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j}$

Example 3

The vectors $\boldsymbol{a}$ and $\mathbf{b}$ are: $a = 6\mathbf{i} + \mathbf{j}$ and $b = 2\mathbf{i} + 7\mathbf{j}$, express (i) $\boldsymbol{a} + \boldsymbol{b}$ (ii) $\boldsymbol{a} - \boldsymbol{b}$ in the form $a\mathbf{i} + b\mathbf{j}$

Solution

(i)

$a + b = (6+2)\mathbf{i} + (1+7)\mathbf{j}$

$a + b = 8\mathbf{i} + 8\mathbf{j}$

(ii)

$a - b = a + -b$

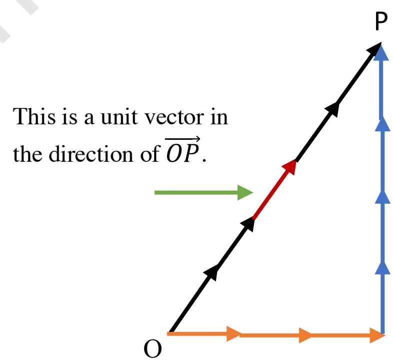
$a - b = (6\mathbf{i} + \mathbf{j}) + (-2\mathbf{i} - 7\mathbf{j})$

$a - b = (6-2)\mathbf{i} + (1-7)\mathbf{j}$

$a - b = 4\mathbf{i} - 6\mathbf{j}$

Unit vectors that are not parallel to the x and y-axes

A unit vector always has a magnitude of one unit, but it can have any direction. We can visualize a unit vector parallel to the vector, $\overrightarrow{OP} = \binom{3}{4}$ as a vector whose magnitude is one and whose direction is the same as OP.



When we divide a vector by its magnitude we obtain a vector which has the same direction but a magnitude of one.

The magnitude of the vector $\binom{3}{4}$ is $\sqrt{3^2 + 4^2}$ =5, hence, $\overrightarrow{OP}$ has a length of 5 units.

The unit vector in the direction of $\binom{3}{4}$ is one-fifth the vector $\binom{3}{4}$ or $\frac{1}{5}\binom{3}{4} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$. Hence, there are 5 vectors, each of length one unit along OP to make up a magnitude of 5.

In general, a unit vector in the direction of $\begin{pmatrix} a \\ b \end{pmatrix}$ is

$$\frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a \\ b \end{pmatrix}$$

Example 4

Find the unit vector in the direction of $\overrightarrow{OM}$, where $\overrightarrow{OM} = 5\boldsymbol{i} + 12\boldsymbol{j}$.

Solution

$|\overrightarrow{\boldsymbol{OM}}| = \sqrt{25+144} = \sqrt{169} = 13$

The unit vector in the direction of $\overline{OM}$ is

$\frac{1}{13}(5\boldsymbol{i} + 12\boldsymbol{j})$

Proofs in vectors

Often when we study vectors, we are asked to prove relationships between two vectors. For example, we may be asked to prove that two vectors are parallel, or collinear.

To prove two vectors are parallel

If we wish to prove that vectors are parallel, we must simply show that either one of them is a scalar multiple of the other. The converse is also true, that is, if a vector can be expressed as a scalar multiple of another, then they are parallel.

For example, if we are given two vectors $\overrightarrow{VW}$ and $\overrightarrow{ST}$ such that, $\overrightarrow{VW} = \alpha\ \overrightarrow{ST}$, where α is a scalar. We can conclude that $\overrightarrow{VW}$ and $\overrightarrow{ST}$ are parallel vectors.

Example 4

The position vectors of the points L, M and K relative to the origin are:

$\overrightarrow{OM} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $\overrightarrow{ON} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$ $\overrightarrow{OK} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

(a) Express as column vectors:

(i) $\overrightarrow{MN}$ (ii) $\overrightarrow{KM}$

(b) The point T is such that $MT = NT$. Use a vector method to determine the position vector of T. Hence, state the coordinates of T.

(c) Hence, prove that $OKMT$ is a parallelogram.

Solution

(a) (i) $\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} + \begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$

(a)(ii) $\overrightarrow{KM} = \overrightarrow{KO} + \overrightarrow{OM} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

(b) Since $MT = NT$, $MT = \frac{1}{2}MN$

$\overrightarrow{OT} = \overrightarrow{OM} + \overrightarrow{MT}$

$= \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 4 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

Hence, the coordinates of T are $(4, 1)$.

(c) In the quadrilateral $OKMT$, $\overrightarrow{OT} = \overrightarrow{KM} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

Hence, the quadrilateral is a parallelogram because it has one pair of opposite sides which are both parallel and equal.

Example 5

In triangle ABC, R and S are the midpoints of AB and BD respectively.

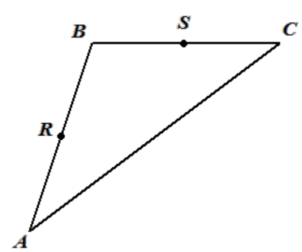
(a) Sketch the diagram to show the points R and S.

(b) Given that $\overrightarrow{AB} = 4\boldsymbol{x}$ and $\overrightarrow{BC} = 6\boldsymbol{y}$, express in terms of $\boldsymbol{x}$ and $\boldsymbol{y}$ (i) $\overrightarrow{AC}$ (ii) $\overrightarrow{RS}$

(c) Hence, show that $\overrightarrow{RS} = \frac{1}{2}\overrightarrow{AC}$.

Solution

(a)



(b) (i) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = 4\boldsymbol{x} + 6\boldsymbol{y}$

(ii) $\overrightarrow{RS} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$

$= \frac{1}{2}(4\boldsymbol{x}) + \frac{1}{2}(6\boldsymbol{y}) = 2\boldsymbol{x} + 3\boldsymbol{y}$

(c) $\overrightarrow{AC} = 2(2\boldsymbol{x} + 3\boldsymbol{y}) = 2\overrightarrow{RS}$

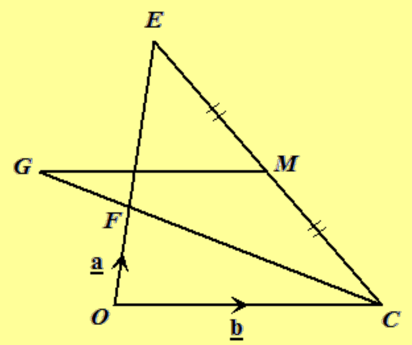
$\overrightarrow{RS} = \frac{1}{2}\overrightarrow{AC}$

Example 6

In the diagram below, M the midpoint of CE, $\overrightarrow{OF} = \underline{a}$, $\overrightarrow{OC} = \underline{b}$ $\overrightarrow{FE} = 2\overrightarrow{OF}$.

(a) Express in terms of $\underline{a}$ and $\underline{b}$ (i) $\overrightarrow{CF}$ (ii) $\overrightarrow{CE}$ (iii) $\overrightarrow{CM}$ (iv) $\overrightarrow{MG}$

(b) Calculate the value of k for which $\overrightarrow{MG} = \overrightarrow{CO}$.



Solution

(a) If $FE = 2OF$, then $FE = 2\underline{a}$ and $OE = \underline{a} + 2\underline{a} = 3\underline{a}$.

(i) CF

$\overrightarrow{CF} = \overrightarrow{CO} + \overrightarrow{OF}$
$= -(\underline{b}) + \underline{a}$
$= \underline{a} - \underline{b}$

(ii) CE

$\overrightarrow{CE} = \overrightarrow{CO} + \overrightarrow{OE}$
$= -\underline{b} + 3\underline{a}$
$= 3\underline{a} - \underline{b}$

(iii) CM

$\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CE}$
$= \frac{1}{2}(3\underline{a} - \underline{b})$
$= 1\frac{1}{2}\underline{a} - \frac{1}{2}\underline{b}$

(iv) MG

$\overrightarrow{CG} = k\overrightarrow{CF} = k(\underline{a} - \underline{b})$

$\overrightarrow{MG} = \overrightarrow{MC} + \overrightarrow{CG}$
$= -\left(1\frac{1}{2}\underline{a} - \frac{1}{2}\underline{b}\right) + k(\underline{a} - \underline{b})$
$= -1\frac{1}{2}\underline{a} + \frac{1}{2}\underline{b} + k\underline{a} - k\underline{b}$
$= \left(k - 1\frac{1}{2}\right)\underline{a} + \left(\frac{1}{2} - k\right)\underline{b}$

(b) To calculate the value of k for which $\overrightarrow{MG} = \overrightarrow{CO}$ we equate the expressions for both vectors found previously.

$$\overrightarrow{MG} = \overrightarrow{CO}$$

$$\left(k - 1\frac{1}{2}\right)\underline{a} + \left(\frac{1}{2} - k\right)\underline{b} = 0\underline{a} + (-\underline{b})$$

Equating components:

$k - 1\frac{1}{2} = 0, \Rightarrow k = 1\frac{1}{2}$

OR

$\frac{1}{2} - k = -1, \Rightarrow k = 1\frac{1}{2}$

To prove three points are collinear

Points are collinear if they lie on the same straight line. To prove A, B and C are collinear, we may prove that two of the line segments, AB, BC or AC are parallel. We do so by proving that one is a scalar multiple of the other. Since there is a common point together with the parallel property, then the three points must be collinear.

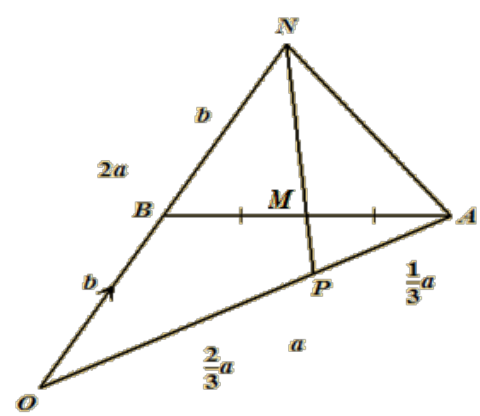
Example 6

In the triangle OAB, the point, P is on OA such that $\overrightarrow{OP} = 2\overrightarrow{PA}$, and M is on BA such that $BM = MA$ and OB is produced to N such that $OB = BN$. Given that $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$,

(a) Draw a diagram showing all the given information.
(b) Express $\overrightarrow{AB}$, $\overrightarrow{PA}$ and $\overrightarrow{PM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.
(c) Prove P, M and N are collinear.

Solution

(a)



(b) $\overrightarrow{AB}$, $\overrightarrow{PA}$ and $\overrightarrow{PM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -(\boldsymbol{a}) + \boldsymbol{b} = -\boldsymbol{a} + \boldsymbol{b}$$

If $\overrightarrow{OP} = 2\overrightarrow{PA}$, then

$$\overrightarrow{OP} = \frac{2}{3}\overrightarrow{OA} = \frac{2}{3}\boldsymbol{a}$$

$$\overrightarrow{PA} = \frac{1}{3}\overrightarrow{OA} = \frac{1}{3}\boldsymbol{a}$$

$$\overrightarrow{PM} = \overrightarrow{PA} + \overrightarrow{AM} = \frac{1}{3}\boldsymbol{a} + \overrightarrow{AM}$$

$$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$$

$$\overrightarrow{PM} = \frac{1}{3}\boldsymbol{a} + \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{b}) = \frac{1}{3}\boldsymbol{a} - \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} = -\frac{1}{6}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$$

c) Required to Prove: P, M and N are collinear.

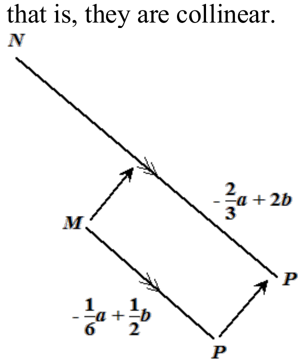
$$\overrightarrow{ON} = 2b$$

$$\overrightarrow{PN} = \overrightarrow{PO} + \overrightarrow{ON} = -\frac{2}{3}\boldsymbol{a} + 2\boldsymbol{b}$$

$$= 4\left(-\frac{1}{6}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right) = 4\overrightarrow{PM}$$

$\overrightarrow{PN} = 4\overrightarrow{PM}$, so $\overrightarrow{PN}$ is a scalar multiple of $\overrightarrow{PM}$

Hence, $\overrightarrow{PN}$ is parallel to $\overrightarrow{PM}$. P is a common point, so M must lie on PN and P, M and N lie on the same straight line, that is, they are collinear.

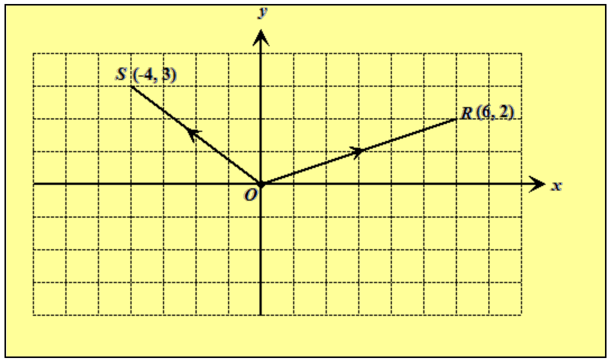


Example 7

The diagram below shows two position vectors $\overline{OR}$ and $\overline{OS}$ such that $R(6, 2)$ and $S(-4, 3)$.

Write as a column vector in the form $\begin{pmatrix} x \\ y \end{pmatrix}$:

(i) $\overline{OR}$ (ii) $\overline{OS}$ (iii) $\overline{SR}$



Solution

(i) $R(6, 2)$, $\overline{OR} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ is of the form $\begin{pmatrix} x \\ y \end{pmatrix}$, where $x = 6$ and $y = 2$.

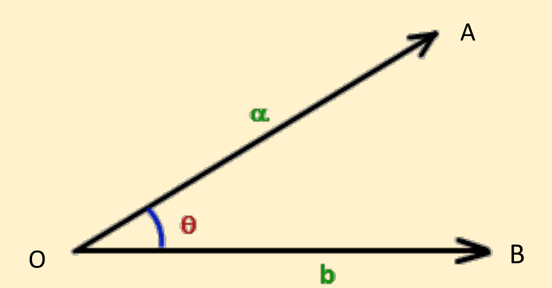
(ii) $S(-4, 3)$, $\overline{OS} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ is of the form $\begin{pmatrix} x \\ y \end{pmatrix}$, where $x = -4$ and $y = 3$.

(iii) $\overline{SR} = \overline{SO} + \overline{OR} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \end{pmatrix}$

is of the form $\begin{pmatrix} x \\ y \end{pmatrix}$, where $x = 10$ and $y = -1$.

The angle between two vectors

By definition, the angle between two vectors, OA and OB is the angle AOB, where both vectors must originate from the fixed point, O and AOB is less than 180^0.



We define the angle between the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as θ , where θ is always less than 180^0.

To calculate the angle between two vectors, we must be given the vectors expressed either as column matrices or in unit vector notation. The angle is calculated using the dot product law. This law states if $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors, such that

$a = a_1 i + b_1 j$ and $b = a_2 i + b_2 j$, then

$$a \bullet b = |a||b|\cos\theta$$

In using this law, we introduce a new concept, called the **scalar product or dot product of two vectors,** defined as follows:

$a \bullet b = a_1a_2 + b_1b_2$ where $\bullet$ is a binary operation.

The result $a_1a_2 + b_1b_2$ is called the dot product or the scalar product of the vectors. The scalar product is useful when calculating the angle between two vectors.

To calculate the angle between the two vectors $\overrightarrow{OP}$ and $\overrightarrow{OM}$ we use the following procedure:

1. We need to first find the dot product, $\overrightarrow{OP} \bullet \overrightarrow{OM}$.
2. Then, we must compute the modulus of each vector, $|OP|$ and $|OM|$.
3. Next, we substitute these values into the formula:
   $\overrightarrow{OP} \bullet \overrightarrow{OM} = |\overrightarrow{OP}||\overrightarrow{OM}|\cos\theta$
4. Solve for θ.

The angle, θ, between two vectors, ***a*** and ***b***, is calculated from the formula:

$$\cos\theta = \frac{\boldsymbol{a} \bullet \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Example 8

Find the scalar product of the vectors:
(i) $a = 2i + 6j$ and $b = 5i - j$
(ii) $p = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$ and $q = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Solution

(i) $a \bullet b = (2\times5) + (6\times-1) = 10 - 6 = 4$

(ii) $p \bullet q = (-5 \times 4) + (3 \times 2) = -20 + 6 = -14$

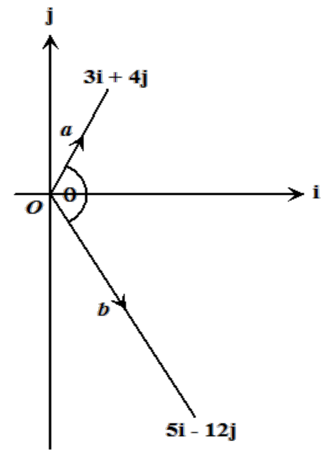
Notice that 4 and -14 are scalar quantities.

Example 9

$a = 3\mathbf{i} + 4\mathbf{j}$ and $b = 5\mathbf{i} - 12\mathbf{j}$
Find the angle between *a* and *b*.

Solution

Let the angle between ***a*** and ***b*** be θ, where θ < 180⁰.



$a \bullet b = (3 \times 5) + (4 \times 12) = 15 - 48 = -33$

$|a| = \sqrt{(3)^2 + (4)^2} = 5$

$|b| = \sqrt{(5)^2 + (-12)^2} = 13$

$-33 = 5 \times 13 \cos\theta$

$\cos\theta = \dfrac{-33}{65}$

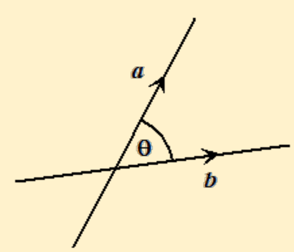
Since θ is negative, it is in the third or fourth quadrant, but it is also less than 180^0, hence it must be in the third quadrant only.

$\theta = 180° - \cos^{-1}\left(\dfrac{+33}{65}\right)$ $\theta = 180° - 59.49°$

$\theta = 120.5°$

Perpendicular vectors

Let *a* and *b* represent two vectors and let θ represent the angle between them.



By the dot product law

$a \bullet b = |a||b|\cos\theta$

When $\theta = 90^0$, $\cos 90° = 0$

$a \bullet b = 0$

We use the dot product law to test for perpendicularity of vectors.

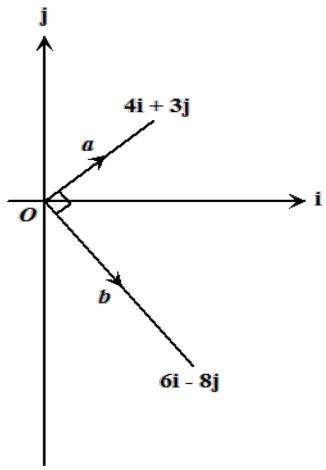
Perpendicular vectors

If *a* is perpendicular to *b*, then $a \bullet b = 0$.
Also, if $a \bullet b = 0$, *a* and *b* are perpendicular.

Example 10

$a = 4\mathbf{i} + 3\mathbf{j}$ and $b = 6\mathbf{i} - 8\mathbf{j}$. Prove that ***a*** and ***b*** are perpendicular.

Solution



$a \bullet b = (4\times 6) + (3\times -8)$

$a \bullet b = 0$

Therefore, a is perpendicular to b.

Example 11

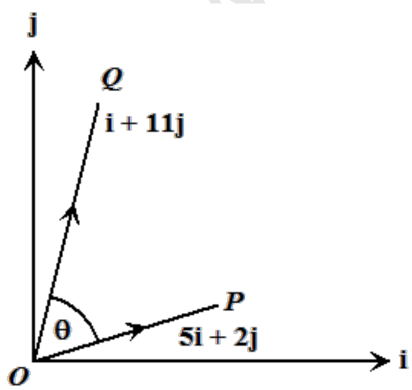
The position vectors of P and Q with respect to O are p and q respectively and where $p = 5\mathbf{i} + 2\mathbf{j}$ and $q = \mathbf{i} + 11\mathbf{j}$. Find

i. $p \bullet q$

ii. $P\hat{O}Q$

iii. $|p + 2q|$

Solution



(i) $p \bullet q$

$$p \bullet q = (1\times 5) + (11\times 2)$$
$$= 5 + 22 = 27$$

(ii) $P\hat{O}Q$

$$|q| = \sqrt{(1)^2 + (11)^2} = \sqrt{122}$$
$$|p| = \sqrt{(5)^2 + (2)^2} = \sqrt{29}$$

$$p \bullet q = |p||q|\cos\theta$$
$$27 = \sqrt{29}\sqrt{122}\ \cos\theta$$
$$\cos\theta = \frac{27}{\sqrt{29}\sqrt{122}}$$
$$\theta = P\hat{O}Q = 63^\circ$$

(iii) $|p + 2q|$

$$p + 2q = 5i + 2j + 2(i + 11j)$$
$$p + 2q = 5i + 2j + 2i + 22j$$
$$p + 2q = 7i + 24j$$
$$\therefore |p + 2q| = \sqrt{(7)^2 + (24)^2}$$
$$|p + 2q| = 25$$

Example 12

If p and q are two vectors where $p = 2\mathbf{i} + 5\mathbf{j}$ and $q = \alpha\mathbf{i} + 4\mathbf{j}$, find α, such that p is perpendicular to q.

Solution

If p is perpendicular to q, then $p \bullet q = 0$.

$$\therefore (2\times\alpha) + (5\times 4) = 0$$
$$2\alpha + 20 = 0$$
$$\alpha = -10$$