FAS-PASS Maths

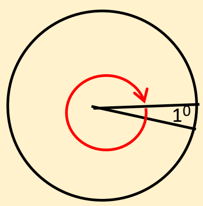
12: TRIGONOMETRY I

Measure of angles

An angle is a measure of the turn of a straight line about a fixed point. Angles are usually measured in degrees or in radians. The difference between these two measures is illustrated below.

1. Degrees

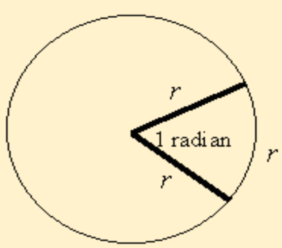
A whole turn = 360^0.



If we were to rotate the radius of a circle about its center so as to form a complete revolution, the angle formed is 360^0. One degree is a measure of turn equal to $\frac{1}{360}$ of a whole revolution.

2. Radians

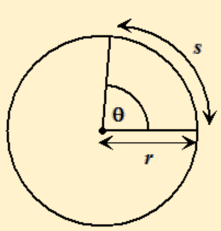
A whole turn = 2π radians.



The angle subtended at the center of a circle by an arc equal in length to r, where r is the radius of the circle, is one radian.

Since, circumference, $C = 2\pi r$, The number of arcs of length r that make up the circumference of a circle is $\frac{2\pi r}{r} = 2\pi$

Arc length, $s = r\theta$



If an arc of length s subtends an angle θ at the center of a circle, then the measure of the angle, θ, in radians is

$$\theta = \frac{s}{r}$$

When $s = r$, $\theta = 1$ rad

The relationship between radians and degrees

We can convert from radians to degrees or degrees to radians using the following relationship.

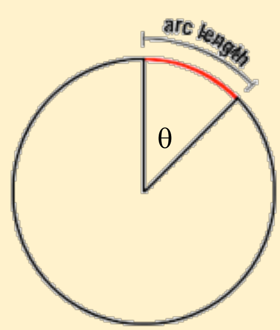
One whole turn = $360^0 = 2\pi$ radians

Radians to degrees	Degrees to radians
2π radians $\equiv 360°$	$360^0 = 2\pi$ radians
π radians $\equiv 180^0$	$1^0 = \frac{2\pi}{360}$ radians
1 radian $\equiv \frac{180^0}{\pi}$	$1^0 = \frac{\pi}{180}$ radians
θ radians $= \frac{\theta}{\pi} \times 180°$	$\alpha° = \frac{\alpha}{180°} \times \pi$ radians

It should be noted that 1 radian is approximately 57.3^0.

Arc length and area of the sector in radians

Using the relationship between radians and degrees we can develop new formulae



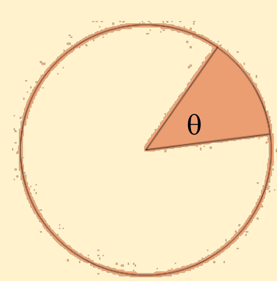
The length of an arc, s, of a circle whose circumference is $2\pi r$ when θ is measured in degrees is

$$s = \frac{\theta^0}{360^0} \times 2\pi r$$

When θ is in radians, $360^0 = 2\pi$

$$s = \frac{\theta}{2\pi} \times 2\pi r$$

$$s = r\theta$$



The area, A, of a sector of a circle, whose area is πr^2 when θ is measured in degrees is $\frac{1}{2} r^2 \theta$

Example 1

(a) Convert 60° to radians

(b) Convert $\dfrac{3\pi}{2}$ radians to degrees.

Solution

Part (a)

$$180° \equiv \pi \text{ radians}$$

$$\therefore 1° \equiv \frac{\pi}{180} \text{ radians}$$

$$\text{So, } 60° \equiv \frac{\pi}{180°} \times 60°$$

$$= \frac{\pi}{3} \text{ radians}$$

Part (b)

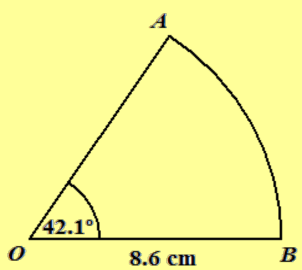
$$\pi \text{ radians} \equiv 180°$$

$$\therefore 1 \text{ radian} \equiv \frac{180°}{\pi}$$

$$\frac{3\pi}{2} \text{ radians} \equiv \frac{180°}{\pi} \times \frac{3\pi}{2}$$

$$= 270°$$

Example 2



The diagram shows a sector of a circle with radius 8.6 cm. If the angle of the sector measures 42.1°, determine the

i. length of the arc *AB*
ii. area of the sector *AOB*.

Solution

i. $42.1° \equiv 42.1^0 \times \dfrac{\pi}{180^0} = 0.735$ radians

Length of arc $= r\theta$, where r = radius and θ = angle in radians

$$\text{Length of arc } AB = (8.6 \times 0.735) \text{ cm}$$

$$= 6.32 \text{ cm (to 2 decimal places)}$$

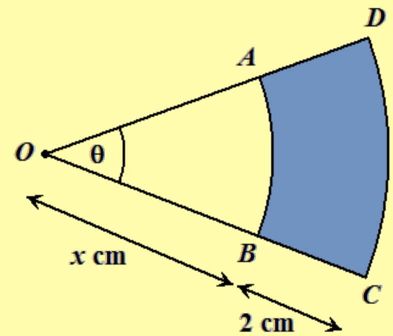
ii. The area of a sector is $\dfrac{1}{2} r^2 \theta$.

$$\text{The area of } AOB = \frac{1}{2}(8.6)^2(0.735)$$

$$= 27.18 \text{ cm}^2$$

Example 3

The following diagram (not drawn to scale) shows two sectors, *AOB* and *DOC*.
OB and *OC* are x cm and $(x + 2)$ cm respectively and angle $AOB = \theta$.



If $\theta = \dfrac{2\pi}{9}$ radians, calculate the area of the shaded region in terms of x

Solution

The area of the sector

$$AOB = \frac{1}{2}(x)^2\left(\frac{2\pi}{9}\right) \text{ cm}^2$$

$$= \frac{2\pi x^2}{2(9)} \text{ cm}^2$$

$$= \frac{\pi x^2}{9} \text{ cm}^2$$

The area of the sector

$$DOC = \frac{1}{2}(x+2)^2\left(\frac{2\pi}{9}\right) \text{ cm}^2$$

$$= \frac{\pi(x+2)^2}{9} \text{ cm}^2$$

Hence, the area of the shaded region

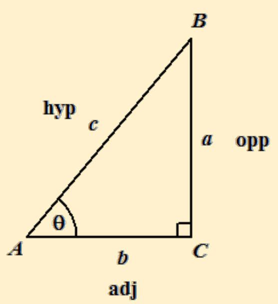
$$= \frac{\pi(x+2)^2}{9} - \frac{\pi x^2}{9}$$

$$= \frac{\pi}{9}\left((x+2)^2 - x^2\right) \text{ cm}^2$$

$$= \frac{\pi}{9}(4x+4) \text{ cm}^2$$

Trigonometric ratios

In the diagram shown below, triangle ABC is right-angled, with an acute angle $A = \theta°$. The side opposite the right angle, AB is called the hypotenuse. Relative to the angle, θ, the side opposite the angle θ, BC is called the opposite side and the third side, AC, is called the adjacent side. For any given right-angled triangle, the ratio of the sides is constant. These ratios are called trigonometric ratios. The three most commonly used ratios are defined below.



Note that opposite and adjacent sides are defined in relation to the given angle, θ.

However, the hypotenuse is always fixed.

The sine ratio (abbreviated sin)

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{a}{c}$$

The cosine ratio (abbreviated cos)

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c}$$

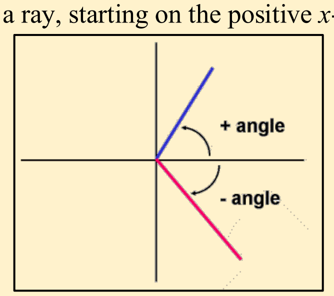
The tangent ratio (abbreviated tan)

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{a}{b}$$

Angles on the Cartesian Plane

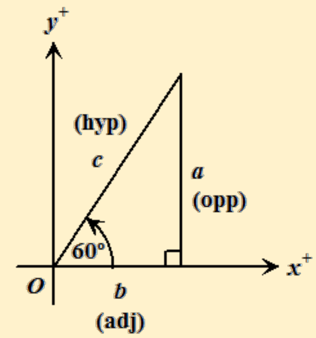
Angles can be positive or negative, depending on the direction of turn on the Cartesian plane, measured from the positive x-axis.

A positive angle is defined as an anticlockwise turn, about the origin of a ray, starting on the positive x-axis.
A negative angle is defined as a clockwise turn, about the origin of a ray, starting on the positive x-axis.



Trigonometric ratios for angles less than 90^0

Consider the right-angled triangle drawn in the first quadrant of the Cartesian plane.
Angles in this quadrant have measures that range from 0^0 to 90^0 and are acute. Using a calculator, we can obtain the values of these ratios for a given angle.



For example, for $\theta = 60°$.

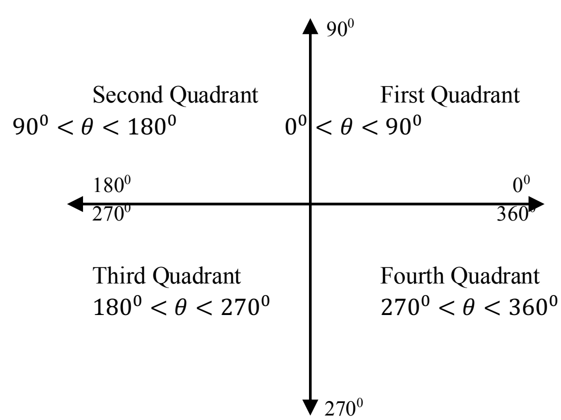
$$\sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{a}{c} \approx 0.866$$

$$\cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c} = 0.5$$

$$\tan 60° = \frac{\text{opp}}{\text{adj}} = \frac{a}{b} \approx 1.732$$

Trigonometric ratios for angles greater than 90^0

To determine the trigonometric ratios for angles greater than 90^0, we must first establish to which quadrant the angle belongs. Every angle, θ, regardless of its measure, is associated with a given quadrant, as shown below.



Basic Acute angle

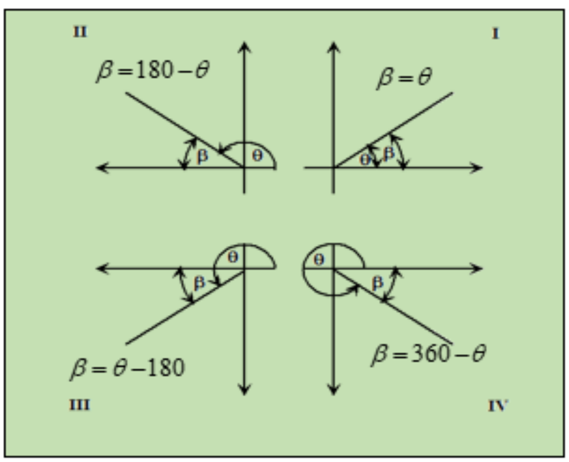
In order to determine the trigonometric ratios for angles in the second, third, and fourth quadrant, we must define the basic acute angle associated with each angle.

Basic Acute Angle

For any angle between 0 and 360 degrees, we define its **basic acute angle** as the acute angle formed between the *x*-axis and the terminal arm of the angle when rotated about the origin, in an anticlockwise direction.

For a given angle, θ, whose ratio we wish to determine, the basic acute angle, β is shown for the angle in each quadrant.

First quadrant- the basic acute angle, $\beta = \theta$.
Second quadrant- the basic acute angle, $\beta = 180^0 - \theta$.
Third quadrant- the basic acute angle, $\beta = \theta - 180^0$
Fourth quadrant- the basic acute angle, $\beta = 360^0 - \theta$.

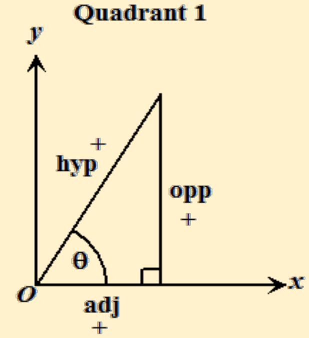


The signs of trigonometric ratios in each quadrant

The signs of the trigonometric ratio can be positive or negative depending on the quadrant in which the angle belongs. When determining the ratios, we consider the sign of the vertical and horizontal in the right-angled triangle.

We may think of the hypotenuse as the radius of a circle, equal to one unit in length. In this sense, the hypotenuse is always positive.

Quadrant 1



The terminal arm of the angle lies in the first quadrant. Hence, the basic acute angle, β is the same as is θ.

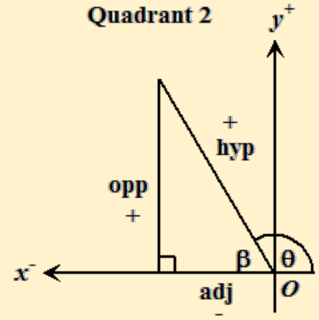
$$\sin\theta = \frac{+}{+} = +$$

$$\cos\theta = \frac{+}{+} = +$$

$$\tan\theta = \frac{+}{+} = +$$

All trigonometric ratios are positive in Quadrant 1.

Quadrant 2



The terminal arm of the angle lies in the second quadrant. Hence, the basic acute angle, β is

$$\beta = 180 - \theta.$$

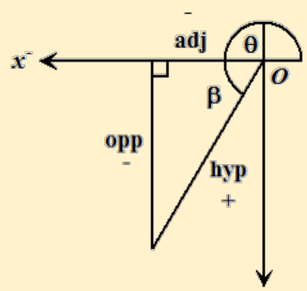
$$\sin\theta = \sin\beta = \frac{+}{+} = +$$

$$\cos\theta = \cos\beta = \frac{-}{+} = -$$

$$\tan\theta = \tan\beta = \frac{+}{-} = -$$

Only the sine ratio is positive in Quadrant 2.

Quadrant 3



The terminal arm of the angle lies in the third quadrant. Hence, the basic acute angle, β is

$$\beta = \theta - 180$$

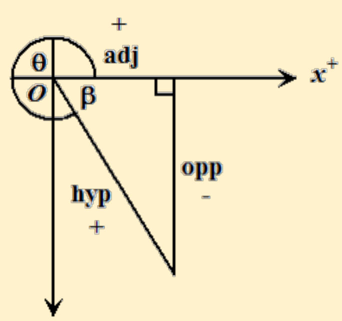
$$\sin\theta = \sin\beta = \frac{-}{+} = -$$

$$\cos\theta = \cos\beta = \frac{+}{-} = -$$

$$\tan\theta = \tan\beta = \frac{-}{-} = +$$

Only the tangent ratio is positive in Quadrant 3.

Quadrant 4



The terminal arm of the angle lies in the fourth quadrant. Hence, the basic acute angle, β is

$$\beta = 360 - \theta$$

$$\sin\theta = \sin\beta = \frac{-}{+} = -$$

$$\cos\theta = \cos\beta = \frac{+}{+} = +$$

$$\tan\theta = \tan\beta = \frac{-}{+} = -$$

Only the cosine ratio is positive in Quadrant 4.

Summary

sin +ve	All +ve
tan +ve	cos +ve

All trigonometric ratios are positive in the first quadrant.
In the second quadrant, only the sine ratio is positive.
In the third quadrant, only the tangent ratio is positive.
In the fourth quadrant, only the cosine ratio is positive.

Example 3

Determine the following trigonometric ratios
(i) sin 174^0 (ii) cos 236^0 (iii) tan 312^0

Solution

(i) sin 174^0

1. ***First, determine the quadrant in which the angle lies.***
   174^0 is an obtuse angle and lies in the second quadrant.
2. ***Determine the sign of the trigonometric ratio in this quadrant***
   The sine ratio is positive in the second quadrant.
3. ***Calculate the basic acute angle for this quadrant***
   In the second quadrant, the basic acute angle,
   $\beta = 180 - \theta$
   $\beta = 180^0 - 174^0 = 6^0$
4. ***Use the basic acute angle to evaluate the trigonometric ratio***.
   In this case, we use 6 degrees as our input.
   sin 174^0 = sin 6^0 = 0.1045 (by calculator).

(ii) cos 236^0

Using the four steps outlined above:

1. 236^0 lies in the third quadrant.
2. The cosine ratio is negative in the third quadrant.
3. In the third quadrant, the basic acute angle,
   $\beta = \theta - 180$
   $\beta = 236^0 - 180^0 = 56^0$
4. Therefore cos $236^0 = -$ cos $56^0 = -$ 0.5592 (by calculator)

(iii) $\tan 312^0$

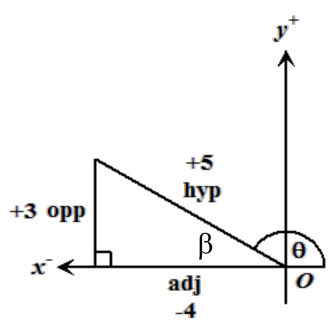
Using the four steps outlined above:

1. 312^0 lies in the fourth quadrant.
2. The tangent ratio is negative in the fourth quadrant.
3. In the fourth quadrant, the basic acute angle, $\beta = 360 - \theta$
   $\beta = 360^0 - 312^0 = 48^0$
4. Therefore $\tan 312^0 = -\tan 48^0 = -1.1106$ (by calculator)

Example 4

Given that $\sin\theta = \frac{3}{5}$ and θ is obtuse, find the exact value of $\cos\theta$.

Solution



Since θ is obtuse, it is in Quadrant 2.
By Pythagoras' Theorem

$$\text{adj} = \sqrt{(5)^2 - (3)^2}$$
$$= \pm 4$$

In Quadrant 2, adj $= -4$

$$\therefore \cos\theta = \cos\beta = \frac{-4}{+5} = -\frac{4}{5}$$

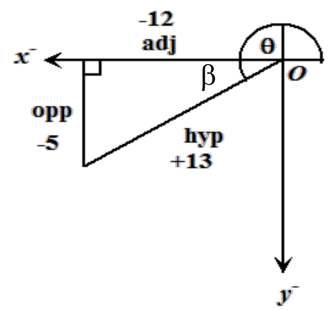
Example 5

Given that $\tan\theta = \frac{5}{12}$ and θ is reflex, calculate the exact value of $\sin\theta + \cos\theta$.

Solution

Since $\tan\theta$ is positive, and θ is reflex, it is in quadrant 3. In this quadrant, the opposite side and the adjacent side are negative.

$$tan\theta = tan\beta = \frac{Opp}{Adj} = \frac{-5}{-12}$$



Using Pythagoras' theorem, the hypotenuse

$$= \sqrt{(-12)^2 + (-5)^2} = 13$$

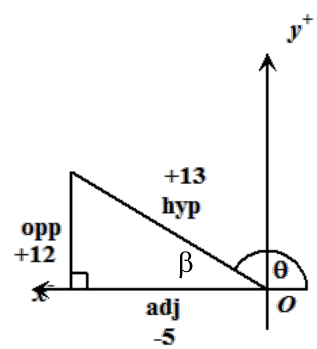
$$\sin\theta + \cos\theta = \sin\beta + \cos\beta = \frac{-5}{13} + \frac{-12}{13} = -\frac{17}{13}$$

Example 6

Given that $\sin\theta = \frac{12}{13}$ and $\frac{\pi}{2} < \theta < \pi$. Show that $\cos\theta = -\frac{5}{13}$.

Solution

Since θ lies in the second quadrant, we can sketch a diagram as shown.



$$\sin\theta = \sin\beta = \frac{12}{13},\ \frac{\pi}{2} < \theta < \pi$$

Using Pythagoras' theorem

$$\text{adj} = \sqrt{(13)^2 - (12)^2}$$
$$= \pm 5$$

In Quadrant 2, $\text{adj} = -5$

$$\therefore \cos\theta = \frac{-5}{+13} = -\frac{5}{13}$$

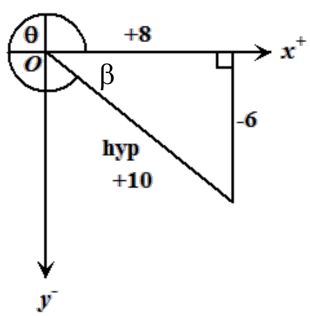
Example 7

Given that $\tan\theta = \frac{-6}{8}$ and θ is reflex, find the exact value of:

(i) $\sin\theta$ (ii) $\cos\theta$

Solution

$\tan\theta$ is negative and θ is reflex. Hence, θ lies in quadrant 4.



$$\tan\theta = \tan\beta = \frac{-6}{8}$$

Using Pythagoras' theorem

$$\text{hyp} = \sqrt{(8)^2 - (-6)^2}$$
$$= +10$$

i. $\sin\theta = \sin\beta = \frac{-6}{10} = -\frac{3}{5}$

ii. $\cos\theta = \cos\beta = \frac{+8}{+10} = \frac{4}{5}$

Trigonometric ratios for special angles

Sometimes we are required to express trigonometric ratios in an exact form. We know that many of these ratios are irrational numbers and when we use a calculator, we usually obtain the result as a decimal. For example, the value of sin 60^0 obtained from a calculator is sin 60^0 = 0.866025403… This is an irrational number and when expressed in decimal form, it is not exact. To obtain an exact value, we leave it in surd form.

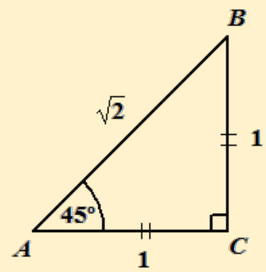
Trigonometric ratios for $\theta = 45°$

Consider the right-angled isosceles triangle with equal sides of unit length. (Any other chosen length would have produced the same results). Triangle *ABC* is an isosceles right-angled triangle, therefore

$$\hat{A} = \hat{B} = \frac{180° - 90°}{2} = 45° \text{ and}$$

$$AB = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

Trigonometric ratios for 45^0



The ratios are as follows;

$$\sin 45° = \frac{1}{\sqrt{2}} \quad \cos 45° = \frac{1}{\sqrt{2}} \quad \tan 45° = \frac{1}{1} = 1$$

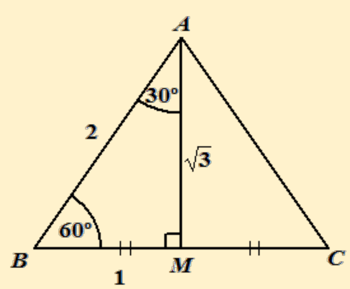
Trigonometric ratios for $\theta = 60^0$ and $\theta = 30^0$

Consider the equilateral triangle *ABC*, shown below, of side 2 units and *M* the midpoint of *BC*. In the right-angled triangle, ABM.

Angle ABM = 60^0, angle BAM = 30^0. BM = 1 unit and by Pythagoras' Theorem

$AM = \sqrt{(2)^2 - (1)^2} = \sqrt{3}$ We can now use triangle ABM to state the values of the trigonometric ratios for both $\theta = 60°$ and $\theta = 30^0$

Trigonometric ratios for 30^0 and 60^0



$$\sin 30° = \frac{1}{2} \quad \cos 30° = \frac{\sqrt{3}}{2} \quad \tan 30° = \frac{1}{\sqrt{3}}$$

$$\sin 60° = \frac{\sqrt{3}}{2} \quad \cos 60° = \frac{1}{2} \quad \tan 60° = \sqrt{3}$$

Summary

The table below presents the ratios for special angles.

Angle / Ratio	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\pm\infty$

The compound angle formula

When we combine two angles using the operations of addition or subtraction, we refer to their sum or difference as a compound angle. The compound angle formulae are shown below.

Compound Angle Formulae

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$
$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$
$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

We use the compound angle formulae to evaluate the exact value of certain angles. If an angle can be expressed as a sum of or difference between two special angles, we can state its value in surd form which is exact.

Example 8

Find the exact value of cos 75^0.

Solution

$$\cos 75^0=\cos(30^0+45^0)$$

Recall: The compound angle formula

$$\cos(30°+45°)=\cos 30°\cos 45°-\sin 30°\sin 45°$$

$$=\frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}}-\frac{1}{2}.\frac{1}{\sqrt{2}}=\frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}=\frac{\sqrt{3}-1}{2\sqrt{2}}\text{ (exact)}$$

Example 9

Find the exact value of $\tan 15°$, expressing the result in its simplest form.

Solution

$$\tan 15°\equiv\tan(45°-30°)$$

$$\tan 15°=\frac{\tan 45°-\tan 30°}{1+\tan 45°\tan 30°}$$

$$\tan 15°=\frac{1-\frac{1}{\sqrt{3}}}{1+1.\frac{1}{\sqrt{3}}}=\frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$\tan 15°=\frac{\sqrt{3}-1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}\quad\text{[Rationalising]}$$

$$\tan 15°=\frac{(\sqrt{3}-1)(\sqrt{3}-1)}{3-1}=\frac{3-2\sqrt{3}+1}{2}=2-\sqrt{3}$$

OR

$$\tan 15°\equiv\tan(60°-45°)$$

$$=\frac{\tan 60°-\tan 45°}{1+\tan 60°\tan 45°}$$

$$=\frac{\sqrt{3}-1}{1+(1)(\sqrt{3})}=\frac{\sqrt{3}-1}{1+\sqrt{3}}$$

$$=\frac{\sqrt{3}-1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}\quad\text{[Rationalising]}$$

$$=\frac{3-2\sqrt{3}+1}{3-1}=\frac{4-2\sqrt{3}}{2}=2-\sqrt{3}$$

Trigonometric identities

Trigonometric identities are equalities that involve trigonometric functions and are true for any value of the variable. They are extremely useful when solving problems in trigonometry.

In mathematics, identities are defined as an equality that is true for all values of the variable. An equation, on the other hand, is true for only a few values of the variable.

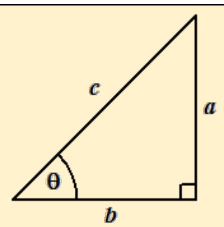
For example, the equation $x+5=7$ is true for $x=2$, while the equation $x^2-4=0$ is true for $x=2$ and for $x=-2$.

On the other hand, the identity $x^2-4=(x+2)(x-2)$ is true for any value of x.

We will now examine two trigonometric identities. The proof of these identities will be derived by considering a right-angled triangle with sides a, b and c and an acute angle, θ

Identity 1

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

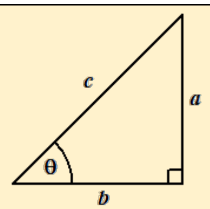


$$\sin\theta = \frac{a}{c} \quad \text{and} \quad \cos\theta = \frac{b}{c}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{c} \times \frac{c}{b} = \frac{a}{b} = \tan\theta$$

Identity 2

$$\sin^2\theta + \cos^2\theta = 1$$



$$\sin \quad = \frac{a}{c} \quad \text{and} \quad \cos \quad = \frac{b}{c}$$

$$\sin^2\theta + \cos^2\theta = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$$

$$\sin^2\theta + \cos^2\theta = \frac{a^2}{c^2} + \frac{b^2}{c^2}$$

$$\sin^2\theta + \cos^2\theta = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1$$

Trigonometric Identities- Summary

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

Double angle formulae

These are derived from the compound angle formulae by substituting $A = B$.

Recall $\sin(A+B) = \sin A\cos B + \cos A\sin B$.

If $A = B$, then $\sin(A+A) = \sin A\cos A + \cos A\sin A$

$$\sin 2A = 2\sin A\cos A$$

Recall $\cos(A+B) = \cos A\cos B - \sin A\sin B$.

If $A = B$, then $\cos(A+A) = \cos A\cos A - \sin A\sin A$

$$\cos 2A = \cos^2 A - \sin^2 A$$

Since $\sin^2 A + \cos^2 A = 1$, we can express $\cos 2A$ in two other forms:

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= (1-\sin^2 A) - \sin^2 A \\ &= 1 - 2\sin^2 A\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= \cos^2 A - (1-\cos^2 A) \\ &= 2\cos^2 A - 1\end{aligned}$$

Recall $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ and $A = B$, then

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A\tan A}$$

And so $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Double angle formulae

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \cos^2 A - \sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

Example 10

Express $\tan 22\frac{1}{2}°$ in exact form.

Solution

Recall: $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ Let $A = 45°$

$$\therefore \tan 45° = \frac{2\tan 22\frac{1}{2}°}{1-\tan^2 22\frac{1}{2}°}$$

Let $t = \tan 22\frac{1}{2}$ and substitute $\tan 45^0 = 1$

$$\therefore 1 = \frac{2t}{1-t^2}$$

$$1 - t^2 = 2t$$

$$t^2 + 2t - 1 = 0$$

$$t=\frac{-2\pm\sqrt{(2)^2-4(1)(-1)}}{2(1)}$$

$$t=\frac{-2\pm\sqrt{8}}{2}=\frac{-2\pm2\sqrt{2}}{2}$$

$$t=-1\pm\sqrt{2}=-1+\sqrt{2},-1-\sqrt{2}$$

$\tan 22\frac{1}{2}°$ is positive, so we reject $-1-\sqrt{2}$

$\therefore \tan 22\frac{1}{2}°=\sqrt{2}-1$

Proofs in trigonometry

Sometimes we are required to prove that a given identity is true. In so doing, we can use any of the three basic identities or the compound/double angle formulae. The following techniques are useful in solving problems on proofs:

1. Choose one side (L.H.S or R.H.S.) and prove it is the same as the other.
2. Prove that the L.H.S and the R.H.S are equal to the same expression.

Example 11

Prove that $\frac{\cos x}{\sin x}+\frac{\sin x}{\cos x}\equiv\frac{2}{\sin 2x}$.

Solution

L.H.S

$$\frac{\cos x}{\sin x}+\frac{\sin x}{\cos x}=\frac{\cos^2 x+\sin^2 x}{\sin x\cos x}$$

[Sub $\sin^2 x+\cos^2 x=1$]

$$=\frac{1}{\sin x\cos x}=\frac{2}{2\sin x\cos x}=\frac{2}{\sin 2x}$$

[Double angle formula]

Example 12

Prove that $\tan x+\frac{1}{\tan x}=\frac{2}{\sin 2x}$.

Solution

Take the L.H.S and substitute $\tan x=\frac{\sin x}{\cos x}$

$$\text{L.H.S.}=\tan x+\frac{1}{\tan x}=\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}$$

$$=\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}$$

Recall: $\sin^2 x+\cos^2 x=1$

$$\text{L.H.S.}=\frac{1}{\sin x\cos x}=\frac{2}{2\sin x\cos x}=\frac{2}{\sin 2x}=\text{R.H.S.}$$

Example 12

Prove that

$$\frac{\sin(\theta+\alpha)}{\cos\theta\cos\alpha}\equiv\tan\theta+\tan\alpha$$

Solution

Consider the left-hand side (LHS)

$$\frac{\sin(\theta+\alpha)}{\cos\theta\cos\alpha}=\frac{\sin\theta\cos\alpha+\cos\theta\sin\alpha}{\cos\theta\cos\alpha}$$

$$=\frac{\sin\theta\cos\alpha}{\cos\theta\cos\alpha}+\frac{\cos\theta\sin\alpha}{\cos\theta\cos\alpha}$$

$$=\frac{\sin\theta}{\cos\theta}+\frac{\sin\alpha}{\cos\alpha}$$

$$=\tan\theta+\tan\alpha$$

$$=\text{R.H.S.}$$

Example 13

Given that $cos30^0=\frac{\sqrt{3}}{2}$ and $sin45^0=\frac{\sqrt{2}}{2}$, without the use of a calculator, evaluate cos 105^0, in surd form, giving your answer in the simplest terms.

Solution

By double angle formula:

$$cos\ 60^0=2cos^2 30^0-1=2\left(\frac{\sqrt{3}}{2}\right)^2-1=\frac{1}{2}$$

$$cos45^0=sin45^0=\frac{\sqrt{2}}{2}$$

$$\cos 105°=\cos(60°+45°)$$

$$=\cos 60°\cos 45°-\sin 60°\sin 45°$$

$$=\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$=\frac{\sqrt{2}-\sqrt{3}\sqrt{2}}{4}$$

$$=\frac{\sqrt{2}(1-\sqrt{3})}{4}$$

$$=\frac{\sqrt{2}(1-\sqrt{3})}{2\sqrt{2}\sqrt{2}}$$

$$=\frac{1-\sqrt{3}}{2\sqrt{2}}$$