FAS-PASS Maths

11: THE REMAINDER AND FACTOR THEOREM

Solving and simplifying polynomials

In our study of quadratics, one of the methods used to simplify and solve was factorisation. For example, we may solve for x in the following equation as follows:

$x^2+5x+6=0$

$(x+3)(x+2)=0$

$x+3=0, \Rightarrow x=-3$

$x+2=0 \Rightarrow x=-2$

Hence, $x = -3$ or -2 are solutions or roots of the quadratic equation.

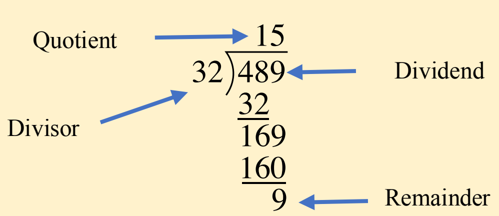
A more general name for a quadratic is a polynomial of degree 2, since the highest power of the unknown is two. The method of factorisation worked for quadratics whose solutions are integers or rational numbers.

For a polynomial of order 3, such as $x^3+4x^2+x-6=0$ the method of factorisation may also be applied. However, obtaining the factors is not as simple as it was for quadratics. We would likely have to write down three linear factors, which may prove difficult. In this section, we will learn to use the remainder and factor theorems to factorise and to solve polynomials that are of degree higher than 2.

Before doing so, let us review the meaning of basic terms in division.

Terms in division

We are familiar with division in arithmetic.



The number that is to be divided is called the ***dividend***. The dividend is divided by the ***divisor***. The result is the ***quotient*** and the ***remainder*** is what is left over.

From the above example, we can deduce that:

489	= (15	× 32)	+ 9
↑	↑	↑	↑
Dividend	Quotient	Divisor	Remainder

Thus, from arithmetic, we know that we can express a dividend as:

Dividend = (Quotient × Divisor) + Remainder

When there is no remainder, $\boldsymbol{R} = 0$, and the divisor is now a factor of the number, so

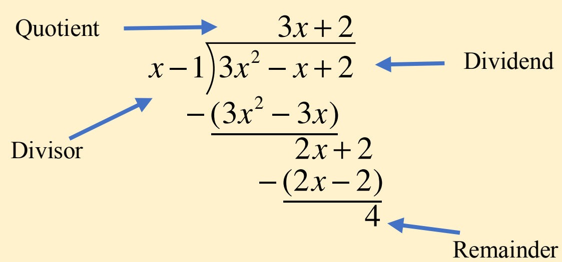
Dividend = Quotient × Factor

The process we followed in arithmetic when dividing is very similar to what is to be done in algebra. Examine the following division problems in algebra and note the similarities.

Division of a polynomial by a linear expression

We can apply the same principles in arithmetic to dividing algebraic expressions. Let the quadratic function $f(x)$ represent the dividend, and $(x-1)$ the divisor, where

$$f(x) = 3x^2 - x + 2$$



From the above example, we can deduce that:

$3x^2 - x + 2$	$= (3x+2)$	$(x-1)$	$+ 4$
↑	↑	↑	↑
Dividend	Quotient	Divisor	Remainder

Consider $(x-1) = 0$,

$$f(x) = (3x+2) \times 0 + 4 = 4$$

But, $(x-1) = 0$ implies that $x = 1$

Therefore, when $x = 1$, $f(x) = 4$ or $f(1) = 4$.

We can conclude that when the polynomial

$$3x^2 - x + 2$$

is divided by $(x-1)$, the remainder is

$$f(1) = 4$$

We shall now perform division using a cubic polynomial as our dividend.

$$
\begin{array}{r}
x^2+14x+25 \\
x-2\overline{)\,x^3+12x^2-3x+4} \\
-\underline{(x^3-2x^2)} \\
14x^2-3x+4 \\
-\underline{(14x-28x)} \\
25x+4 \\
\underline{-(25x-50)} \\
54
\end{array}
$$

From the above example, we can deduce that:

$x^3+12x^2-3x+4=(x^2+14x+25)(x-2)+54$

↑ Dividend ↑ Quotient ↑ Divisor ↑ Remainder

Consider $(x-2)=0$,

$f(x)=(x^2+14x+25)\times 0 \ + \ 54=54$

But, $(x-2)=0$ implies that $x=2$

Therefore, when $x=2$, $f(x)=54$ or $f(2)=54$.

We can conclude that when the polynomial

x^3+12x^2-3x+4

is divided by $(x-2)$, the remainder is

$f(2)=54$

Now consider another example of a cubic polynomial divided by a linear divisor.

$$
\begin{array}{r}
2x^2+7x+18 \\
x+2\overline{)\,2x^3-3x^2+4x+5} \\
-\underline{(2x^3+4x^2)} \\
-7x^2+4x+5 \\
-\underline{(7x^2-14x)} \\
18x+5 \\
\underline{-(18x+36)} \\
-31
\end{array}
$$

From the above example, we can deduce that:

$2x^3-3x^2+4x+5=(2x^2+x+18)(x+2)-31$

↑ Dividend ↑ Quotient ↑ Divisor ↑ Remainder

Consider $(x+2)=0$,

$f(x)=(2x^2+x+18)\times 0 \ - \ 31=-31$

But, $(x+2)=0$ implies that $x=-2$

Therefore, when $x=-2$, $f(x)=-31$ or $f(-2)=-31$.

We can conclude that when the polynomial

$2x^3-3x^2+4x+5$

is divided by $(x+2)$, the remainder is

$f(-2)=-31$

The Remainder Theorem for divisor $(x-a)$

From the above examples, we saw that a polynomial can be expressed as a product of the quotient and the divisor plus the remainder:

$3x^2-x+2=(3x+2)(x-1)+4$

$x^3+12x^2-3x+4=(x^2+14x+25)(x-2)+54$

$2x^3-3x^2+4x+5=(2x^2+x+18)(x+2)-31$

We can now formulate the following expression where, $f(x)$ is a polynomial whose quotient is $Q(x)$ and whose remainder is R when divided by $(x-a)$.

$f(x)=Q(x)\times(x-a)+R$

If we were to substitute $x=a$ in the above expression, then our result will be equal to R, the remainder when the divisor, $(x-a)$ is divided by the polynomial, $f(x)$.

We are now able to state the remainder theorem.

The Remainder Theorem
If $f(x)$ is any polynomial and $f(x)$ is divided by $(x-a)$, then the remainder is $f(a)$.

The validity of this theorem can be tested in any of the equations above, for example:

1. When $3x^2-x+2$ was divided by $(x-1)$, the remainder was 4.

According to the remainder theorem, the remainder can be computed by substituting $x=1$ in $f(x)$

$f(x)=3x^2-x+2$

$f(1)=3(1)^2-1+2$

$f(1)=4$

2. When x^3+12x^2-3x+4 was divided by $(x-2)$, the remainder was 54.

According to the remainder theorem, the remainder can be computed by substituting $x=2$ in $f(x)$

$f(x)=x^3+12x^2-3x+4$

$f(2)=(2)^3+12(2)^2-3(2)+4$

$f(2)=8+48-6+4$

$f(2)=54$

3. When $2x^3 - 3x^2 + 4x + 5$ was divided by $(x + 2)$ the remainder was -31.

According to the remainder theorem, the remainder can be computed by substituting $x = -2$ in $f(x)$

$f(x) = 2x^3 - 3x^2 + 4x + 5$
$f(-2) = 2(-2)^3 - 3(-2)^2 + 4(-2) + 5$
$f(-2) = -16 - 12 - 8 + 5$
$f(-2) = -31$

Example 1

Find the remainder when $f(x) = 3x^3 + x^2 - 4x - 1$ is divided by $(x - 2)$.

Solution

By the Remainder Theorem, the remainder is $f(2)$.
$f(x) = 3x^3 + x^2 - 4x - 1$
$f(2) = 3(2)^3 + (2)^2 - 4(2) - 1$
$f(2) = 24 + 4 - 8 - 1$
$f(2) = 19$
Hence, the remainder is 19

The Factor Theorem for divisor $(x - a)$

Now, consider the following examples when there is no remainder.

$3x+2$ $x-1\overline{)3x^2-x-2}$ $-(3x^2-3x)$ $2x-2$ $-(2x-2)$ 0	$8x^2-2x-3$ $x-1\overline{)8x^3-10x^2-x+3}$ $-(8x^3-8x^2)$ $-2x^2-x+3$ $-(-2x^2+2x)$ $-3x+3$ $-(3x+3)$ 0

We can express the dividend as a product of the divisor and the quotient only, since $R = 0$.

$$f(x) = 3x^2 - x - 2 = (3x + 2)(x - 1)$$

$$f(x) = 8x^3 - 10x^2 - x + 3 = (8x^2 - 2x - 3)(x - 1)$$

We can now formulate the following expression where $f(x)$ is a polynomial, $Q(x)$ is the quotient and $(x - a)$ is a **factor** of the polynomial.

$$f(x) = Q(x) \times (x - a)$$

The above rule is called the Factor Theorem, it is a special case of the Remainder Theorem, when $R = 0$. The validity of this theorem can be tested by substituting $x = 1$ in each of the above functions.

$f(x) = 3x^2 - x - 2$ $f(1) = 3(1)^2 - 1 - 2$ $f(1) = 0$	$f(x) = 8x^3 - 10x^2 - x + 3$ $f(1) = 8(1)^3 - 10(1)^2 - 1 + 3$ $f(1) = 0$

In the above examples, when we let $(x - 1) = 0$, or $x = 1$, $f(x) = 0$ because the remainder, $R = 0$.

The Factor Theorem
If $f(x)$ is any polynomial and $f(x)$ is divided by $(x-a)$, and the remainder $f(a) = 0$ then $(x-a)$ is a factor of $f(x)$

Example 2

Show that $(x - 2)$ is a factor of $f(x) = 3x^3 + x^2 - 14x$

Solution
By the Factor Theorem, if $(x - 2)$ is a factor of the remainder is zero. We now compute the remainder, $f(2)$.
$f(x) = 3x^3 + x^2 - 14x$
$f(2) = 3(2)^3 + (2)^2 - 14(2)$
$f(2) = 24 + 4 - 28$
$f(2) = 0$

Hence, $(x - 2)$ is a factor of $f(x)$.

The Remainder and Factor Theorem for divisor $(ax + b)$

When the divisor is not in the form, $(x-a)$, but in the general linear form $(ax + b)$, the remainder can no longer be $f(a)$. This is because the coefficient of x is not equal to one.

Consider the following example, where the divisor is of the form, $(ax + b)$.

Let $(2x + 3)$ be a divisor of
$$f(x) = 2x^3 + 7x^2 + 2x + 9$$
We perform the division as shown below and note that the remainder is 15.

$$\begin{array}{r} x^2+2x-2 \\ 2x+3\overline{)2x^3+7x^2+2x+9} \\ -(2x^3+3x^2) \\ 4x^2+2x+9 \\ -(4x^2+6x) \\ -4x+9 \\ -(4x-6) \\ 15 \end{array}$$

We can deduce that:

$2x^3 + 7x^2 + 2x + 9 = (x^2 + 2x - 2)(2x + 3) + 15$

Consider $(2x + 3) = 0$,

$$f(x) = (x^2 + 2x - 2) \times 0 + 15 = 15$$

But, $(2x + 3) = 0$ implies that $x = -\frac{3}{2}$.

Therefore, when $x = -\frac{3}{2}$, $f(x) = 15$ or $f\left(-\frac{3}{2}\right) = 15$.

We can conclude that when the polynomial $2x^3 + 7x^2 + 2x + 9$ is divided by $(2x + 3)$, the remainder is $f\left(-\frac{3}{2}\right) = 15$

Now consider the example below.

$$\begin{array}{r} 3x^2+6x-1 \\ 3x-1\overline{)9x^3+15x^2-9x+1} \\ -(9x^3-3x^2) \\ 18x^2-9x+1 \\ -(18x^2-6x) \\ -3x+1 \\ -(-3x+1) \\ 0 \end{array}$$

We can deduce that:

$9x^3 + 15x^2 - 9x + 1 = (3x^2 + 6x - 1)(3x - 1)$

Consider $(3x - 1) = 0$,

$$f(x) = (3x^2 + 6x - 1) \times 0 = 0$$

But, $(3x - 1) = 0$ implies that $x = \frac{1}{3}$.

Therefore, when $x = \frac{1}{3}$, $f(x) = 0$ or $f\left(\frac{1}{3}\right) = 0$.

We can conclude that when the polynomial $9x^3 + 15x^2 - 9x + 1$ is divided by $(3x - 1)$, the remainder is $f\left(\frac{1}{3}\right) = 0$.

So, $(3x - 1)$ is a factor of $9x^3 + 15x^2 - 9x + 1$

From the above examples, we can formulate the following expression:

$$f(x) = Q(x) \times (ax + b) + R$$

where, $f(x)$ is a polynomial whose quotient is $Q(x)$ and the remainder is R when divided by $(ax + b)$.

By setting $(ax + b) = 0$, the above polynomial will become

$$f(x) = R$$

When $(ax + b) = 0$, $x = -\frac{b}{a}$.

Now, since

$f(x) = R$, when $x = -\frac{b}{a}$, we conclude that

$$f\left(-\frac{b}{a}\right) = R$$

We are now in a position to restate the remainder theorem when the divisor is of the form $(ax+b)$.

The Remainder Theorem

If $f(x)$ is any polynomial and $f(x)$ is divided by $(ax+b)$ then the remainder is $f\left(-\frac{b}{a}\right)$.

If $f\left(-\frac{b}{a}\right) = 0$, then $(ax+b)$ is a factor of $f(x)$.

We apply the Remainder Theorem to obtain the remainder when $f(x) = 2x^3 + 7x^2 + 2x + 9$ was divided by $(2x + 3)$.

By the Remainder Theorem, the remainder is $f\left(-\frac{3}{2}\right)$.

$$f\left(-\frac{3}{2}\right) = 2\left(-\frac{3}{2}\right)^3 + 7\left(-\frac{3}{2}\right)^2 + 2\left(-\frac{3}{2}\right) + 9$$

$$f\left(-\frac{3}{2}\right) = -\frac{27}{4} + \frac{63}{4} - 3 + 9$$

$$f\left(-\frac{3}{2}\right) = 15$$

We can apply the Factor Theorem to show that $(3x - 1)$ is a factor of $f(x)$, where

$$f(x) = 9x^3 + 15x^2 - 9x + 1$$

By the Factor Theorem $(3x - 1)$ is a factor of $f(x)$. if $f\left(\frac{1}{3}\right) = 0$,

$$f(x) = 9x^3 + 15x^2 - 9x + 1$$

$$f\left(\frac{1}{3}\right) = 9\left(\frac{1}{3}\right)^3 + 15\left(\frac{1}{3}\right)^2 - 9\left(\frac{1}{3}\right) + 1$$

$$f\left(\frac{1}{3}\right) = \frac{1}{3} + \frac{5}{3} - 3 + 1 = 0$$

Hence, $(3x - 1)$ is a factor of $f(x)$.

Instead of performing long division, we can apply the remainder theorem to find the remainder when a polynomial is divided by a linear expression of the form $(ax + b)$. The remainder, $R = f\left(-\frac{b}{a}\right)$ when the polynomial is divided by the linear factor.

We can use the factor theorem to show that a linear expression of the form ($ax + b$) is a factor of a polynomial. In this case, we show that the remainder $R = f\left(-\frac{b}{a}\right)$ is zero when the polynomial is divided by the linear factor.

Example 3

Find the remainder when $f(x) = 4x^3 + 2x^2 - 3x + 1$ is divided by $(2x-1)$.

Solution

In this example $(2x-1)$ is of the form $(ax+b)$, where $a = 2$ and $b = -1$

The remainder would be

$$f\left(\frac{-(-1)}{2}\right) = f\left(\frac{1}{2}\right).$$

$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 + 2\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) + 1$$

$$= \frac{1}{2} + \frac{1}{2} - 1\frac{1}{2} + 1 = \frac{1}{2}$$

Alternatively, we could have used long division to show that the remainder is one half.

$$\begin{array}{r} 2x^2 + 2x - \frac{1}{2} \\ 2x-1\overline{)\,4x^3 + 2x^2 - 3x - 1} \\ -\ \underline{4x^3 - 2x^2} \\ 4x^2 - 3x \\ -\ \underline{4x^2 - 2x} \\ -x + 1 \\ -\ \underline{-x + \frac{1}{2}} \\ \underline{\frac{1}{2}} \end{array}$$

Example 4

State the quotient and the remainder when $6x^3 - 8x + 5$ is divided by by $2x - 4$

Solution

In this example, the dividend has no terms in x^2. It is advisable to add on such a term to maintain consistency between the quotient and the divisor. This is done by inserting a term in x^2 as shown below.

$$\begin{array}{r} 3x^2 + 6x + 8 \\ 2x-4\overline{)\,6x^3 + 0x^2 - 8x + 5} \\ -\underline{(6x^3 - 12x^2)} \\ 12x^2 - 8x + 5 \\ -\underline{(12x^2 - 24x)} \\ 16x + 5 \\ -\underline{(16x - 32)} \\ 37 \end{array}$$

The quotient is $3x^2 + 6x + 8$

The Remainder is 37

Factorising and Solving Polynomials

We can use the factor theorem to factorise or solve a polynomial. However, to factorise a polynomial of the form $ax^3 + bx^2 + cx + d$ it would be helpful to know one linear factor. Then we can obtain the other factors by the process of long division.

Example 5

Show that ($2x + 3$) is a factor of $f(x) = 2x^3 + 3x^2 - 2x - 3$.

Solution

When $f(x)$ is divided by $(2x+3)$, the remainder is

$$f(-\tfrac{3}{2}) = 2(-\tfrac{3}{2})^3 + 3(-\tfrac{3}{2})^2 - 2(-\tfrac{3}{2}) - 3 = 0$$

Hence, $(2x+3)$ is a factor of $f(x)$ since the remainder is 0.

Example 6

Factorise, $f(x) = 2x^3 + 3x^2 - 29x - 60$ and hence solve $f(x) = 0$.

Solution

Let $f(x) = 2x^3 + 3x^2 - 29x - 60$.

To obtain the first factor, we use the remainder theorem to test for $f(1)$, $f(-1)$ and so on, until we obtain a remainder of zero.

We found that,

$$f(4) = 2(4)^3 + 3(4)^2 - 29(4) - 60 = 128 + 48 - 116 - 6$$

$f(4) = 0$

Therefore ($x - 4$) is a factor of $f(x)$.

Now that we have found a first factor, we divide $f(x)$ by ($x - 4$).

$$
\begin{array}{r}
2x^2+11x+15 \\
x-4\overline{)2x^3+3x^2-29x-60} \\
-\underline{2x^3-8x^2} \\
11x^2-29x \\
-\underline{11x^2-44x} \\
15x-60 \\
-\underline{15x-60} \\
0
\end{array}
$$

$2x^2+11x+15=(x+3)(2x+5)$

$\therefore f(x)=(2x+5)(x+3)(x-4)$

If $f(x)=0$, then $(2x+5)(x+3)(x-4)=0$.

So $x=-3, 2\frac{1}{2}$ and 4.

Example 7

Factorise $x^3-x^2+4x-12$ and hence show that $x^3-x^2+4x-12=0$ has only one solution.

Solution

To get the first factor, we apply the remainder theorem as follows:

$f(1)=(1)^3-(1)^2+4(1)-12\neq 0$

$f(-1)=(-1)^3-(-1)^2+4(-1)-12\neq 0$

$f(2)=(2)^3-(2)^2+4(2)-12=0$

Hence, $(x-2)$ is a factor of $f(x)$.

Dividing by the factor $(x-2)$

$$
\begin{array}{r}
x^2+x+6 \\
x-2\overline{)x^3-x^2+4x-12} \\
-\underline{x^3-2x^2} \\
x^2+4x-12 \\
-\underline{x^2-2x} \\
6x-12 \\
-\underline{6x-12} \\
0
\end{array}
$$

$x^3-x^2+4x-12$

$=(x-2)(x^2+x+6)$

$\Rightarrow (x-2)=0$ or $(x^2+x+6)=0$

$\Rightarrow x=2$

The quadratic x^2+x+6 has no real roots because the discriminant

$b^2-4ac=(1)^2-4(1)(6)=1-25=-24<0$

Hence, $x^3-x^2+4x-12=0$ has only one solution, $x=2$

Example 8

If $(x+3)$ and $(x-4)$ are both factors of $2x^3+3x^2-29x-60$, find the third factor.

Solution

Let the third factor be $(ax+b)$.
We can write the expression as a product of three linear factors as shown.

$2x^3+3x^2-29x-60=(x+3)(x-4)(ax+b)$

$x\times x\times ax=2x^3$

$\therefore a=2$

And

$3\times -4\times b=-60$

$\therefore b=5$

Hence, $(ax+b)$, the third factor, is $(2x+5)$.

$(x+3)(x-4)=x^2-x-12$

Alternatively, we may divide $2x^3+3x^2-29x-60$ by x^2-x-12.

$$
\begin{array}{r}
2x+5 \\
x^2-x-12\overline{)2x^3+3x^2-29x-60} \\
-\underline{2x^3-2x^2-24x} \\
5x^2-5x-60 \\
-\underline{5x^2-5x-60} \\
0
\end{array}
$$

$\therefore$ The third factor is $(2x+5)$.

Example 9

Solve for x in $x^3-2x^2-7x+12=0$, giving the answer to 2 decimal places where necessary.

Solution

Recall: If $f(x)$ is any polynomial and $f(x)$ is divided by $(x-a)$, then the remainder is $f(a)$.

If the remainder $f(a)=0$, then $(x-a)$ is a factor of $f(x)$.

First test to see if $(x-1)$ is a factor.

Let $f(x)=x^3-2x^2-7x+12$

$f(1)=(1)^3-2(1)^2-7(1)+12\neq 0$

$\therefore (x-1)$ is not a factor of $f(x)$.

Next, test to see if $(x+1)$ is a factor.

$f(-1) = (-1)^3 - 2(-1)^2 - 7(-1) + 12 \neq 0$
$\therefore (x+1)$ is not a factor of $f(x)$.

Now, try $(x-2)$
$f(2) = (2)^3 - 2(2)^2 - 7(2) + 12 \neq 0$
$\therefore (x-2)$ is not a factor of $f(x)$.

Try $(x+2)$
$f(-2) = (-2)^3 - 2(-2)^2 - 7(-2) + 12 \neq 0$
$\therefore (x+2)$ is not a factor of $f(x)$.

Try $(x-3)$
$f(3) = (3)^3 - 2(3)^2 - 7(3) + 12 = 0$
$\therefore (x-3)$ is a factor of $f(x)$.

Now we divide $f(x)$ by $(x-3)$.

$$
\begin{array}{r}
x^2 + x - 4 \\
x-3 \overline{\big) x^3 - 2x^2 - 7x + 12} \\
- \underline{x^3 - 3x^2} \qquad\qquad \\
x^2 - 7x + 12 \\
- \underline{x^2 - 3x} \qquad \\
-4x + 12 \\
- \underline{-4x + 12} \\
0 \\
\end{array}
$$

Since we were asked to give our answers correct to 2 decimal places, we deduce that $x^2 + x - 4 = 0$ would not have exact roots. We, therefore, employ the quadratic equation formula to find the roots.

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

where $a = 1, b = 1$ and $c = -4$.

$$
\begin{aligned}
x &= \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(-4)}}{2(1)} \\
&= \frac{-1 \pm \sqrt{17}}{2} \\
&= 1.561 \text{ or } -2.561 \\
&= 1.56 \text{ or } -2.56 \text{ (correct to 2 decimal places)}
\end{aligned}
$$

Hence, $x = 3, 1.56$ or -2.56.