

## 9: LOGARITHMS

### Introducing Logarithms

Many years ago, complex computations had to be done without a calculator and so it was necessary to find a strategy to simplify such tasks. Logarithms (often shortened to logs) were first used to simplify computations involving very large and very small numbers. We shall not offer a full explanation of how logarithms facilitated computation here. But, to illustrate the principle, consider the following:

$$100\,000 \times 10\,000 = 10^5 \times 10^4 = 10^9$$

$$100\,000 \div 10\,000 = 10^5 \div 10^4 = 10$$

We know from the laws of indices that in performing the **multiplication** we can **add** the indices, and in performing **division**, we **subtract** indices. Applying these laws, multiplication was performed using the addition of indices and division was performed using subtraction of indices.

To compute tedious calculations, we would use logs to convert the numbers to index form. For example, the number 234586 in log form would be  $10^{5.3703029}$ . (we may express it at the required level of accuracy). Then, we would add the indices when multiplying and subtract the indices when dividing. The accuracy of the answers depended on the number of decimal places we chose when the numbers were converted to logs.

In the following computations, when the numbers were expressed in log form, the indices were rounded to two decimal places and so, answers would be approximations and not exact.

|                                |                                    |
|--------------------------------|------------------------------------|
| $234\,586 \times 525$          | $0.5614 \div 0.4365$               |
| $= 10^{5.37} \times 10^{2.72}$ | $= 10^{-0.25} \div 10^{-0.36}$     |
| $= 10^{5.37+2.72} = 10^{8.09}$ | $= 10^{-0.25-(-0.36)} = 10^{0.11}$ |
| $= 123\,026\,877$              | $= 1.2882$                         |

### Defining the logarithm of a number

Any number can be expressed in log form provided we know the base. The base can be any number. When a number is expressed in index form, the base is defined as follows:

$$1000 = 10^3, \text{ the base is } 10$$

$$16 = 2^4, \text{ the base is } 2.$$

$$125 = 5^3, \text{ the base is } 5.$$

### Index and log Form

When numbers are written in index form, it is easy to interpret their logs. The table below illustrates how this is done.

| Index Form                 | Log Form                                  | Interpretation                                    |
|----------------------------|---|---|
| $10^2 = 100$               | $\log_{10} 100 = 2$                       | The log of 100 to the base 10 is 2.               |
| $2^5 = 32$                 | $\log_2 32 = 5$                           | The log of 32 to the base 2 is 5.                 |
| $5^3 = 125$                | $\log_5 125 = 3$                          | The log of 125 to the base 5 is 3.                |
| $2^{-4} = \frac{1}{16}$    | $\log_2 \left( \frac{1}{16} \right) = -4$ | The log of $\frac{1}{16}$ to the base 2 is -4.    |
| $10^{-3} = \frac{1}{1000}$ | $\log_{10} \frac{1}{1000} = -3$           | The log of $\frac{1}{1000}$ to the base 10 is -3. |
| $b^a = N$                  | $\log_b N = a$                            | The log of N to the base b is a.                  |

### Definition

The log of a number is defined as the power to which its base is raised to give the number. We write:

$$\log_b N = a$$

where  $b$  is the base,  $N$  is the number and  $a$  is the log of the number. Hence, by definition,  $b^a = N$ .

We may think of the log (short for logarithmic) function as a mapping of numbers,  $N$ , onto their powers  $a$ , for a given base,  $b$ .

### Properties of logarithms

The following properties hold for all logarithmic functions.

1. We can only find the log of a positive number, so  $N > 0$ . This is because negative numbers cannot be written in index form.
2. The log of any number that lies between 0 and 1 (proper fractions) has a negative value.
3. The log of a number greater than 1 is positive.
4. The log of 1 to any base is 0, since  $b^0 = 1 \Rightarrow \log_b 1 = 0$ .
5. The log of a number,  $b$  to the base  $b$  is 1, since  $b^1 = b \Rightarrow \log_b b = 1$  and so  $\log_b b^x = x$

### Evaluating the log of a number in base 10

We can easily say that  $\log_{10}1000 = 3$  since we know  $10^3 = 1000$ . But in general, we use tables or the electronic calculator to give us this value. Usually, we read off the result and round off to any desired degree of accuracy.

Using a book of tables or calculators, we can obtain the log of any number to the base of 10. For example,  $\log_{10}2000 = 3.3$ , when expressed to 1 decimal place.

In other words,  $10^{3.3} \approx 2000$ , (since we rounded the value to 3.3).

### Common and natural logs

The log of a number, written to the base of 10, is usually written as  $lg$  10 instead of  $\log_{10}$ .

So,  $\log_{10}3.5$  is written as  $lg3.5$ . Logs expressed in base 10 are also called common logs.

Another popular base is the Natural or Napier's constant,  $e \approx 2.71828$ . In this case,  $\log_e$  is written as  $ln$ .

So,  $\log_e3.8$  is written as  $ln3.8$ . Logs in base  $e$  are called natural logs.

Note that we only used these shortened forms ( $lg$  and  $ln$ ) for the base of 10 and of  $e$ . For any other base, we need to state the base. For example, for base 2, we write  $\log_2x$ .

### The graph of the log function

Consider the table of values for the exponential function  $f(x) = 2^x$ .

|       |                |               |               |               |   |   |   |
|-------|----------------|---------------|---------------|---------------|---|---|---|
| $x$   | -4             | -3            | -2            | -1            | 0 | 1 | 2 |
| $2^x$ | $\frac{1}{16}$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 |

Now consider the inverse function  $f^{-1}(x)$  which maps  $2^x$  onto  $x$ . The set of ordered pairs for this function is:

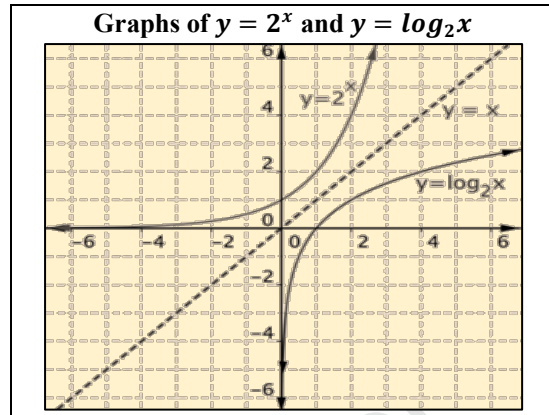
$$\left(\frac{1}{16}, -4\right), \left(\frac{1}{8}, -3\right), \left(\frac{1}{4}, -2\right), \left(\frac{1}{2}, -1\right), (1, 0), (2, 1), (4, 2)$$

This inverse function maps a number onto its logarithm in base 2. For example,

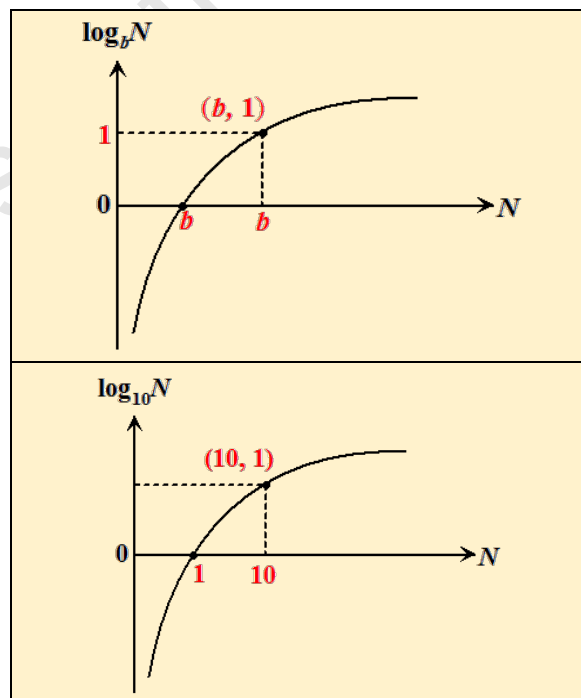
$$\log_2 \frac{1}{16} = -4, \quad \log_2 \frac{1}{8} = -3 \quad \log_2 \frac{1}{4} = -2$$

By definition, the logarithm function is the inverse of the exponential function.

This relationship is shown below for base 2.



The characteristic shape for any log graph,  $\log_b N$  and  $\log_{10} N$  is shown below for base,  $b$ . The graph will always pass through  $(b, 1)$ . The graph for  $lg_{10}$  is also shown.



### Antilogarithms (Antilogs)

The logarithmic function maps a number onto its log and so its inverse function (antilog) maps the log of a number back onto the number. For example, in base ten, the log of 100 is 2, and so, the antilog of 2 is 100. Using a calculator, we can find the log of any number in base 10, for example,

$$\log_{10}130 = 2.114$$

Using the inverse function (shift on the log function), we can get back the number if we input 2.114.

$$\text{antilog } 2.114 = 10^{2.1} = 130$$

We say that the antilog of 2.114 is 130.

Note that in evaluation the antilog, the calculator displays  $10^?$ , verifying that the antilog is an exponential function, which we already established as the inverse of the log function.

Therefore, if we know the antilog of some unknown number, we can obtain the number using the calculator. Suppose we wish to find out what number,  $N$ , in base 10, has a log of 3.2, then the number we are looking for is the antilog of 3.2.

This is obtained directly from a calculator by pressing the shift button on the log function and entering 3.2 on the display. The calculator will evaluate  $10^{3.2} = 1584.9$ .

So  $10^{3.2}$  is called the antilog of 3.2.

### Example 1

- (i) Evaluate  $\lg 785$ .  
 (ii) The log of a number in base 10 is 2.099. What is the number?

#### Solution

- i)  $\lg 785 = \log_{10} 785 = 2.895$  (to 3 dp by the calculator)  
 (ii)  $\log_{10} N = 2.099$   
 $10^{2.099} = N$   
 $N = 125.8$  (By calculator)  
 We say that the antilog of 2.099 is 125.8

### Example 2

- (i) Evaluate  $\ln 76$ .  
 (ii) The log of a number in base  $e$  is 0.655. What is the number?

#### Solution

- i)  $\ln 76 = \log_e 76 = 4.331$  (to 3 dp by the calculator)  
 (ii)  $\log_e N = 0.655$   
 $N = e^{0.655}$   
 $N = 1.925$  (By calculator)  
 We say that the antilog of 0.655 is 1.925

### Laws of logs

The laws of logs are stated below. These laws may have some resemblance to the laws of indices.

$$1. \log_b (MN) = \log_b M + \log_b N$$

$$\log_8 (18.2 \times 4.3) = \log_8 18.2 + \log_8 4.3$$

Conversely,

$$\log_2 7.4 + \log_2 3.6 = \log_2 (7.4 \times 3.6)$$

Similarly,

$$\log_b (MNP) = \log_b M + \log_b N + \log_b P$$

$$2. \log_b \left( \frac{M}{N} \right) = \log_b M - \log_b N$$

$$\log_{10} \left( \frac{12.2}{4.6} \right) = \log_{10} 12.2 - \log_{10} 4.6$$

$$\text{Similarly, } \log_4 6.3 - \log_4 2.8 = \log_4 \left( \frac{6.3}{2.8} \right)$$

$$3. \log_b N^a = a \times \log_b N$$

$$\log_{10} (6.4)^3 = 3 \log_{10} 6.4$$

$$\text{Similarly, } 4 \log_{10} 18.4 = \log_{10} (18.4)^4$$

$$4. \log_b \sqrt[a]{N} = \log_b N^{\frac{1}{a}} = \frac{1}{a} \log_b N$$

$$\log_{10} \sqrt{6.3} = \frac{1}{2} \log_{10} 6.3$$

$$\log_8 \sqrt[3]{16.4} = \frac{1}{3} \log_8 16.4$$

$$\text{Similarly, } \frac{1}{5} \log_8 7.5 = \log_8 \sqrt[5]{7.5}$$

We can combine these laws in any calculation. For example,

$$\log_b \left( \frac{MN}{P} \right) = \log_b M + \log_b N - \log_b P$$

$$\log_b (M^a \times \sqrt[n]{N}) = a \log_b M + \frac{1}{n} \log_b N$$

It is also useful to know that in solving equations if both sides of the equation have logs are such that

$$\lg(4x + 3) = \lg 100$$

Then we can rewrite the equation as

$$4x + 3 = 100$$

because if two expressions are equal their logs are also equal, provided that the base is the same. The converse is also true.

### Example 3

Express  $y$  in terms of  $x$  when

$$\lg(x+2y) = \lg 5 + \lg 3$$

**Solution**

$$\lg(x+2y) = \lg 5 + \lg 3$$

$$\lg(x+2y) = \lg(5 \times 3)$$

$$\lg(x+2y) = \lg 15$$

Hence,  $x+2y = 15$  and  $y = \frac{15-x}{2}$ .

**Example 4**

$$\text{Solve for } x \text{ in } \lg(3x+2) + 6\lg 2 = 2 + \lg(2x+1).$$

**Solution**

To simplify the equation, all the terms should be log terms and expressed to the same base. This will be a log equation and can now be simplified by using the laws of logs seen above.

In this example, all the terms are log terms, and to the same base of 10, except for the term, 2.

So, we need to replace 2 by  $\lg 10^2$  to maintain the same base.

$$\lg(3x+2) + 6\lg 2 = \lg 10^2 + \lg(2x+1)$$

We now have a log equation and proceed to simplify it.

$$\lg\{(3x+2)(2^6)\} = \lg\{100(2x+1)\}$$

$$64(3x+2) = 100(2x+1)$$

$$192x + 128 = 200x + 100$$

$$8x = 28$$

$$x = 3 \frac{1}{2}$$

**A word of caution**

When we solve equations involving logs and more than one solution is obtained, it is wise to check the solutions to ensure that they are valid. We learned earlier that the logarithmic function has certain properties and these properties must not be violated. We must check to verify that our solutions do not give rise to undefined logs such as:

$$\log_b(0) \text{ or } \log_b(-ve).$$

**Example 5**

$$\text{Solve for } x \text{ given, } \log_b(x^2 - 30) = \log_b(x)$$

**Solution**

Since the logs have the same base, we can equate

to get

$$x^2 - 30 = x$$

$$x^2 - x - 30 = 0$$

$$(x-6)(x+5) = 0$$

$$x = 6, -5$$

However, when  $x = -5$  one or more of the terms of the equation will contain the log of a negative term. This is NOT so when  $x = 6$ .

So, we reject  $x = -5$  and conclude that  $x = 6$  only.

**Change of base**

Sometimes we may need to change the log of a number from one base to another. For this, we use the 'change of base formula'. In this example, we are changing from base  $b$  to base  $a$ .

$$\log_b N = \frac{\log_a N}{\log_a b}$$

For example, we wish to convert 25 written to the base of 10 to the base of 6,

$$\log_{10} 25 = \frac{\log_6 25}{\log_6 10}$$

**Solving equations involving logs**

Logarithms provide an alternative way of writing expressions that involve exponents. If we wish to solve for an exponent in an equation, then we can convert the equation from exponential to log form. For example,

If  $a = b^c$  and we wish to find  $c$ , then we can convert to log form,  $\log_b a = c$ . Now,  $c$  can be easily found.

Alternatively, we can take logs on both sides of an equation since if two numbers are equal their logs are equal.

If  $a = b^c$  then taking logs to the base  $b$

$$\log_b a = \log_b b^c$$

$$\Rightarrow \log_b a = c \log_b b$$

$$\Rightarrow \frac{\log_b a}{\log_b b} = c$$

$$\Rightarrow \log_b a = c \quad [\log_b b = 1]$$

Thus, the same result is obtained when we converted to log form.

**Example 6**

$$\text{Solve for } x \text{ given that } 2^x = 31.$$

**Solution**

Take logs to base 10 on both sides

$$\lg 2^x = \lg 31$$

$$x \lg 2 = \lg 31$$

$$x = \frac{\lg 31}{\lg 2}$$

$$x = 4.95 \text{ (2 decimal places)}$$

**Example 7**

Solve for  $x$  in  $2^{x+1} + 2^x = 9$ .

**Solution**

$$2^{x+1} + 2^x = 9$$

$$2^x \times 2 + 2^x = 9$$

$$3 \times 2^x = 9$$

$$2^x = 3$$

Take logs on both sides

$$x \lg 2 = \lg 3$$

$$x = \frac{\lg 3}{\lg 2}$$

$$x = 1.58 \text{ (2 decimal places)}$$

**Application of Logarithms- Linear form**

Equations involving exponents of the form such as  $y = ax^b$  and  $y = ab^x$  can be solved using logarithms.

By taking logs on both sides of the equation, we can obtain an equation that is similar to a linear equation.

We then solve graphically to obtain the unknowns.

The procedure will now be illustrated.

$$y = ax^b \text{ where } a \text{ and } b \text{ are constants.}$$

If we take logs on both sides,

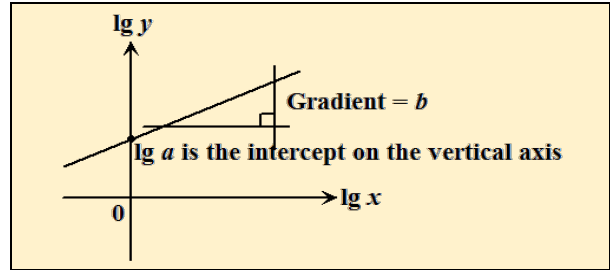
$$\lg y = \lg(ax^b)$$

$$\lg y = \lg a + b \lg x$$

$$\lg y = b \lg x + \lg a$$

In this form, we can see the similarity to,  $Y = MX + C$ , the general equation of a straight line, where  $Y$  is the variable  $\lg y$ ,  $M$  is the constant  $b$ ,  $X$  is the variable  $\lg x$  and  $C$  is the constant  $\lg a$ .

So, a graph of  $\lg y$  versus  $\lg x$  will produce a straight line whose gradient is  $b$  and with  $y$ -intercept will be  $\lg a$ .



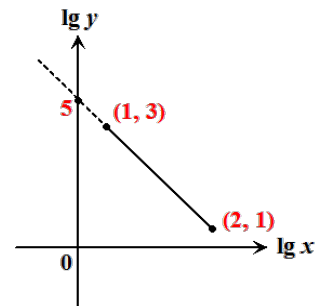
**Example 8**

Two variables are related such that  $y = kx^n$ , where  $k$  and  $n$  are constants. When  $\lg y$  is plotted against  $\lg x$ , a straight line passing through  $(1, 3)$  and  $(2, 1)$  is obtained. The graph is shown below. Find the value of  $k$  and of  $n$ .

**Solution**

If  $y = kx^n$ , then by taking logs to the base of 10, we obtain,  $\lg y = n \lg x + \lg k$ , which is of the form  $Y = MX + C$ , where

$$Y = \lg y, M = n, X = \lg x \text{ and } C = \lg k.$$



$$\text{The gradient of the line is } = \frac{3-1}{1-2} = \frac{2}{-1} = -2$$

$$\text{Therefore, } n = -2$$

$$\text{The equation of the line is: } \frac{y-3}{x-1} = -2$$

$$y = -2x + 5$$

$$\text{Therefore, } C = 5. \text{ But } C = \lg k$$

$$\Rightarrow 5 = \lg k$$

$$k = 10^5.$$

$$\text{Hence, } n = -2 \text{ and } k = 10^5$$