FAS-PASS Maths

4: QUADRATICS

Linear and Quadratic Functions

In our study of linear functions, we recall that a linear function is written or can be reduced to the form $y = mx + c$ where x and y are variables and m and c are constants. A graph of a linear function is always a straight line with gradient m and whose intercept on the y-axis is c.

We will now introduce another function whose general form and graph differ from the linear function. This function is called a quadratic function. Its special features and characteristics will be explored fully in this section.

Quadratic Expressions

We must first learn to recognise quadratic expressions.

> A quadratic expression is one in which the highest power of the variable is two (hence, degree two). A quadratic expression in x, has a general form: $ax^2 + bx + c$, where a, b and c are real numbers, $a \neq 0$.

Quadratic expressions may take the following forms

$3x^2 + 2x + 5$	where $a = 3$, $b = 2$ and $c = 5$
$5x^2 - 2x$	where $a = 5$, $b = -2$ and $c = 0$
$-x^2 + 4$	where $a = -1$, $b = 0$ and $c = 4$
$-2x^2$	where $a = -2$, $b = 0$ and $c = 0$

It is the non-zero term in x^2 which identified the quadratic expressions in each of the examples.

Features of the quadratic function

When we speak of a function, we are referring to a relationship between two variables or unknowns, usually denoted by x and y.

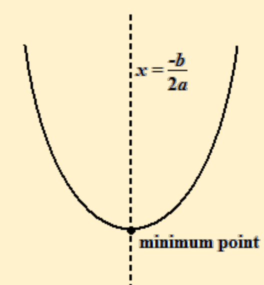
> A quadratic function is a second degree polynomial of the form
> $f(x) = ax^2 + bx + c$ or
> $y = ax^2 + bx + c$, where a, b and $c \in R$

1. The graph of a quadratic function has a characteristic shape called a parabola. This is a curve with a single maximum or a minimum point and also a single axis of symmetry that passes through the maximum or minimum point. For $y = ax^2 + bx + c$ this axis of symmetry is a vertical line with equation is $x = \frac{-b}{2a}$.

2. The sign of the constant, a, in the quadratic function, can be used as an indicator for whether the parabola has a maximum or a minimum point. For any $a > 0$, the parabola has a minimum point and for any $a < 0$, the parabola has a maximum point.

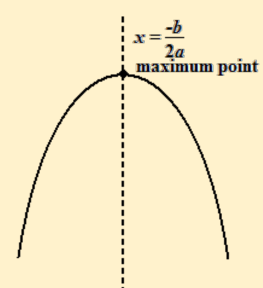
The above features are illustrated in the following graphs.

$y = ax^2 + bx + c$, for $a > 0$



Axis of symmetry $x = \frac{-b}{2a}$

$y = ax^2 + bx + c$, for $a < 0$



Axis of symmetry $x = \frac{-b}{2a}$

Sketching a quadratic function
A simple sketch of a quadratic function can be obtained from an analysis of its equation.

Example 1

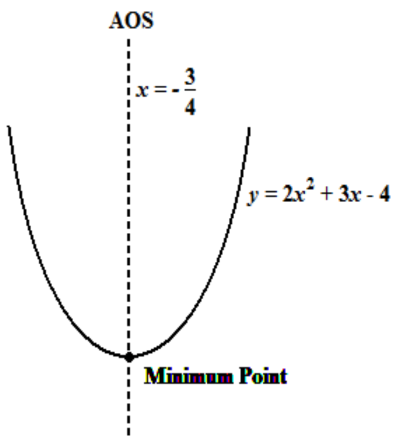
Sketch the graph of the function $y = 2x^2 + 3x - 4$,

Solution

By inspection, $a = 2$, $b = 3$ and $c = -4$.
Since $a > 0$, the curve has a minimum point.

The axis of symmetry is $x = \dfrac{-b}{2a}$

$x = \dfrac{-(3)}{2(2)}$, $x = -\dfrac{3}{4}$



Quadratic Equations

Quadratic equations in x take the general form, $ax^2 + bx + c = 0$ where a, b and c are real numbers, $a \neq 0$. We can solve quadratic equations using graphical or algebraic methods. These methods are illustrated in the following examples.

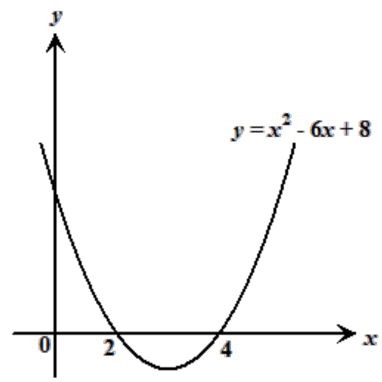
Solving quadratic equations - graphical method
Since a quadratic function is a second-degree polynomial, a quadratic equation may have at most two solutions. Some may have only one solution while others may not have any solutions at all. These solutions are also called the roots of the quadratic equation.
The solution by graphical methods is obtained by drawing the graph and examining the nature of its roots and which are the points where it cuts the x-axis or the horizontal axis.

Example 2

Using a graphical method, solve the quadratic equation $x^2 - 6x + 8 = 0$.

Solution
We draw up a table of values as shown:

x	0	1	2	3	4	5
y	8	3	0	−1	0	3



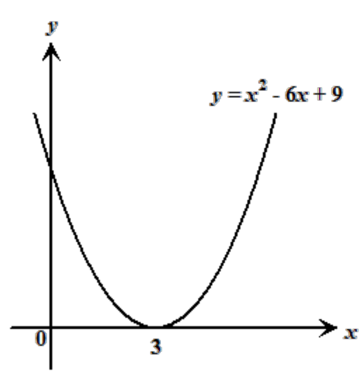
The graph of $y = x^2 - 6x + 8$ cuts the x-axis at $x = 2$ and $x = 4$. These are the two solutions of the quadratic equation, $y = x^2 - 6x + 8$.
Since the graph intersects the x-axis at two different points, we say that the solutions or roots are real and distinct.

Example 3

Using a graphical method, solve the quadratic equation $y = x^2 - 6x + 9$.

Solution
We draw up a table of values as shown:

x	0	1	2	3	4	5
y	9	4	1	0	1	4



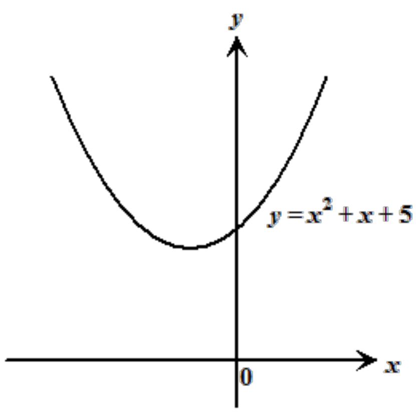
The graph $y = x^2 - 6x + 9$ only touches the x-axis at $x = 3$. Hence, the equation $x^2 - 6x + 9 = 0$ has only one solution or root, $x = 3$.
Since the graph touches the x-axis at one point, the solutions or roots are said to be real and equal and so there is really only **one** solution or root.

Example 4

Using a graphical method, solve the quadratic equation $x^2 + x + 5 = 0$.

Solution
We draw up a table of values as shown:

x	-3	-2	-1	0	1	2
y	11	7	5	5	7	11



The curve neither touches nor cuts the x-axis and hence the equation $x^2 + x + 5 = 0$ and so has no solutions. There are no values of x that satisfy the given equation and so the solutions or roots are not real. Such roots are sometimes called unreal or imaginary.

Solving quadratic equations using the method of factorisation

The method of factorisation works only when the solutions are integers or simple fractions. So, one should note that a even though a quadratic equation is non-factorisable, it may have solutions or roots.

Example 5

Solve for x in the equation, $x^2 - 5x - 6 = 0$.

Solution

$$x^2 - 5x - 6 = 0$$
$$(x-6)(x+1) = 0$$

So $x - 6 = 0$
or
$x + 1 = 0$
$\therefore x = 6$ or $x = -1$
The graph of $y = x^2 - 5x - 6$, cuts the x – axis at $x = 6$ and $x = -1$.
There are two distinct solutions of the quadratic equation $x^2 - 5x - 6 = 0$.

Example 6

Solve for x in the equation $4x^2 - 12x + 9 = 0$.

Solution

$$4x^2 - 12x + 9 = 0$$
$$(2x-3)(2x-3) = 0$$

So $2x - 3 = 0$ or $2x - 3 = 0$

$\therefore x = \frac{3}{2}$ or $x = \frac{3}{2}$

Notice that both solutions are the same, that is, both roots are equal.
So we conclude that the one solution or root is $x = \frac{3}{2}$ only.

Solving Quadratic Equations using the formula
There are quadratic equations which cannot be factorised but do have solutions. In such a case, we may choose to use the formula. The formula can be used to solve any quadratic equation and will give the roots or solutions to any desired level of accuracy. It will even indicate when the quadratic equation has only one solution or does not have any solutions.

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 7

Solve for x in the equation $-2x^2-7x+4=0$.

Solution

$-2x^2-7x+4=0$ is of the form $ax^2+bx+c=0$ where $a=-2$, $b=-7$ and $c=4$.

Hence, substituting in the above formula,

$$x=\frac{-(-7)\pm\sqrt{(-7)^2-4(-2)(4)}}{2(-2)}$$

$$x=\frac{7\pm\sqrt{49-(-32)}}{-4}$$

$$x=\frac{7\pm\sqrt{81}}{-4}$$

$$x=\frac{7+9}{-4} \text{ or } x=\frac{7-9}{-4}$$

$$x=-4 \text{ or } x=\frac{1}{2}$$

Determining the nature of the roots of a quadratic equation

In solving quadratic equations, we note that there can be two roots or one root or no roots. The formula gives the value of x, the roots, where

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Each root is obtained by substituting for a, b and c in:

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ or } x=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

The nature of the roots depends on the value of the term b^2-4ac in the formula. The term, b^2-4ac, is called the **discriminant**, D, so we can replace b^2-4ac by D into the formula to obtain,

$$x=\frac{-b\pm\sqrt{D}}{2a}$$

We can now find each root using

$$x=\frac{-b+\sqrt{D}}{2a} \text{ or } x=\frac{-b-\sqrt{D}}{2a}.$$

To determine the nature of the roots we examine the value of the discriminant, D.

There are three possibilities for the value of D, these are:

1. $D=0$, this occurs when $b^2=4ac$. The parabola touches the x-axis at $x=\frac{-b\pm 0}{2a}$ $=\frac{-b}{2a}$ and there is only one root.
2. $D>0$, this occurs when $b^2>4ac$. The parabola cuts the x-axis at $x=\frac{-b+\sqrt{D}}{2a}$, and $x=\frac{-b-\sqrt{D}}{2a}$ and there are two real and distinct roots.
3. $D<0$, this occurs when $b^2<4ac$, the parabola does not cut or touch the x-axis because $x=\frac{-b\pm\sqrt{D}}{2a}$. But, $\sqrt{-D}$ is an imaginary number. So, we say the roots are not real or do not exist or the equation has imaginary roots.

Solving quadratic equations by completing the square

In arithmetic when a number can be expressed as a product of two whole numbers, it is called a **perfect square**. For example, 25 is a perfect square since 25 = 5 × 5.

Algebraic expressions are perfect squares if they can be expressed as a product of two identical linear factors.

For example, $x^2+8x+16=(x+4)(x+4)=(x+4)^2$

So the expression $x^2+8x+16$ is called a **perfect square**. We can illustrate this as shown below

Other examples of perfect squares are:

$$x^2+10x+25=(x+5)^2$$
$$x^2+6x+9=(x+3)^2$$
$$x^2+2x+1=(x+1)^2$$

In general, $(x+h)^2=x^2+2hx+h^2$

Re-arranging

$x^2+2hx=(x+h)^2-h^2$

One half the coefficient of x

Completing the square

A quadratic expression such as x^2+6x+2 cannot be written as a product of two identical factors and is not a perfect square. To solve a quadratic equation such as $x^2+6x+2=0$, it would be convenient to convert it to the form [(x + ?)^2 + some constant]. When expressed in such a form, it can be readily solved.

In this case, we know that x^2+6x+9 is a perfect square because it can be written as $(x+3)^2$. Hence, if we were to add 7 to x^2+6x+2 it would take a form that we desire. Since we do not wish to alter the expression we add 7 and subtract 7 as follows.

$$x^2+6x+2=$$
$$(x^2+6x+2)+(7-7)=(x^2+6x+9)-7=(x+3)^2-7$$

In converting the algebraic expression to this form, we had to determine what must be added to x^2+6x in order to make it a complete square. This process is called completing the square.

The technique of converting an algebraic expression of the form ax^2+bx+c to the form, $a\left(x+h\right)^2+k=0$, where a, h and k are constants, is called completing the square.

Example 8

Solve the quadratic equation $x^2+12x+16=0$ by completing the square.

Solution

We know that $x^2+2hx=(x+h)^2-h^2$. Therefore $x^2+12x=(x+6)^2-6^2$ where $2h = 12$ and $h = 6$

We solve our quadratic equation as follows:

$$x^2+12x+16=0$$
$$(x+6)^2-6^2+16=0$$
$$(x+6)^2-36+16=0$$
$$(x+6)^2-20=0$$
$$(x+6)^2=20$$
$$x+6=\sqrt{20}$$
$$x=-6\pm\sqrt{20}$$

Example 9

Solve the quadratic equation $x^2-4x+1=0$ by the method of completing the square.

Solution

We know that $x^2+2hx=(x+h)^2-h^2$, therefore $x^2-4x=(x-2)^2-(-2)^2$ where $2h=-4$, and so $h=-2$

We solve our quadratic equation as follows:

$$(x-2)^2-4+1=0$$
$$(x-2)^2-3=0$$
$$(x-2)^2=3$$
$$x-2=\sqrt{3}$$
$$x=2\pm\sqrt{3}$$
$$x=2+\sqrt{3} \quad \text{or} \quad 2\text{-}\sqrt{3}$$

x = 2.73 or 0.27 (to 2 d.p)

Example 10

Solve the quadratic equation $4x^2-3x-1=0$ by the method of completing the square.

Solution

When the coefficient of x^2 is **NOT** one. We use the algebraic technique of factorisation to ensure that the coefficient of x^2 is one.

In this example, we factor out the four by dividing each term in the equation by 4. When this is done we work with the expression inside the brackets. In this example, we completed the square for the expression, $(x^2+\frac{3}{4}x)$ and then proceeded to solve the equation.

$$4(x^2-\frac{3}{4}x)-1=0$$
$$4[(x-\frac{3}{8})^2-\frac{9}{64}]-1=0$$
$$4(x-\frac{3}{8})^2-\frac{9}{16}-1=0$$
$$4(x-\frac{3}{8})^2-\frac{25}{16}=0$$
$$4(x-\frac{3}{8})^2=\frac{25}{16}$$
$$(x-\frac{3}{8})^2=\frac{25}{64}$$
$$(x-\frac{3}{8})=\sqrt{\frac{25}{64}}$$
$$x=\frac{3}{8}\pm\frac{5}{8}$$

Where $2h=\frac{3}{8}$, and $h=\frac{3}{16}$, so

$$h^2=\frac{9}{64}$$

Example 11

Solve the quadratic equation $1-4x-2x^2=0$ by the method of completing the square.

Solution

$$1-4x-2x^2=0$$
$$1-2(2x+x^2)=0$$
$$1-2[x^2+2x]=0$$
$$1-2[(x+1)^2-1]=0$$
$$1-2(x+1)^2+2=0$$
$$3-2(x+1)^2=0$$
$$-2(x+1)^2=-3$$
$$(x+1)^2=1.5$$
$$x+1=\sqrt{1.5}$$
$$x=-1\pm 1.22$$
$$x=-2.22 \quad \text{or} \quad 0.22$$

(correct to 2 decimal places)

Maximum and minimum Points

We can find the maximum or the minimum point of a quadratic function $y=ax^2+bx+c$ by three methods:

1. Using the equation axis of symmetry, $x=\frac{-b}{2a}$, which passes through the maximum or minimum point.
2. Converting the quadratic to the form $a(x+h)^2+k$, which readily gives the coordinates of the maximum point.
3. Drawing the graph and reading the coordinates of the maximum or minimum point.

The first two methods are algebraic in nature while the third is graphical. The result obtained by the third method involves drawing the graph and obtaining such results by reading off these values, by eyesight. The graphical method is subject to 'human limitations' and so cannot yield highly accurate results. We will illustrate methods 1 and 2.

Example 12

Find the coordinates of the minimum point on the curve $y=x^2-6x+8$.

Solution

We know that there is a minimum point on the curve. Let us name the minimum point, P.

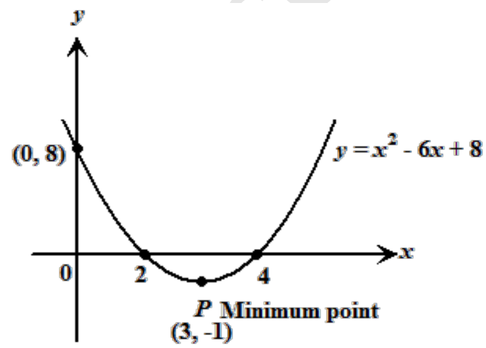
The axis of symmetry passes through P and has the equation:

$$x=\frac{-b}{2a}, \quad x=\frac{-(-6)}{2(1)}=3$$

Therefore the x-coordinate of the minimum point is 3.

When $x=3$

$$y=(3)^2-6(3)+8$$
$$y=9-18+8$$
$$y=-1$$



$\therefore P(3,-1)$ is the minimum point.

Example 13

Find the coordinates of the minimum point of the function x^2+4x-5.

Solution

We need to find an equivalent expression for x^2+4x-5 that is of the form $(x+h)^2+k$.

$x^2+4x=(x+2)^2-2^2$ where $2h=4, h=2$

$$x^2+4x-5=[(x+2)^2-4]-5$$
$$=(x+2)^2-9$$

Now, $(x+2)^2\geq 0$ for all values of x.

Therefore the minimum value of the expression will occur when $(x+2)^2=0$ and which is at $x=-2$.

$f(x)_{\min}=0-9=-9$ Hence, the minimum value is −9.

If the graph of $y=x^2+4x-5$ is sketched, the minimum point would be (−2, −9).

Example 14

Find the maximum value of the function $f(x)=3+8x-2x^2$ and the value of x at which it occurs.

Solution

$f(x)=3+8x-2x^2$

$f(x)=3-2(x^2-4x)$

Now, $x^2 - 4x = (x - 2)^2 - 4$, by the process of completing the square

$$\begin{aligned} f(x) &= 3-2(x^2-4x) \\ &= 3-2[(x-2)^2-4] \\ &= 3-2(x-2)^2+8 \\ &= -2(x-2)^2+11 \end{aligned}$$

And so $f(x)=3+8x-2x^2 \equiv -2(x-2)^2+11$

OR $11 - 2(x-2)^2$

$2(x-2)^2 \geq 0 \quad \forall x$

$\therefore f(x)_{\max} = 11-(0) = 11$

This maximum value occurs when, $2(x-2)^2=0$ and $x=2$.

If the graph of $f(x)$ was sketched, then the point (2, 11) would be the coordinates of the maximum point.

In general,

If a quadratic expression is written in the form $a(x+h)^2+k$, the maximum or the minimum value is always k and it will occur at $x=-h$.

Expressing a quadratic function in the form $a(x+h)^2+k$ by equating coefficients

Instead of completing the square, we can obtain the values of a, h and k using an alternative method. The method of equating coefficients is illustrated in the examples below.

Example 15

Express x^2+4x+1 in the form $(x+h)^2+k$.

Solution

$$\begin{aligned} x^2+4x+1 &= (x+h)^2+k \\ &= x^2+2hx+h^2+k \end{aligned}$$

By equating coefficients of x:

$2h = 4$ and $h = 2$

Equating the constant term:

$$h^2+k=1$$

We substitute $h = 2$ to get

$(2)^2+k=1$

$k=-3$

Hence, $x^2+4x+1=(x+2)^2-3$.

Example 16

Express $3x^2+4x-5$ in the form $a(x+h)^2+k$.

Solution

$$\begin{aligned} 3x^2+4x-5 &= a(x+h)^2+k \\ &= a(x^2+2hx+h^2)+k \\ &= ax^2+2ahx+ah^2+k \end{aligned}$$

Equating coefficients of x^2:

$a=3$

Equating coefficients of x:

$2ah=4$

$2(3)h=4$

$h=\frac{4}{6}=\frac{2}{3}$

Equating the constants:

$ah^2+k=-5$

$a=3$ and $h=\frac{2}{3}$

$3\times\frac{4}{9}+k=-5$

$k=-5-\frac{4}{3}=-6\frac{1}{3}$

$y=3x^2+4x-5=3\left(x+\frac{2}{3}\right)^2-6\frac{1}{3}$

Example 17

Express $10-4x-x^2$ in the form $c-(b+x)^2$.

Hence, state the maximum value of the expression and the value of x for which the maximum occurs.

Solution

$10-4x-x^2=c-\left(b^2+2bx+x^2\right)$

$=c-b^2-2bx-x^2$

Equating the coefficients of x:

$-4=-2b$

$b=2$

Equating the constants terms

$10=c-b^2$

$10=c-(2)^2$

$c=14$

Hence, $10-4x-x^2=14-(2+x)^2$.

$(2+x)^2\geq 0\ \forall x$

Hence, the maximum value of the expression,

$10-4x-x^2=14-(2+x)^2$ is 14 when $(2+x)^2=0$,

that is, at $x=-2$.

Points of intersection of a line and a curve

So far we have been solving quadratic equations to find the roots or the points of intersection of the graph with the x-axis. For example, when we solve a quadratic equation of the form $ax^2+bx+c=0$, we actually solved the following pair of equations simultaneously:

$y=ax^2+bx+c$ (1) and

$y=0$ (2)

where the equation $y=0$ is the equation of the x-axis.

Let us now solve a pair of equations where one is quadratic and the other is a linear equation of the form $y=mx+c$.

Example 18

Find the points of intersection of the line with equation $y=3x+7$ and the curve with equation $y=x^2+5x+4$.

Solution

To find the points of intersection of a curve and a line, we solve their equations simultaneously.

Let $y=x^2+5x+4$…(1)

and $y=3x+7$ ….(2)

Equating (1) and (2) we get $x^2+5x+4=3x+7$

$x^2+2x-3=0$

$(x-1)(x+3)=0$

Hence, $x=1$ or $x=-3$.

We substitute these values of x in any of the equations to find y

When $x=1$, $y=3(1)+7=10$

When $x=-3$, $y=3(-3)+7=-2$

And so the points of intersection of the line and the curve are

(1, 10) and (–3, –2).

Interpreting solutions

In the above example, the curve and the line intersect at two distinct points, (1, 10) and (–3, –2). This is not always the case. Consider the following examples:

Example 19

Find the points of intersection of the line with equation $y=2x-3$ and the curve with equation $y=x^2-2x+1$.

Solving simultaneously, we have

$y=x^2-2x+1$ (1)

$y=2x-3$ (2)

Equating equations (1) and (2)

$x^2-2x+1=2x-3$

$x^2-4x+4=0$

$(x-2)(x-2)=0$

$x=2$

When $x=2$, $y=2(2)-3=1$

We obtain only one solution.

In this case, the line just touches the curve at (2, 1).

Therefore the line is a tangent to the curve at (2, 1).

Recall, in the equation $x^2-4x+4=0$, the discriminant, $D=(-4)^2-4(1)(4)=0$.

Hence, there is only one root.

Example 20

Find the points of intersection of the line with equation $y=-x-5$ and the curve with equation $y=x^2+x+2$.

Solving simultaneously, we have

$y=x^2+x+2$ (1)

$y=-x-5$ (2)

Equating equations (1) and (2)

$x^2+x+2=-x-5$

$x^2+2x+7=0$

In attempting to solve the equation $x^2 + 2x + 7 = 0$, we examine the value of the discriminant,

$$D = b^2 - 4ac$$
$$D = (2)^2 - 4(1)(7) = -24.$$

A negative value indicates that there are no solutions. In this case, the line does not meet the curve.

Disguised quadratic equations

Quadratics expressions or equations may also appear in forms that are quite different from the standard form, $ax^2 + bx + c = 0$. To solve these 'disguised' quadratics, we perform algebraic manipulations to reduce them to the standard quadratic form.

Example 21

Solve for x in $x = 5 - \frac{6}{x}$.

Solution

The given form is difficult to solve.

$$x = 5 - \frac{6}{x}$$

If we multiply the equation by x,

$$x^2 = 5x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x-2)(x-3) = 0$$

And so $x = 2$ or $x = 3$.

Example 22

Solve for x in the polynomial, $x^4 - 13x^2 + 36 = 0$.

Solution

This is a polynomial of degree 4 and is not likely that this will be recognised as a quadratic. A simple substitution reduces it to a quadratic equation.

Let $t = x^2$

$$\therefore (x^2)^2 - 13(x^2) + 36 = 0$$

A quadratic is now created and which is of the form

$$t^2 - 13t + 36 = 0$$
$$(t-4)(t-9) = 0$$
$$t = 4 \text{ or } t = 9$$

Hence, $x^2 = 4$ or $x^2 = 9$

When $x^2 = 4$, $x = \pm 2$

When $x^2 = 9$, $x = \pm 3$

Hence, $x = -2$ or -3 or 3 or 2.

Example 23

Solve for x, $x + 2 = 3\sqrt{x}$.

Solution

$$x + 2 = 3\sqrt{x}$$
$$\therefore x - 3\sqrt{x} + 2 = 0$$
$$\left(x^{\frac{1}{2}}\right)^2 - 3\left(x^{\frac{1}{2}}\right) + 2 = 0$$

This is now actually reduced to a quadratic in $x^{\frac{1}{2}}$.

In this case, we can factorise to obtain $\left(x^{\frac{1}{2}} - 1\right)\left(x^{\frac{1}{2}} - 2\right) = 0$.

We obtain

$x^{\frac{1}{2}} = 1$ or $x^{\frac{1}{2}} = 2$

$\therefore x = (1)^2 = 1$ **OR** $x = (2)^2 = 4$

Hence, the solutions are $x = 1$ or $x = 4$.

Example 24

Solve for x in $x = \sqrt{x+9} + 3$.

Solution

$$x = \sqrt{x+9} + 3$$
$$x - 3 = \sqrt{x+9}$$

Squaring both sides:

$$(x-3)^2 = x + 9$$
$$x^2 - 6x + 9 = x + 9$$
$$x^2 - 7x = 0$$
$$x(x-7) = 0$$
$$x = 0 \text{ or } 7$$

Since we squared the equation, it is wise to re-check the solutions for false or spurious solutions which may have surfaced as a result of squaring.

Test when $x = 0$,

$\sqrt{0+9} + 3 = 3 \neq 0, \therefore x \neq 0$

When $x = 7$, $7 = \sqrt{7+9} + 3 = 4 + 3$

Therefore, $x = 7$ only