FAS-PASS Maths

3. SURDS

Examples of surds

We know that a perfect square has an exact square root. However, most numbers do not have exact square roots, that is, their square roots are not whole numbers. If we use a calculator to obtain the square root of numbers that are not perfect squares, we notice that the set of numbers after the decimal point continues infinitely without any pattern of reoccurrence because these numbers are irrational. We cannot obtain their exact result in a decimal form. So, we may choose to round them off to any number of decimal places, as shown below. The following are examples of surds.

$\sqrt{5} = 2.236$ (to 3 decimal places)
$\sqrt{12} = 3.4641$ (to 4 decimal places)
$\sqrt{27} = 5.196152$ (to 6 decimal places)

These numbers are also examples of surds because their decimal equivalent continues indefinitely without any pattern of reoccurrence.

$\sqrt[3]{15} = 2.4662$ (to 4 decimal places)
$\sqrt[4]{10} = 1.778279$ (to 6 decimal places)
$\sqrt[5]{29} = 1.961$ (to 3 decimal places)

Non-examples of surds

Some numbers do have exact roots. They may have exact square roots, cube roots, fourth roots and so on. For example, the numbers 1, 4, 9, 16, 25 and 36 have exact square roots and are called perfect squares.

$$\sqrt{1} = 1, \sqrt{4} = 2, \sqrt{9} = 3, \sqrt{16} = 4, \sqrt{25} = 5, \sqrt{36} = 6$$

We can also see that, 27 has an exact cube root, 625 has an exact fourth root and 32 has an exact fifth root. These are examples of perfect cubes.

$$\sqrt[3]{27} = 3,\ \sqrt[4]{625} = 5, \sqrt[5]{32} = 2$$

Notice that in the above examples the result is an exact number. So, these numbers are not examples of surds, even though they can be expressed using root signs.

Defining a surd

A root of a positive real quantity is called a surd if its value cannot be exactly determined. Surds have infinite non-recurring decimals and are actually irrational numbers. So, when a root is irrational, it is a surd. But, not all roots are surds.

Rules for operations on surds

Rules for surds are the same as the rules for simplifying roots and involve the same rules of operation as basic algebra.

1. $\sqrt{ab} = \sqrt{a}\sqrt{b}$

Applying this rule, we have

$$\sqrt{5} \times \sqrt{15} = \sqrt{5} \times \left(\sqrt{3}\sqrt{5}\right) = 5\sqrt{3}$$

2. $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

Applying this rule, we have

$$\sqrt{\frac{75}{15}} = \frac{\sqrt{75}}{\sqrt{15}} = \frac{\sqrt{5\times5\times3}}{\sqrt{5}\sqrt{3}} = \frac{\sqrt{5}\sqrt{5}\sqrt{3}}{\sqrt{5}\sqrt{3}} = \sqrt{5} \text{ (exact)}$$

$$\begin{aligned}\sqrt{72} - \sqrt{32} + \sqrt{50} &= \sqrt{36\times2} - \sqrt{16\times2} + \sqrt{25\times2} \\ &= 6\sqrt{2} - 4\sqrt{2} + 5\sqrt{2} = 7\sqrt{2}\end{aligned}$$

$$\frac{\sqrt{98}}{\sqrt{128}} = \frac{\sqrt{49}\sqrt{2}}{\sqrt{64}\sqrt{2}} = \frac{7}{8}$$

3. $m\sqrt{a} \pm n\sqrt{a} = (m \pm n)\sqrt{a}$

Applying this rule, we have

$$5\sqrt{2} \pm 3\sqrt{2} = (5 \pm 3)\sqrt{2}$$

4. $m\sqrt{a} \times n\sqrt{b} = mn\sqrt{a}\sqrt{b}$

Applying this rule, we have

$$\begin{aligned}2\sqrt{5} \times 3\sqrt{3} &= 2\times3\times\sqrt{5}\times\sqrt{3} \\ &= 2\times3\times\sqrt{5\times3} = 6\sqrt{15}\end{aligned}$$

5. $\dfrac{m\sqrt{a}}{n\sqrt{b}}=\dfrac{m}{n}\sqrt{\dfrac{a}{b}}$

Applying this rule, we have

$$\frac{5\sqrt{3}}{2\sqrt{2}}=\frac{5}{2}\sqrt{\frac{3}{2}}$$

Rationalising a surd

To rationalise a surd, we remove all surds from the denominator of the expression, without altering its numerical value. In the process, surds may now appear in the numerator of the expression, where there may not have even had any from before.

Example 1

Rationalise $\dfrac{3}{\sqrt{2}}$.

Solution

$$\frac{3}{\sqrt{2}}=\frac{3}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{3\sqrt{2}}{2}$$

Example 2

Rationalise $\dfrac{6+\sqrt{10}}{\sqrt{2}}$.

Solution

$$\begin{aligned}\frac{\sqrt{6}+\sqrt{10}}{\sqrt{2}}&=\frac{\sqrt{6}+\sqrt{10}}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{6\times2}+\sqrt{10\times2}}{2}\\&=\frac{\sqrt{12}+\sqrt{20}}{2}=\frac{\sqrt{4}\times\sqrt{3}+\sqrt{4}\times\sqrt{5}}{2}\\&=\frac{2\sqrt{3}+2\sqrt{5}}{2}=\sqrt{3}+\sqrt{5}\end{aligned}$$

Example 3

Rationalise $\dfrac{4-\sqrt{2}}{3-\sqrt{5}}$.

Solution

We multiply by the conjugate (same expression with opposite signs separating the terms) of the denominator.

$$\begin{aligned}\frac{4-\sqrt{2}}{3-\sqrt{5}}&=\frac{4-\sqrt{2}}{3-\sqrt{5}}\times\frac{3+\sqrt{5}}{3+\sqrt{5}}\\&=\frac{12-3\sqrt{2}+4\sqrt{5}-\sqrt{10}}{9-3\sqrt{5}+3\sqrt{5}-5}=\frac{12-3\sqrt{2}+4\sqrt{5}-\sqrt{10}}{4}\end{aligned}$$

Example 4

Rationalise $\dfrac{1+\sqrt{2}}{2+\sqrt{2}}$.

Solution

$$\begin{aligned}\frac{1+\sqrt{2}}{2+\sqrt{2}}&=\frac{1+\sqrt{2}}{2+\sqrt{2}}\times\frac{2-\sqrt{2}}{2-\sqrt{2}}\\&=\frac{2+2\sqrt{2}-\sqrt{2}-2}{4-2\sqrt{2}+2\sqrt{2}-2}=\frac{\sqrt{2}}{2}\end{aligned}$$