2. INDICES

From arithmetic, we know, for example, that:

$8 = 2\times 2\times 2 = 2^3$

$16 = 2\times 2\times 2\times 2 = 2^4$

$32 = 2\times 2\times 2\times 2\times 2 = 2^5$, and so on.

When we write 2^5, we refer to 2 as the base and 5 as the index (or the power or the exponent).

Hence, if m is a positive integer, then $a\times a\times a\times \ldots\times a$, written a total of m times, is expressed as a^m, where a is the base and m is the index.

Laws of Indices

1. Law 1- Multiplication

Consider,

$a^3\times a^2 = (a\times a\times a)\times(a\times a) = a^5$

$b\times b^5 = b\times(b\times b\times b\times b\times b) = b^6$

We can deduce that

$$a^3\times a^2 = a^{3+2} = a^5$$
$$b\times b^2 = b^{1+5} = b^6$$

It follows that,

$$x^4\times x^3 = x^{4+3} = x^7$$
$$y^7\times y^3 = y^{10}$$

In general, for any real numbers, m and n, and a common base, a

$$a^m\times a^n = a^{m+n}$$

2. Law 2 - Division

Consider,

$a^6\div a^2 = \dfrac{a\times a\times a\times a\times a\times a}{a\times a} = a^4$

$b^4\div b = \dfrac{b\times b\times b\times b}{b} = b^3$

We can deduce that

$$a^6\div a^2 = a^{6-2} = a^4$$
$$b^4\div b = b^{4-1} = b^3$$

It follows that

$$x^7\div x^3 = x^{7-3} = x^4$$
$$y^8\div y^6 = y^{8-6} = y^2$$

In general, for any real numbers, m and n, and a common base, a

$$a^m\div a^n = a^{m-n}$$

Note that for the above laws to be applicable, the bases of both of the numbers to be multiplied or divided must be ***the same***.

3. Law 3 – Power of a power

Consider,

$\left(a^4\right)^3 = a^4\times a^4\times a^4 = a^{12}$

$\left(b^5\right)^2 = b^5\times b^5 = b^{10}$

We can deduce that

$$(a^4)^3 = a^{4\times 3} = a^{12}$$
$$(b^5)^2 = b^{5\times 2} = b^{10}$$

It follows that

$(x^6)^4 = x^{6\times 4} = x^{24}$

$(y^3)^5 = y^{3\times 5} = y^{15}$

In general, the expression a^m, raised to a power n, is equal to a^{mn}.

$$\left(a^m\right)^n = a^{m\times n}$$

4. Law 4 - Zero Index

Consider

$$a^6\div a^6 = \frac{a\times a\times a\times a\times a\times a}{a\times a\times a\times a\times a\times a} = 1$$

But by Law 2,

$$a^6\div a^6 = a^{6-6} = a^0$$

Therefore

$$a^0 = 1$$

This law applies to bases that are whole numbers, fractions, positive, negative, or even irrational numbers.

It follows that

$5^0 = 1$

$(-4)^0 = 1$

$\left(\frac{3}{4}\right)^0 = 1$

$\left(\sqrt{2}\right)^0 = 1$

In general, any base raised to the power of zero is equal to one.

$$a^0 = 1$$

5. Law 5 – Negative Indices

Consider
$$a^2 \div a^3 = \frac{a \times a}{a \times a \times a} = \frac{1}{a}$$
But by Law 2,
$$a^2 \div a^3 = a^{2-3} = a^{-1}$$
Therefore,
$$\frac{1}{a} = a^{-1}$$
Consider
$$b \div b^4 = \frac{b}{b \times b \times b \times b} = \frac{1}{b^3}$$
But by Law 2,
$$b \div b^4 = b^{1-4} = b^{-3}$$
Therefore,
$$\frac{1}{b^3} = b^{-3}$$
This law is useful when we wish to convert from fractional form to index form. For example,
$$3^{-1} = \frac{1}{3} \qquad \frac{1}{x^4} = x^{-4} \qquad \frac{4}{b^2} = 4b^{-2}$$
In general,
$$\frac{1}{a^m} = a^{-m}$$

6. Law 6 - Fractional indices (numerator = 1)

Consider
$$a^2 \times a^2 = a^4 \Longrightarrow \sqrt{a^4} = a^2$$
But by law 3, $(a^4)^{\frac{1}{2}} = a^{4\times\frac{1}{2}} = a^2$

Therefore, $\boldsymbol{\sqrt{a^4} = (a^4)^{\frac{1}{2}}}$

Similarly,
$$a \times a = a^2 \Longrightarrow \sqrt{a^2} = a$$
But by law 3, $(a^2)^{\frac{1}{2}} = a^{2\times\frac{1}{2}} = a$

Therefore, $\boldsymbol{\sqrt{a^2} = (a^2)^{\frac{1}{2}}}$

We can deduce that the square root is equivalent to a power of one-half.

So, $\boldsymbol{\sqrt{a} = a^{\frac{1}{2}}}$

We can repeat this process for cube root and fourth root and derive the following:
$$\sqrt[3]{a} = a^{\frac{1}{3}} \qquad \sqrt[4]{a} = a^{\frac{1}{4}}$$
The n^{th} root of a number can be expressed as the number raised to the power of $\frac{1}{n}$.
$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

7. Fractional indices (numerator ≠ 1)

When the numerator in the fraction is not one, the base is raised to the power $\frac{m}{n}$. This is interpreted as the nth root of the base raised to the power of m, and the simplification can be done in any order.
$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \text{ or } \left(\sqrt[n]{a}\right)^m$$
Applying the above rule:
$$(8)^{\frac{2}{3}} = \sqrt[3]{(8)^2} \text{ or } \left(\sqrt[3]{8}\right)^2 = 4$$
$$(81)^{\frac{3}{4}} = \sqrt[4]{(81)^3} \text{ or } \left(\sqrt[4]{81}\right)^3 = 27$$
$$(-32)^{\frac{2}{5}} = \sqrt[5]{(-32)^2} \text{ or } \left(\sqrt[5]{-32}\right)^2 = 4$$

Example 1

Simplify $\dfrac{(8)^{\frac{2}{3}} \times \sqrt{4}}{\left(2^2\right)^2}$.

Solution
$$\frac{(8)^{\frac{2}{3}} \times \sqrt{4}}{\left(2^2\right)^2} = \frac{\sqrt[3]{8^2} \times \sqrt{4}}{2^4} = \frac{\sqrt[3]{64} \times 2}{16} = \frac{4 \times 2}{16} = \frac{1}{2}$$

Example 2

Simplify $\dfrac{x^{\frac{2}{3}} \times x^{\frac{1}{2}}}{\sqrt{x^3}}$.

Solution
$$\frac{x^{\frac{2}{3}} \times x^{\frac{1}{2}}}{\sqrt{x^3}} = \frac{x^{\frac{2}{3}} \times x^{\frac{1}{2}}}{x^{\frac{3}{2}}} = x^{\frac{2}{3}+\frac{1}{2}-\frac{3}{2}} = x^{-\frac{1}{3}} = \frac{1}{x^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{x}}$$

Example 3

Simplify $\dfrac{\left(y^2\right)^3 \times \sqrt[3]{y^3}}{2y^0 \times y^{-4}}$.

Solution

$$\frac{\left(y^2\right)^3 \times \sqrt[3]{y^3}}{2y^0 \times y^{-4}} = \frac{y^{2\times 3} \times y^{3\times\frac{1}{3}}}{2(1)\times y^{-4}} = \frac{y^6 \times y^1}{2y^{-4}} = \frac{y^{6+1+4}}{2}$$

$$= \frac{1}{2}y^{11}$$

Solving equations when the unknown is an index

We can apply the laws of indices to solve equations when the unknown is an index. In such cases, we need to have a common base on both sides of the equation and then we can equate indices. The following examples illustrates these procedures.

Example 4

Solve the equation, $4^x \times 3^{2x} = 6$.

Solution

$4^x \times 3^{2x} = 6$

$2^{2x} \times 3^{2x} = 6$

$6^{2x} = 6$

$2x = 1$ [equating indices]

$x = \frac{1}{2}$

Example 5

Solve for x: $3^{2x} - 9\left(3^{-2x}\right) = 8$.

Solution

$3^{2x} - 9\left(3^{-2x}\right) = 8$

$3^{2x} - \frac{9}{3^{2x}} - 8 = 0$

$\left(3^{2x}\right)^2 - 8\left(3^{2x}\right) - 9 = 0 \quad [\times 3^{2x}]$

$\left(3^{2x} - 9\right)\left(3^{2x} + 1\right) = 0$

Either $(3^{2x} - 9) = 0$ or $(3^{2x} + 1) = 0$

When $(3^{2x} - 9) = 0$

$3^{2x} = 3^2$

$2x = 2$ (Equating indices)

$x = 1$

When $(3^{2x} + 1) = 0$

$3^{2x} = -1$, x has no real solutions

$\therefore x = 1$ only

Example 6

Solve $9^{x+1} - 28(3)^x + 3 = 0$.

Solution

$9^{x+1} - 28(3^x) + 3 = 0$

$(3^2)^{x+1} - 28(3^x) + 3 = 0$

$3^{2x+2} - 28(3^x) + 3 = 0$

$3^{2x}3^2 - 28(3^x) + 3 = 0$

$9(3^x)^2 - 28(3^x) + 3 = 0$

$9(3^x)^2 - 23(3^x) + 3 = 0$

Substitute $y = 3^x$

$9y^2 - 23y + 3 = 0$

$(9y - 1)(y - 3) = 0$

$y = \frac{1}{9}$, or $y =$

When $y = \frac{1}{9}$, $3^x = \frac{1}{9}$

$3^x = 3^{-2} \Rightarrow x = -2$

When $y = 3$, $3^x = 3$

$\Rightarrow x = 1$

$\therefore x = -2$ or $x = 1$