FAS-PASS Maths

CSEC ADD MATHS 2018

SECTION I

Answer BOTH questions.

ALL working must be clearly shown.

1. (a) (i) Given that $f(x) = x^2 - 4$ for $x \geq 0$, find the inverse function, stating its domain.

SOLUTION:

Data: $f(x) = x^2 - 4$ for $x \geq 0$

Required to Find: $f^{-1}(x)$, stating its domain.

Solution:

Let $y = f(x)$

$\therefore y = x^2 - 4$

$y + 4 = x^2$

$x^2 = y + 4$

$x = \sqrt{y+4}$

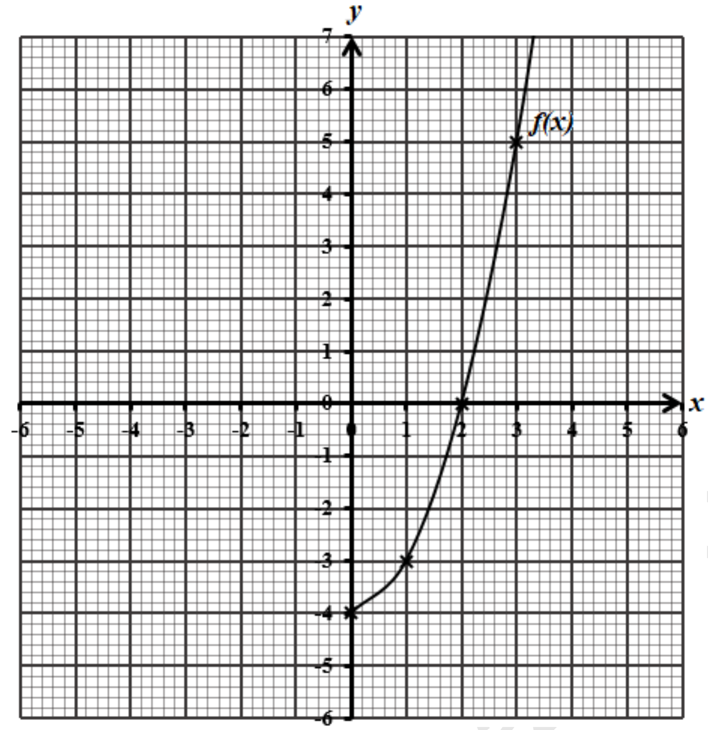
Replace y by x, we obtain:

$f^{-1}(x) = \sqrt{x+4}$

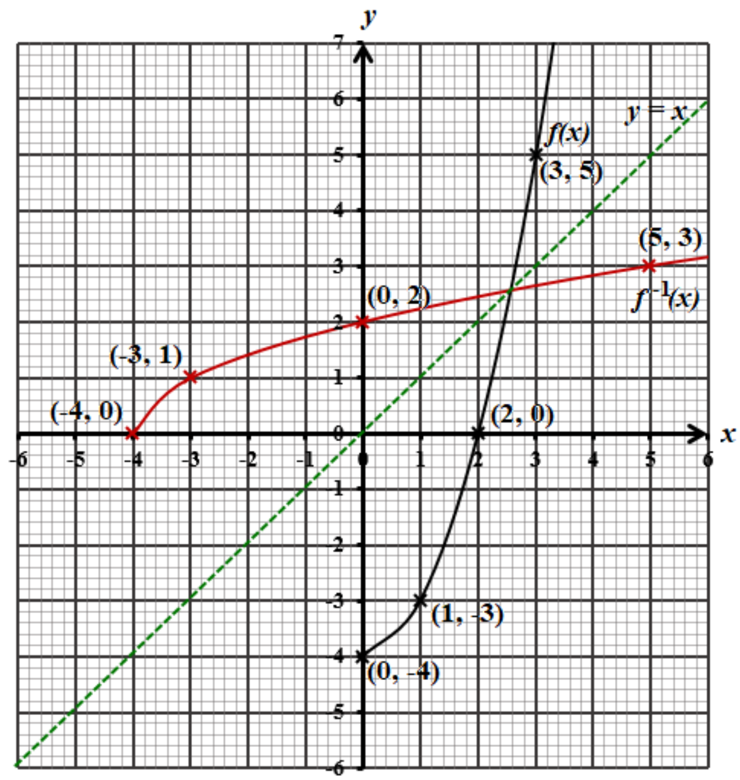
$\nexists \sqrt{\text{-ve}}$

So, $f^{-1}(x) = \sqrt{x+4}$ for $x \geq -4$.

(ii) On the grid provided below, sketch $f^{-1}(x)$.



SOLUTION:
Data: Graph showing $f(x) = x^2 - 4$
Required To Draw: The graph of $f^{-1}(x)$
Solution:



(iii) State the relationship between $f(x)$ and $f^{-1}(x)$.

SOLUTION:
Required to state: The relationship between $f(x)$ and $f^{-1}(x)$
Solution:
$f(x) \xrightarrow{\text{Reflection in } y=x} f^{-1}(x)$
The domain of *f* is the co-domain of f^{-1}. The co-domain of *f* is the domain of f^{-1}.
If (a, b) is a point in $f(x)$ then (b, a) will be the corresponding point on $f^{-1}(x)$.

(b) Derive the polynomial, $P(x)$, of degree 3 which has roots equal to 1, 2 and -4.

SOLUTION:
Required to derive: Polynomial, $P(x)$, of degree 3 with roots equal to 1, 2 and -4.
Solution:
If 1, 2 and -4 are roots of the polynomial $P(x)$, then according to the Remainder and Factor Theorem, $(x-1)$, $(x-2)$ and $(x-(-4))$ will be three factors of $P(x)$.

$$\begin{aligned}\therefore P(x) &= (x-1)(x-2)(x+4) \\ &= (x^2-3x+2)(x+4) \\ &= x^3-3x^2+2x+4x^2-12x+8 \\ &= x^3+x^2-10x+8\end{aligned}$$

Hence, $P(x)=x^3+x^2-10x+8$ is the required polynomial of degree 3.

(c) An equation relating *V* and *t* given by $V = ka^t$ where *k* and *a* are constants.

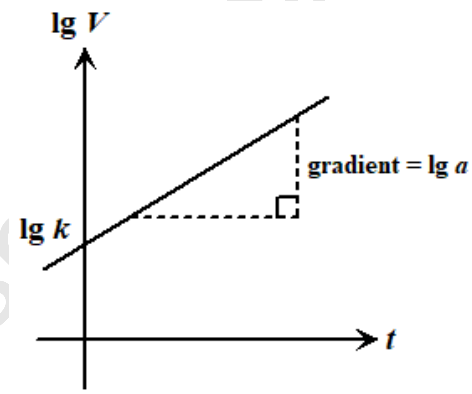
(i) Use logarithms to derive an equation of the form $y = mx + c$ that can be used to find the values of *k* and *a*.

SOLUTION:
Data: *V* and *t* are related by the equation $V = ka^t$, where *k* and *a* are constants.
Required to express: The equation in the form of a straight line

Solution:
$V = ka^t$
Taking lg:
$\lg V = \lg\left(ka^t\right)$
$\lg V = \lg k + \lg a^t$
$\lg V = \lg k + t\lg a$
This is of the form $y = mx + c$, where $y = \lg V$ (a variable), $m = \lg a$ (a constant), $x = t$ (a variable) and $c = \lg k$ (a constant).

(ii) If a graph of *y* versus *x* from the equation in Part (c) (i) is plotted, a straight line is obtained. State an expression for the gradient of the graph.

SOLUTION:
Data: The graph of *y* versus *x* from the equation in Part (c) (i) is a straight line.
Required to state: An expression for the gradient of the graph
Solution:
$\lg V = (\lg a)t + \lg k$



The above diagram gives an indication of what the sketch may look like. When *y* vs *x* is drawn, a straight line of gradient *m* is obtained and which cuts the vertical axis at *c*.
So, when the equivalent form of $\lg V$ vs *t* is drawn, the straight line obtained will have a gradient of $\lg a$. (The intercept on the vertical axis will be $\lg k$.)

2. (a) (i) Given that $g(x) = -x^2 + x - 3$, express $g(x)$ in the form $a(x+h)^2 + k$, where *a*, *h* and *k* are constants.

SOLUTION:
Data: $g(x) = -x^2 + x - 3$

Required to express: $g(x)$ in the form $a(x+h)^2+k$, where *a*, *h* and *k* are constants.
Solution:
$g(x)=-x^2+x-3$
$g(x)=-(x^2-x)-3$

Half the coefficient of *x* is $\frac{1}{2}(-1)=-\frac{1}{2}$

So, $g(x)=-\left(x-\frac{1}{2}\right)^2+*$, where * is to be determined

Consider

$$\begin{aligned}
-\left(x-\frac{1}{2}\right)^2 &= -\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2}\right)\\
&= -\left(x^2-x+\frac{1}{4}\right)\\
&= -x^2+x-\frac{1}{4} \quad +\\
&\quad\ \ -2\frac{3}{4}\\
&\underline{-x^2+x-3}
\end{aligned}$$

So, $*=-2\frac{3}{4}$

Hence, $g(x)=-\left(x-\frac{1}{2}\right)^2-2\frac{3}{4}$ is of the form $a(x+h)^2+k$, where $a=-1$, $h=-\frac{1}{2}$ and $k=-2\frac{3}{4}$.

Alternative Method:

$$\begin{aligned}
a(x+h)^2+k &= a(x+h)(x+h)+k\\
&= a\left(x^2+2hx+h^2\right)+k\\
&= ax^2+2ahx+ah^2+k
\end{aligned}$$

So $ax^2+2ahx+ah^2+k=-x^2+x-3$
Equating coefficients:
$a=-1$

$2(-1)h = -1$

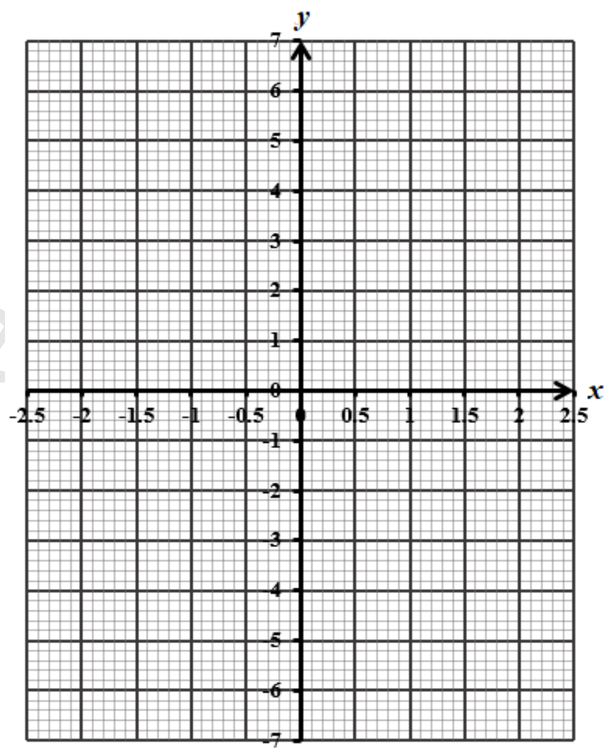
$\therefore h = -\dfrac{1}{2}$

$(-1)\left(-\dfrac{1}{2}\right)^2 + k = -3$

$k = -2\dfrac{3}{4}$

So, $g(x) = -\left(x - \dfrac{1}{2}\right)^2 - 2\dfrac{3}{4}$ and which s of the required form where $a = -1$, $h = -\dfrac{1}{2}$ and $k = -2\dfrac{3}{4}$.

(ii) On the grid provided below, sketch the graph of $g(x)$, showing the maximum point and the *y*-intercept.



SOLUTION:
Required to sketch: The graph of $g(x)$, showing the maximum point and the *y*-intercept.
Solution:

$$g(x) = -\left(x - \frac{1}{2}\right)^2 - 2\frac{3}{4}$$

$$\left(x - \frac{1}{2}\right)^2 \geq 0 \ \forall x$$

So $g(x)$ has a maximum value of $-(0) - 2\frac{3}{4} = -2\frac{3}{4}$ at $-\left(x - \frac{1}{2}\right)^2 = 0$

i.e. $x = \frac{1}{2}$

So on the graph of $g(x)$, $\left(\frac{1}{2}, -2\frac{3}{4}\right)$ is the maximum point.

When $x = 0$, $g(0) = -(0)^2 + (0) - 3 = -3$

$\therefore g(x)$ cuts the vertical axis at $(0, -3)$.

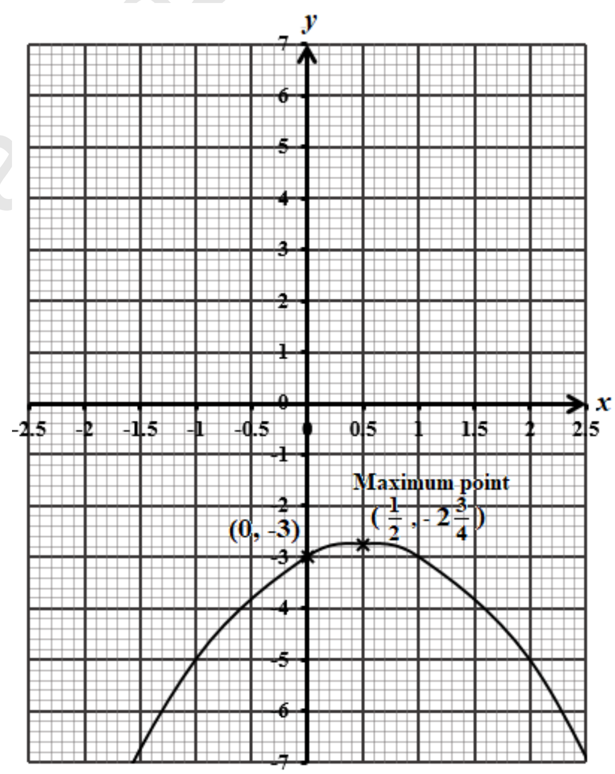
Let $g(x) = 0$

$\therefore -x^2 + x - 3 = 0$ is of the form $ax^2 + bx + c = 0$, where
$a = -1, b = 1, c = -3$

$b^2 = (1)^2 = 1$

and $4ac = 4(-1)(-3) = 12$

Notice $b^2 < 4ac$, hence, $g(x)$ has no real solutions and so does not cut the horizontal axis.



(b) In a geometric progression, the 3rd term is 25 and the sum of the 1st and 2nd terms is 150. Determine the sum of the first four terms, given that $r > 0$.

SOLUTION:
Data: Geometric series with the 3rd term $= 25$ and the sum of the first and second terms $= 150$. The common ratio, $r > 0$.
Required to calculate: The sum of the first four terms

Calculation:
Let $T_n = n^{\text{th}}$ term for the geometric progression
$T_n = ar^{n-1}$, where $a = 1^{\text{st}}$ term
Hence, $T_3 = ar^2 = 25$ (data)

$T_1 = a$ and $T_2 = ar$
So $a + ar = 150$ (data)

Let $ar^2 = 25$ …❶
$a + ar = 150$ …❷
From ❶:
$$a = \frac{25}{r^2}$$

Substitute into ❷:
$$\frac{25}{r^2} + \frac{25}{r^2} \times r = 150$$
$\times r^2$
$$25 + 25r = 150r^2$$
$$1 + r = 6r^2$$
$$6r^2 - r - 1 = 0$$
$$(3r+1)(2r-1) = 0$$
So $r = -\dfrac{1}{3}$ or $\dfrac{1}{2}$

$r > 0$ (data), so $r = \dfrac{1}{2}$, Substitute $r = \dfrac{1}{2}$ in ❶:
$$a\left(\frac{1}{2}\right)^2 = 25$$
$$a = 100$$

$$S_n = \frac{a\left(1-r^n\right)}{1-r}, |r|<1$$

So $$S_4 = \frac{100\left(1-\left(\frac{1}{2}\right)^4\right)}{1-\frac{1}{2}}$$

$$= 200\left(1-\frac{1}{16}\right)$$

$$= 200 \times \frac{15}{16}$$

$$= 187\frac{1}{2}$$

Alternative Solution

Since we need to sum only the first four terms we can write the sequence up to $n = 4$, and sum the terms as follows:

$A = 100, r = \frac{1}{2}$

$$S_4 = 100 + 100\left(\frac{1}{2}\right) + 100(\frac{1}{2})^2 + 100(\frac{1}{2})^2$$

$$S_4 = 100 + 50 + 25 + 12\frac{1}{2}$$

$$S_4 = 187\frac{1}{2}$$

(c) If α and β are the roots of the equation $2x^2 - 5x + 3 = 0$, determine the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

SOLUTION:
Data: α and β are the roots of the equation $2x^2 - 5x + 3 = 0$

Required to calculate: $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

Calculation:
Recall: If $ax^2 + bx + c = 0$

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

If α and β are the roots of the equation, then

$$(x-\alpha)(x-\beta)=0$$

i.e. $x^2-(\alpha+\beta)x+\alpha\beta=0$

Equating coefficients:

$\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

So in $2x^2-5x+3=0$

$$\alpha+\beta=\frac{-(-5)}{2}$$

$$=\frac{5}{2}$$

And $\alpha\beta=\frac{3}{2}$

Now $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\beta^2+\alpha^2}{\alpha^2\beta^2}$

$$=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}$$

$$(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta$$

So $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

Hence, $\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$

$$=\frac{\left(\frac{5}{2}\right)^2-2\left(\frac{3}{2}\right)}{\left(\frac{3}{2}\right)^2}$$

$$=\frac{\frac{25}{4}-3}{\frac{9}{4}}=\frac{6\frac{1}{4}-3}{\frac{9}{4}}=\frac{3\frac{1}{4}}{\frac{9}{4}}$$

$$=\frac{13}{4}\times\frac{4}{9}$$

$$=\frac{13}{9}=1\frac{4}{9}$$

FAS-PASS Maths

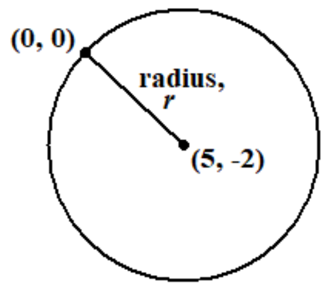
SECTION II

Answer BOTH questions.

ALL working must be clearly shown.

3. (a) Determine the equation of the circle that has center $(5, -2)$ and passes through the origin.

SOLUTION:
Data: Circle has center $(5, -2)$ and passes through the origin.
Required to find: The equation of the circle.
Solution:



The equation of a circle with center (a, b) and radius r is given by $(x-a)^2+(y-b)^2=r^2$.

The radius of the circle $=\sqrt{(5-0)^2+(-2-0)^2}$
$=\sqrt{29}$

So, the equation of the circle is $(x-5)^2+(y-(-2))^2=(\sqrt{29})^2$
$$(x-5)^2+(y+2)^2=29$$

We may express this in another form as $(x-5)(x-5)+(y+2)(y+2)-29=0$
$$x^2-10x+25+y^2+4y+4-29=0$$
$$x^2+y^2-10x+4y=0$$

(b) Determine whether the following pair of lines is parallel.

$$x+y=4$$
$$3x-2y=-3$$

SOLUTION:
Required to determine: Whether or not the lines $x+y=4$ and $3x-2y=-3$ are parallel.
Solution:

$x+y=4$
$y=-x+4$ is of the form $y=mx+c$, where $m=-1$ is the gradient.

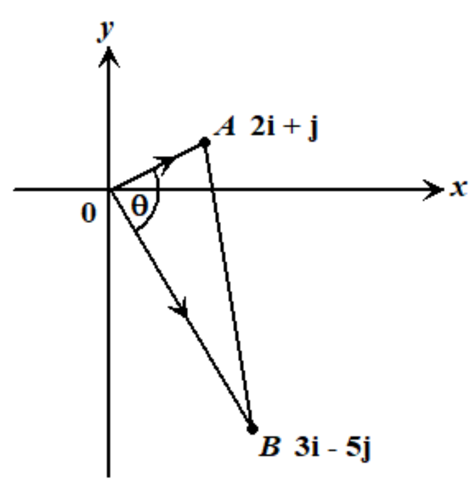
$3x-2y=-3$
$2y=3x+3$
$y=\frac{3}{2}x+1$ is of the form $y=mx+c$, where $m=\frac{3}{2}$ is the gradient.

$-1\neq\frac{3}{2}$
The gradient of the lines are not the same and so they are not parallel since parallel lines have equal gradients.

(c) The position vectors of two points, *A* and *B*, relative to a fixed origin, *O*, are given by $\mathbf{OA}=2\mathbf{i}+\mathbf{j}$ and $\mathbf{OB}=3\mathbf{i}-5\mathbf{j}$, where **i** and **j** represent the unit vectors in the *x* and *y* directions respectively. Calculate:

(i) the magnitude of $\mathbf{AB}$

SOLUTION:
Data: Position vectors of *A* and *B*, relative to *O* are $\mathbf{OA}=2\mathbf{i}+\mathbf{j}$ and $\mathbf{OB}=3\mathbf{i}-5\mathbf{j}$.
Required to find: $\left|\mathbf{AB}\right|$
Solution:

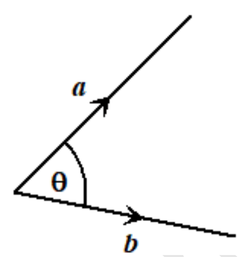


$$\mathbf{AB} = \mathbf{AO} + \mathbf{OB}$$
$$= -(2\mathbf{i} + \mathbf{j}) + (3\mathbf{i} - 5\mathbf{j})$$
$$= \mathbf{i} - 6\mathbf{j}$$

$$|\mathbf{AB}| = \sqrt{(1)^2 + (-6)^2}$$
$$= \sqrt{1 + 36}$$
$$= \sqrt{37}$$

(ii) the angle $A\hat{O}B$, giving your answer to the nearest whole number.

SOLUTION:
Required To Calculate: $A\hat{O}B$
Calculation:
Let θ be $A\hat{O}B$
Recall:



$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$
$$\mathbf{OA}.\mathbf{OB} = |\mathbf{OA}||\mathbf{OB}|\cos\theta$$

$$\mathbf{OA}.\mathbf{OB} = (2 \times 3) + (1 \times -5) = 6 - 5 = 1$$

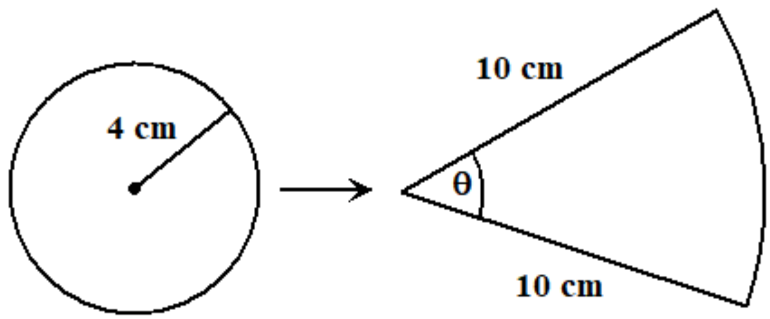
$$|\mathbf{OA}| = \sqrt{(2)^2 + (1)^2} = \sqrt{5}$$
$$|\mathbf{OB}| = \sqrt{(3)^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$$

Hence, $1 = \sqrt{5}\sqrt{34}\cos\theta$

$$\cos\theta = \frac{1}{\sqrt{5}\sqrt{34}}$$
$$= \frac{1}{\sqrt{170}}$$
$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{170}}\right)$$
$$= 85.6^\circ$$
$$\approx 86^\circ \text{ to the nearest degree}$$

4. (a) A wire in the form of a circle with radius 4 cm is reshaped in the form of a sector of a circle with radius 10 cm. Determine, in radians, the angle of the sector, giving your answer in terms of π.

SOLUTION:
Data: A circle of radius 4 cm is formed into a sector of radius 10 cm.
Required to calculate: The angle of the sector
Calculation:
Circle of radius 4 cm $\longrightarrow$ Sector of radius 10 cm



The circumference of the circle will be equal to the perimeter of the sector.

$$\begin{aligned}\text{Circumference of circle} &= 2\pi r\\ &= 2\pi(4)\\ &= 8\pi \text{ cm}\end{aligned}$$

The perimeter of the sector $= r + r + r\theta$ (where θ is the angle of the sector in radians)

$$8\pi = 10 + 10 + 10\theta$$
$$8\pi = 20 + 10\theta$$
$$8\pi - 20 = 10\theta$$
$$\frac{8\pi - 20}{10} = \theta$$
$$\theta = \frac{2(4\pi - 10)}{10}$$
$$\theta = \frac{4\pi - 10}{5} \text{ radians}$$

(b) Solve the equation $\sin^2\theta + 3\cos 2\theta = 2$ for $0 \le \theta \le \pi$. Give your answer(s) to 1 decimal place.

SOLUTION:
Required to solve: $\sin^2\theta + 3\cos 2\theta = 2$ for $0 \le \theta \le \pi$

Solution:

$\sin^2\theta + 3\cos 2\theta = 2$

Recall: $\cos 2\theta = 1 - 2\sin^2\theta$

$\sin^2\theta + 3\left(1 - 2\sin^2\theta\right) = 2$

$$\sin^2\theta + 3 - 6\sin^2\theta = 2$$
$$-5\sin^2\theta = 2 - 3$$
$$-5\sin^2\theta = -1$$
$$5\sin^2\theta = 1$$
$$\sin^2\theta = \frac{1}{5}$$
$$\sin\theta = \pm\sqrt{\frac{1}{5}}$$

$\sin\theta = \sqrt{\frac{1}{5}}$

$\theta = 0.46$

Sin is positive in quadrants 1 and 2.

$\theta = 0.46$

and

$\theta = \pi - 0.46$

$= 2.67$

$\sin\theta = -\sqrt{\frac{1}{5}}$

$\theta = -0.46$

Sin is negative in quadrants 3 and 4.
So there are no solutions for the required range

$\theta = 0.46,\ 2.67$ for $0 \le \theta \le \pi$

$\theta = 0.5$ radians, 2.7 radians correct to 1 decimal place for $0 \le \theta \le \pi$

(c) Prove the identity $\dfrac{1}{1-\sin x} - \dfrac{1}{1+\sin x} \equiv \dfrac{2\tan x}{\cos x}$.

SOLUTION:

Required to prove: $\dfrac{1}{1-\sin x} - \dfrac{1}{1+\sin x} \equiv \dfrac{2\tan x}{\cos x}$

Proof:

L.H.S.:

$$\frac{1}{1-\sin x} - \frac{1}{1+\sin x}$$

$$\frac{(1+\sin x)-(1-\sin x)}{(1-\sin x)(1+\sin x)} = \frac{2\sin x}{(1-\sin x)(1+\sin x)}$$

$$= \frac{2\sin x}{1-\sin^2 x}$$

Recall: $\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x$

$$So\ \frac{1}{1-\sin x}-\frac{1}{1+\sin x}=\frac{2\sin x}{\cos^2 x}$$

Recall: $\tan x=\dfrac{\sin x}{\cos x}$

$$Hence\ \frac{1}{1-\sin x}-\frac{1}{1+\sin x}=\frac{2\sin x}{\cos x}.\frac{1}{\cos x}$$
$$=\frac{2\tan x}{\cos x}$$
$$=\text{R.H.S.}$$

Q.E.D.

SECTION III

Answer BOTH questions.

ALL working must be clearly shown.

5. (a) Given that $y=x^3+2x^2-1$, determine

(i) the coordinates of the stationary points

SOLUTION:
Data: $y=x^3+2x^2-1$
Required to determine: The coordinates of the stationary points
Solution:

Let the gradient function be $\dfrac{dy}{dx}$.

$$\frac{dy}{dx}=3x^{3-1}+2\left(2x^{2-1}\right)-0$$
$$=3x^2+4x$$

At a stationary point, $\dfrac{dy}{dx}=0$

Let $3x^2+4x=0$

$x(3x+4)=0$

$$\therefore x=0 \text{ and } x=-\frac{4}{3}$$

When $x=0$: $y=(0)^3+2(0)^2-1$
$$=-1$$

When $x=-\frac{4}{3}$: $y=\left(-\frac{4}{3}\right)^3+2\left(-\frac{4}{3}\right)^2-1$

$$=-\frac{64}{27}+\frac{32}{9}-1$$

$$=\frac{5}{27}$$

$\therefore$ The stationary points are $(0,\,-1)$ and $\left(-\frac{4}{3},\,\frac{5}{27}\right)$.

(ii) the nature of EACH stationary point

SOLUTION:
Required to find: The nature of each stationary point
Solution:

$$\frac{d^2y}{dx^2}=3\left(2x^{2-1}\right)+4(1)$$
$$=6x+4$$

When $x=0$: $\frac{d^2y}{dx^2}=6(0)+4$
$$=4$$
$$>0$$

So $(0,\,-1)$ is a minimum point.

When $x=-\frac{4}{3}$: $\frac{d^2y}{dx^2}=6\left(-\frac{4}{3}\right)+4$
$$=-4$$
$$<0$$

So $\left(-\frac{4}{3},\,\frac{5}{27}\right)$ is a maximum point.

Alternative Method:
We choose x values just higher and just lower than the x – coordinate of the stationary point and observe the sign change of the gradient.

x	–0.1	0	0.1
$\frac{dy}{dx}$	-ve	0	+ve

So $(0,\,-1)$ is a minimum point.

x	-1.4	$-\frac{4}{3}$	-1.3
$\frac{dy}{dx}$	+ve	0	-ve

So $\left(-\frac{4}{3}, \frac{5}{27}\right)$ is a maximum point.

(b) Differentiate $y = 2x\sqrt{(4-8x)}$ with respect to x, simplifying your answer.

SOLUTION:

Required to differentiate: $y = 2x\sqrt{(4-8x)}$

Solution:

$y = 2x\sqrt{(4-8x)}$ is of the form $y = uv$, where

$u = 2x$ $\quad \frac{du}{dx} = 2$ and

$v = \sqrt{4-8x}$ $\quad$ so $\frac{dv}{dx} = \frac{-4}{\sqrt{4-8x}}$

Let $t = 4 - 8x \Rightarrow \frac{dt}{dx} = -8$

So $v = t^{\frac{1}{2}} \Rightarrow \frac{dv}{dt} = \frac{1}{2}t^{-\frac{1}{2}}$

$$\frac{dv}{dx} = \frac{dv}{dt} \times \frac{dt}{dx} (chain - rule)$$

$$= \frac{1}{2}t^{-\frac{1}{2}} \times -8$$

$$= -4t^{-\frac{1}{2}}$$

$$= -\frac{4}{t^{\frac{1}{2}}}$$

$$= -\frac{4}{\sqrt{4-8x}}$$

Recall: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ $\quad$ (Product law)

$$= \sqrt{4-8x} \times 2 + 2x \times \frac{-4}{\sqrt{4-8x}}$$

$$= \frac{2\sqrt{4-8x}}{1} - \frac{8x}{\sqrt{4-8x}}$$

$$= \frac{2(4-8x)-8x}{\sqrt{4-8x}}$$

$$= \frac{8-16x-8x}{\sqrt{4-8x}}$$

$$= \frac{8-24x}{\sqrt{4-8x}}$$

$$= \frac{8(1-3x)}{\sqrt{4-8x}}$$

NOTE: Some candidates may be knowledgeable of methods beyond the constraints of the syllabus.

Alternative Method:

$$y = 2x\sqrt{4-8x}$$

$$y^2 = 4x^2(4-8x)$$

$$y^2 = 16x^2 - 32x^3$$

Differentiating implicitly w.r.t x:

$$2y\frac{dy}{dx} = 32x - 96x^2$$

$$\frac{dy}{dx} = \frac{32x(1-3x)}{2\left(2x\sqrt{4-8x}\right)}$$

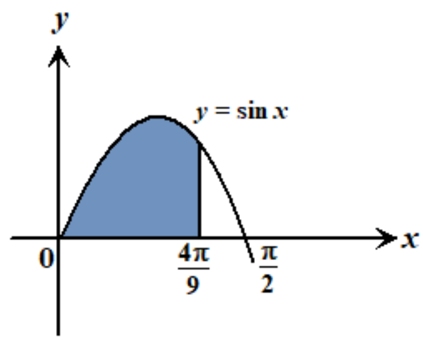
$$= \frac{8(1-3x)}{\sqrt{4-8x}}$$

6. (a) Show, using integration, that the finite area of the curve $y = \sin x$ in the first quadrant bounded by the line $x = \frac{4\pi}{9}$ is smaller than the finite region of $y = \cos x$ in the same quadrant and bounded by the same line.

SOLUTION:

Required to prove: The finite area bounded by $y = \sin x$ and $x = \frac{4\pi}{9}$ is smaller than the finite region bounded by $y = \cos x$ and $x = \frac{4\pi}{9}$.

Proof:



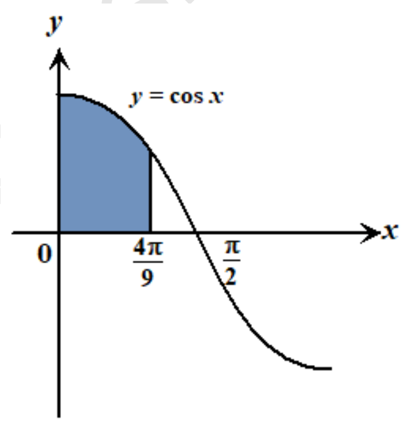
The finite region bounded by $y = \sin x$ and $x = \dfrac{4\pi}{9}$, in the first quadrant, is shown shaded.

Area of the shaded region $= \int_0^{\frac{4\pi}{9}} \sin x \; dx$

$$= \left[-\cos x + C\right]_0^{\frac{4\pi}{9}} \quad C \text{ is a constant}$$

$$= \left(-\cos\frac{4\pi}{9}\right) - \left(-\cos 0\right)$$

$$= 0.826 \text{ correct to 3 decimal places}$$



The finite region bounded by $y = \cos x$ and $x = \dfrac{4\pi}{9}$ in the first quadrant is shown by the shaded region.

Area of the shaded region $= \int_0^{\frac{4\pi}{9}} \cos x \; dx$

$$= \left[\sin x + K\right]_0^{\frac{4\pi}{9}} \quad K \text{ is constant}$$

$$= \sin\frac{4\pi}{9} - \sin 0$$

$$= 0.985 \text{ correct to 3 decimal places}$$

$0.826 < 0.985$

Hence, the area bounded by $y = \sin x$ and $x = \frac{4\pi}{9}$ in the first quadrant is less than the area bounded by $y = \cos x$ and $x = \frac{4\pi}{9}$ in the first quadrant.

Q.E.D.

(b) The finite region in the first quadrant bounded by the curve $y = x^2 + x + 3$, the *x*-axis and the line $x = 4$ is rotated completely about the *x*-axis. Determine the volume of the solid of revolution formed.

SOLUTION:

Data: The region bounded by $y = x^2 + x + 3$, the *x*-axis and $x = 4$ is rotated completely about the *x*-axis.

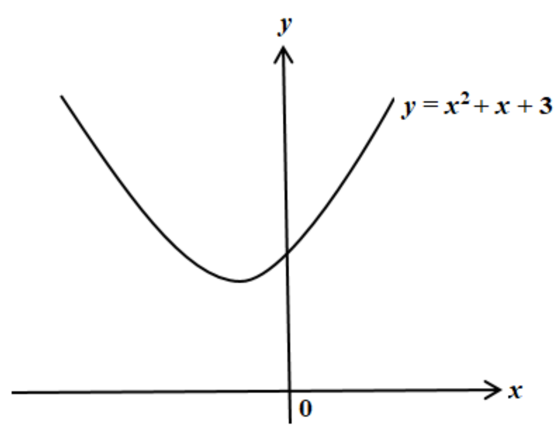
Required to calculate: The volume of the solid generated.

Calculation:

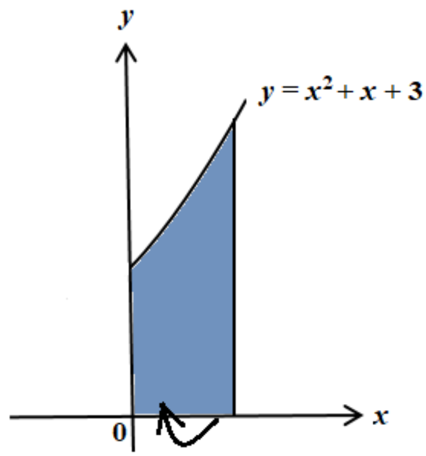
$(1)^2 < 4(1)(3)$

$b^2 < 4ac$ in $y = x^2 + x + 3$, where $a = 1$, $b = 1$ and $c = 3$

So $y = x^2 + x + 3$ is above the *x*-axis.

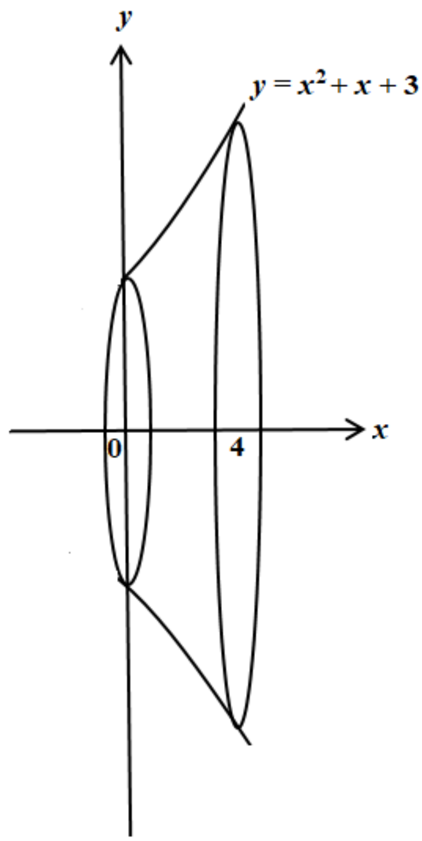


The above diagram shows part of part of $y = x^2 + x + 3$:



The shaded region illustrates the region that is to be rotated 360° or 2π radians or one complete rotation about the x-axis.

The diagram below(not required or necessary) illustrates the shape of the solid generated.



The volume of the solid generated

$= \pi \int_{x_1}^{x_2} y^2 \; dx$

$$= \pi \int_0^4 \left(x^2 + x + 3\right)^2 \, dx$$

$$= \pi \int_0^4 \left(x^4 + 2x^3 + 7x^2 + 6x + 9\right) \, dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{x^4}{2} + \frac{7x^3}{3} + 3x^2 + 9x + C \right]_0^4 \quad C \text{ is a constant}$$

$$= \pi \left\{ \frac{(4)^5}{5} + \frac{(4)^4}{2} + \frac{7(4)^3}{3} + 3(4)^2 + 9(4) \right\} - (0)$$

$$= \pi \left(\frac{1024}{5} + 128 + \frac{448}{3} + 48 + 36 \right)$$

$$= \frac{8492\pi}{15} \text{ cubic units}$$

(c) A curve which has a gradient of $\frac{dy}{dx} = 3x - 1$ passes through the point $A\,(4,\, 1)$. Find the equation of the curve.

SOLUTION:
Data: $\frac{dy}{dx} = 3x - 1$ and the curve passes through $A\,(4,\, 1)$
Required to find: The equation of the curve.
Solution:

$$y = \int \frac{dy}{dx} \, dx$$

$$y = \int (3x - 1) \, dx$$

$$y = \frac{3x^2}{2} - x + C \qquad \text{where } C \text{ is a constant}$$

$A\,(4,\, 1)$ lies on the curve.

$$1 = \frac{3(4)^2}{2} - 4 + C$$

$$1 = 24 - 4 + C$$

$$C = -19$$

$\therefore$ The equation of the curve is $y = \frac{3x^2}{2} - x - 19$.

SECTION IV

Answer only ONE question.

ALL working must be clearly shown.

7. (a) The number of runs scored by a cricketer for 18 consecutive innings is illustrated in the following stem-and-leaf diagram.

0	2	3	6	7	
1	0	3	5	8	9
2	4	4	6	8	
3	1	4	5		
4	5	7			

Key 0|6 means 6

(i) Determine the median score.

SOLUTION:
Data: Stem and leaf diagram showing the runs scored by a cricketer in 18 innings.
Required to find: The median score
Solution:
The middle of 18 is 9 and 10.
From the diagram
The 9^{th} score is 19.
The 10^{th} score is 24.
So the median is

$$\begin{array}{ccc} 9^{th} & & 10^{th} \\ 19 & & 24 \\ & \uparrow & \\ & \text{Median} & \end{array}$$

$$\text{Median } = \frac{19+24}{2}$$
$$= \frac{43}{2}$$
$$= 21.5$$

(ii) Calculate the interquartile range of the scores.

SOLUTION:
Required to calculate: The interquartile range of the scores
Calculation:
To locate the lower and upper medians we include the 9^{th} and the 10^{th}

scores as neither were the median score.
The lower quartile, Q_1, is the middle value from the 1st to the 9th value and which is the 5th value.
Lower quartile, $Q_1 = 10$

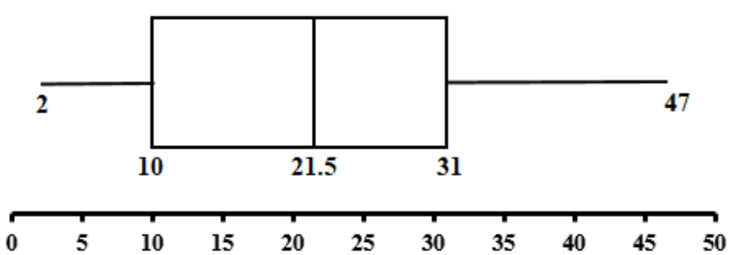
The upper quartile, Q_3, is the middle value from the 10th to the 18th value and which is the 14th value.
Upper quartile, $Q_3 = 31$

The interquartile range, (I.Q.R.) $= \text{Upper quartile} - \text{Lower quartile}$

$$= Q_3 - Q_1$$
$$= 31 - 10$$
$$= 21$$

(iii) Construct a box-and-whisker plot to illustrate the data and comment on the shape of the distribution.

SOLUTION:
Required to construct: A box-and-whisker plot to illustrate the data and comment its shape.
Solution:

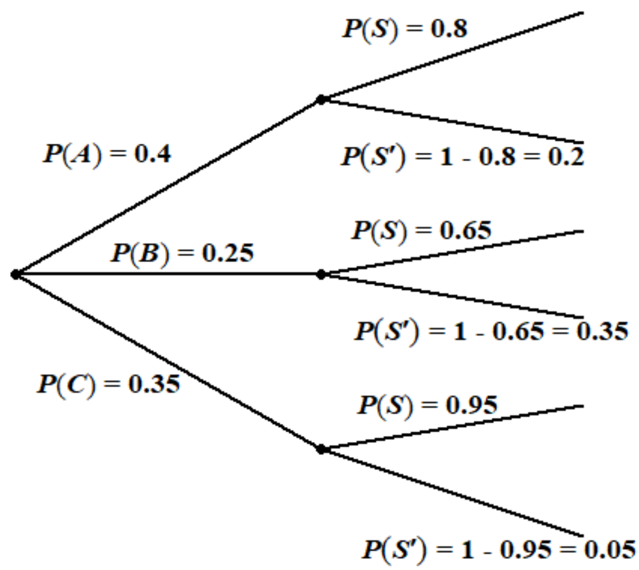


The median is located just around the middle of the box and which indicates that the data is almost symmetric. However, the fourth quartile which is the right whisker is noticeably longer than the other three quartiles. This indicates that there is more variability among the larger scores than among the smaller scores.

(b) Insecticides A, B or C are applied on lots Q, R and S. The same crop is planted on each lot and the lots are of the same size. The probability that a group of farmers will select A, B or C is 40%, 25% and 35% respectively. The probability that insecticide A is successful is 0.8, that B is successful is 0.65 and that C is successful is 0.95.

(i) Illustrate this information on a tree diagram showing ALL the probabilities on ALL branches.

SOLUTION:
Data: The data tells of the types of insecticides A, B and C used on lots P, Q and R and their success rate.
Required To Illustrate: The data given on a tree diagram
Solution:
Based on the data, we let the probability $P(X)$ be the probability of choosing X and $P(S)$ be the probability of success.



(ii) An insecticide is selected at random, determine the probability that is unsuccessful.

SOLUTION:
Required to calculate: P(Insecticide chosen at random is unsuccessful)
Calculation:

$$\begin{aligned} P(\text{Insecticide is unsuccessful}) &= P\left(A\cap S' \text{ or } B\cap S' \text{ or } C\cap S'\right) \\ &= P\left(A\cap S'\right)+P\left(B\cap S'\right)+P\left(C\cap S'\right) \\ &= \left(0.4\times 0.2\right)+\left(0.25\times 0.35\right)+\left(0.35\times 0.05\right) \\ &= 0.08+0.0875+0.0175 \\ &= 0.185 \end{aligned}$$

(c) A regular six-sided die is tossed 2 times.

(i) Calculate the probability of obtaining a 5 on the 2^{nd} toss, given that a 5 was obtained on the 1^{st} toss.

SOLUTION:
Data: A die is tossed two times.

Required to calculate: The probability of obtaining a '5' on the second toss given that a '5' was obtained on the first toss.
Calculation:
Let X represent the event that a '5' is obtained on the second toss.
Let Y represent the event that a '5' was obtained on the first toss.

$$
\begin{aligned}
P(X \text{ given } Y) &= P(X \mid Y) \\
&= \frac{P(X \cap Y)}{P(Y)} \\
&= \frac{\frac{1}{6} \times \frac{1}{6}}{\frac{1}{6}} \\
&= \frac{1}{6}
\end{aligned}
$$

(ii) Determine the probability that a 5 is obtained on both tosses.

SOLUTION:
Required to calculate: The probability of obtaining a '5' on both tosses.
Calculation:

$$
\begin{aligned}
P(5 \text{ and } 5) &= P(5) \times P(5) \\
&= \frac{1}{6} \times \frac{1}{6} \\
&= \frac{1}{36}
\end{aligned}
$$

(iii) Explain why the answers in (c) (i) and (c) (ii) are different.

SOLUTION:
Required to explain: Why the answers to (i) and (ii) are different.
Solution:
In (ii) $P(5 \text{ and } 5)$, we are combining two independent events, that is, events in which the occurrence of the first does not affect the outcome of the second. The combined probability is calculated using the multiplication rule, where $P(X \cap Y) = P(X)P(Y)$.

In (i) $P(X|Y)$ is a conditional probability for independent events. It requires the probability of X only given that Y has occurred before. Since the occurrence of a 5 on the second toss is not affected by the occurrence of 5 on the first toss, $P(X|Y)=P(X)$, as shown below

$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)} = \frac{P(X)P(Y)}{P(Y)} = P(X)$$

8. (a) A particle moves in a straight line so that its distance, s metres, after t seconds, measured from a fixed point, O, is given by the function $s = t^3 - 2t^2 + t - 1$.

Determine

(i) its velocity when $t = 2$

SOLUTION:
Data: Particle moves in a straight line so that its distance, s m, from O, after t s is given by $s = t^3 - 2t^2 + t - 1$.
Required to calculate: Velocity when $t = 2$
Calculation:
Let velocity at the time, t, be v.

$v = \frac{ds}{dt}$ (Definition)

$$v = 3t^{3-1} - 2\left(2t^{2-1}\right) + 1 - 0$$

$$v = 3t^2 - 4t + 1$$

When $t = 2$

$$v = 3(2)^2 - 4(2) + 1 \text{ ms}^{-1}$$
$$= 12 - 8 + 1$$
$$= 5 \text{ ms}^{-1}$$

(ii) the values of t when the particle is at rest

SOLUTION:
Required to calculate: t when the particle is at rest
Calculation:
When the particle is at rest, $v = 0$

Let $3t^2 - 4t + 1 = 0$

$$(3t - 1)(t - 1) = 0$$

$$\therefore t = \frac{1}{3} \text{ or } 1$$

So the particle is at rest when $t = \frac{1}{3}$ and again at $t = 1$.

(iii) the distance between the rest points

SOLUTION:
Required to calculate: The distance between the two rest points
Calculation:
When $t = \frac{1}{3}$

$$s = \left(\frac{1}{3}\right)^3 - 2\left(\frac{1}{3}\right)^2 + \frac{1}{3} - 1$$
$$= \frac{1}{27} - \frac{2}{9} + \frac{1}{3} - 1$$
$$= -\frac{23}{27} \text{ m}$$

When $t = 1$
$$s = (1)^3 - 2(1)^2 + 1 - 1$$
$$= 1 - 2 + 1 - 1$$
$$= -1\text{m}$$

The distance between the rest points = $-\frac{23}{27} - (-1) = \frac{4}{27}$ m

The distance between the rest points = $\frac{4}{27}$ m

(iv) the time at which the maximum velocity occurs.

SOLUTION:
Required to calculate: The time at which maximum velocity occurs.
Calculation:
At maximum velocity, acceleration $= 0$
Let a be the acceleration at time t.

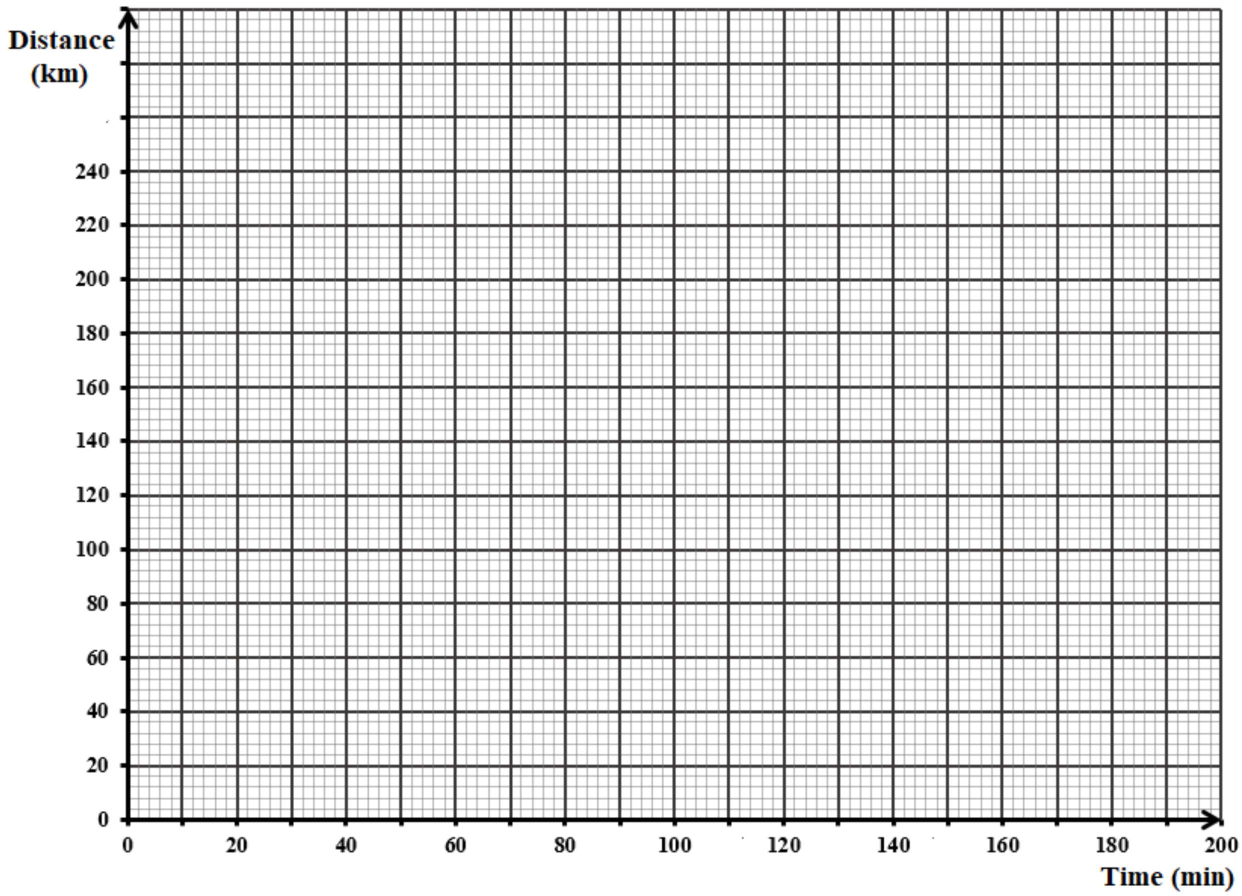
$$a = \frac{dv}{dt} \quad \text{(Definition)}$$
$$a = 3\left(2t^{2-1}\right) - 4(1)$$
$$a = 6t - 4$$

When $a = 0$
$$6t - 4 = 0$$
$$6t = 4$$
$$t = \frac{2}{3}$$

$\therefore$ Maximum velocity occurs at $t = \dfrac{2}{3}$.

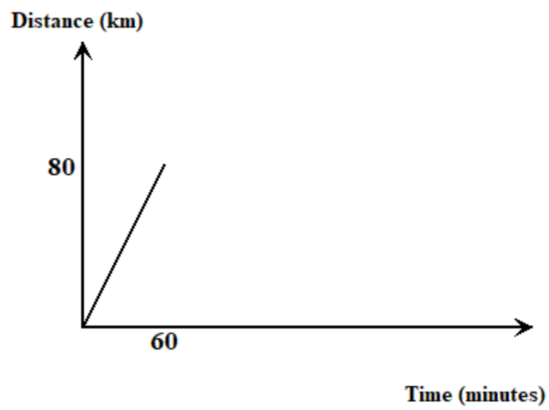
(b) A bus starts from rest at Station A and travels a distance of 80 km in 60 minutes to Station B. Since the bus arrived at Station B early, it remained there for 20 minutes then started the journey to Station C. The time taken to travel from Station B to Station C was 90 minutes at an average speed of 80 kmh^{-1}.

(i) On the grid provided, draw a distance-time graph to illustrate the motion of the bus.

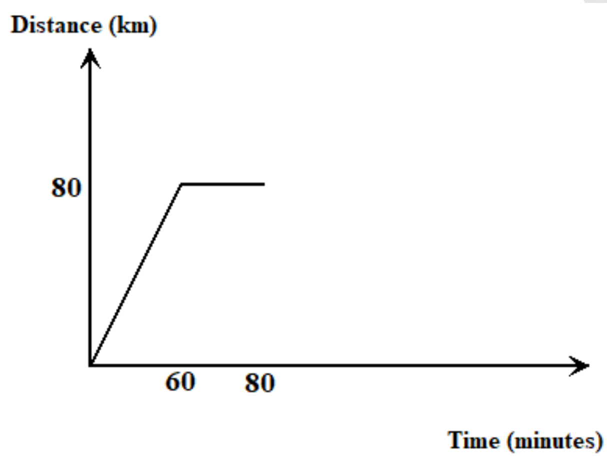


Distance vs Time

SOLUTION:
Data: The journey of a bus from Station A to Station C.
Required to draw: A distance-time graph to illustrate the journey of the bus.
Solution:
Phase from A to B:
The bus starts from rest at A, therefore, initial velocity $= 0\ \text{ms}^{-1}$ at A.
Distance after 60 minutes is 80 km.



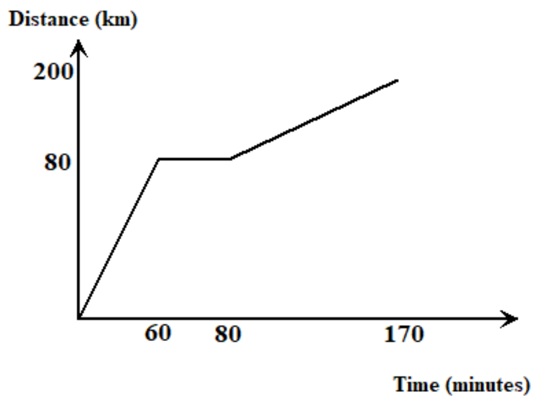
The branch will be a straight line if we **assume** constant velocity.
The bus remains for 20 minutes at B.
Hence, from 60 to $60+20=80$ minutes, the distance covered is still 80 km.



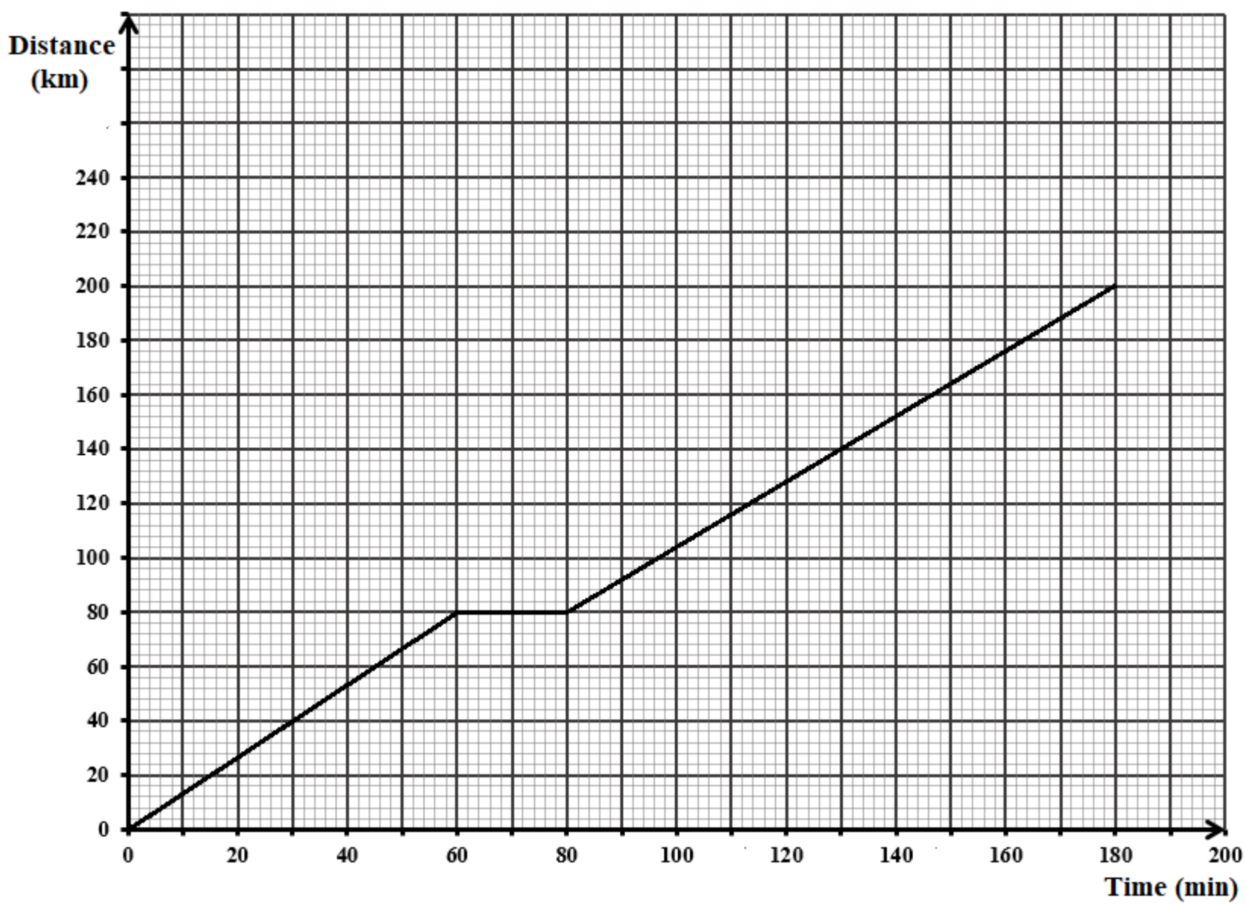
Phase from B to C:

$$90 \text{ minutes} = \frac{90}{60} \text{ hours}$$
$$= 1\frac{1}{2} \text{ hours}$$

$$\text{Distance covered} = \text{Average speed} \times \text{Time}$$
$$= 80 \text{ kmh}^{-1} \times 1\frac{1}{2} \text{ hours}$$
$$= 120 \text{ km}$$



The completed distance-time graph looks like:



Distance vs Time

(ii) Determine the distance from Station B to Station C.

SOLUTION:
Required to calculate: The distance from B to C
Calculation:
This had to be done is part (i) of the question so as to accurately complete

the distance-time graph.

(iii) Determine the average speed from Station A to Station B, in kmh^{-1}.

SOLUTION:
Required To Find: The average speed from Station A to Station B in kmh^{-1}.
Solution:

The average speed from A to B in $\text{kmh}^{-1} = \dfrac{\text{Distance in km}}{\text{Time in hours}}$

$$= \frac{80 \text{ km}}{60 \text{ mins}}$$

$$= \frac{80 \text{ km}}{1 \text{ hour}}$$

$$= 80 \text{ kmh}^{-1}$$

